INTRODUCTION TO COSMOLOGY
FINAL EXAM

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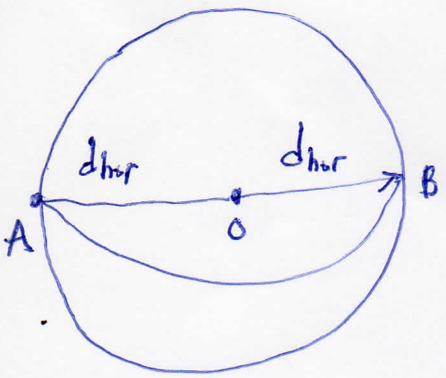
1) a) What is the flatness problem (of the expanding universe)?

The density parameter $\Omega(t)$ defined by $1-\Omega(t) = -k\dfrac{c^2}{R_0^2 a(t)^2 H(t)^2}$ was very close to zero, or $\Omega(t)$ was very close to one. Today $|1-\Omega_0| \lesssim 0.005$. Why is Ω_0 not much less than or more than one, but close to one?

This is the flattness problem

b) What is the horizon problem?



Points A and B have a distance of $2d_{hor}(t_0)$ today. So, they were not in causal contact in the early universe. But the temperatures T_A & T_B are the same to one part in 10^5. What made the temperatures so close?

c) What is the monopole problem?

Grand unified theories predict the existance of magnetic monopoles. Their predicted number per cubic meter is quite large. However, we do not observe the monopoles at all? What suppressed their abundance?

d) What is the theory that solves these problems called?

The Inflationary Universe theory.

2) a) For a homogeneous scalar field $\phi(t)$ write down the expressions for the pressure and the energy density.

$$P(t) = \frac{1}{2}\dot{\phi}(t)^2 - V(\phi)$$

$$\mathcal{E}(t) = \frac{1}{2}\dot{\phi}(t)^2 + V(\phi)$$

b) Starting from the conservation of energy equation, obtain the equation satisfied by $\phi(t)$ in an expanding universe.

$$dE + p\,dV = 0$$

$$d\left[\left(\frac{1}{2}\dot{\phi}^2 + V\right)a^3\right] + \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)da^3 = 0$$

$$\left(\dot{\phi}\ddot{\phi} + \dot{V}\right)a^3 + 3a^2\dot{a}\left(\frac{1}{2}\dot{\phi}^2 + \cancel{V}\right) + \left(\frac{1}{2}\dot{\phi}^2 - \cancel{V}\right)3a^2\dot{a} = 0$$

$$\dot{V} = \frac{dV}{dt} = \frac{dV}{d\phi}\frac{d\phi}{dt} = V'\dot{\phi}.$$ Dividing by $a^3\dot{\phi} \Rightarrow$

$$\boxed{\ddot{\phi} + V' + 3\frac{\dot{a}}{a}\dot{\phi} = 0}$$

c) What is the equation of state for $\phi(t)$, and what is its approximate value? (Show your work.)

$$P = w_\phi \mathcal{E}$$

$$w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}, \quad \text{when } \dot{\phi}^2 \ll V$$

$$w_\phi \simeq -1.$$

3) a) In a region of space of radius R full of gas, what is meant by hydrostatic equilibrium?

The inward gravitational force is opposed by an outward pressure gradient. When the two are equal no gravitational collapse occurs.

b) What is the time it takes for the pressure gradient to build up?

$$t_{pre} \sim \frac{R}{c_s},$$

where c_s is the local speed of sound; $c_s = c\left(\frac{dP}{d\mathcal{E}}\right)^{1/2} = c\sqrt{w}$

c) i) Write down the expression for the dynamical time for collapse?

$$t_{dyn} = 1/\sqrt{4\pi G\rho}$$

ii) Obtain the Jeans length from part b) and ci)

$$\lambda_J = 2\pi c_s t_{dyn} = \cancel{2}\pi c_s \frac{\sqrt{c^2}}{\cancel{2}\sqrt{\pi G\mathcal{E}}} = c_s\left(\frac{\pi c^2}{G\mathcal{E}}\right)^{1/2}$$

d) What is the physical significance of Jeans length for collapse?
Overdense regions larger than λ_J collapse under their own gravity. Overdense regions smaller than λ_J oscillate in density, they constitute stable standing sound waves.

e) Why do not density fluctuations in the air collapse?
The average ~~size~~ of the atmosphere is ~ 12 km which is much less than λ_J.

4) In a multiple-component universe the density fluctuation δ is due to the fluctuation in the density of matter alone.

a) Define δ in terms of $\mathcal{E}_m$ and $\bar{\mathcal{E}}_m$

$$\delta = \frac{\mathcal{E}_m - \bar{\mathcal{E}}_m}{\bar{\mathcal{E}}_m}$$

b) Write down the equation satisfied by δ.

$$\ddot{\delta} + 2H\dot{\delta} = \frac{4\pi G}{c^2}\bar{\mathcal{E}}_m \delta \;;\; \Omega_m = \frac{\bar{\mathcal{E}}_m}{\mathcal{E}_c} = \frac{8\pi G \bar{\mathcal{E}}_m}{3c^2 H^2}$$

$$\Rightarrow \ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0$$

c) In the cosmological constant dominated phase, obtain the solution for $\delta(t)$.

In this era $H = H_\Lambda =$ constant and $\Omega_m \ll 1$ so that

$$\ddot{\delta} + 2H_\Lambda \dot{\delta} \approx 0$$

$$\text{solution: } \delta(t) \approx C_1 + C_2 e^{-2H_\Lambda t}$$

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5 a) Write down the Fourier integral expression for density fluctuation $\delta(\vec{r})$ in terms of δ_k in a comoving volume V.

$$\delta(\vec{r}) = \frac{V}{(2\pi)^3}\int \delta_k e^{-i\vec{k}\cdot\vec{r}} d^3k$$

b) Write down the expression satisfied by δ_k.

$$\delta_k = \frac{1}{V}\int \delta(\vec{r})\, e^{i\vec{k}\cdot\vec{r}} d^3r$$

c) Write down the definition of the power spectrum $P(k)$

$$P(k) = \langle |\delta_k|^2 \rangle$$

In inflationary density fluctuations

d) What is the power law expression that describes the power spectrum well?

$$P(k) \propto k^n$$

e) What is the prediction of inflation for the power spectrum?

Inflation predicts $n = 1$.

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6) a) What is a Harrison-Zeldovich spectrum?

A power spectrum ~~with~~ $P(k) \propto k^n$ with $n=1$ exactly is referred to as a Harrison-Zeldovich spectrum.

b) What is a scale invariant Hárrison-Zeldovich spectrum?

Potential fluctuations $\delta \Phi \sim$ const.

7) Suppose that the universe inflates as the scalar field ϕ rolls slowly down the potential $V(\phi) = \frac{1}{2} m^2 \phi^2$.

a) Obtain from the Friedmann equation the time dependence of the scale factor.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \varepsilon = \frac{4\pi G}{3c^2} m^2 \phi^2 = H_\phi^2 \approx \text{constant}$$

$$\frac{\dot{a}}{a} = H_\phi \;, \quad \int_{a_i}^{a} \frac{da}{a} = H_\phi \int_{t_i}^{t} dt \;; \quad \ln\left(\frac{a}{a_i}\right) = H_\phi (t - t_i)$$

$$\Rightarrow \quad a(t) = a_i \, e^{H_\phi (t - t_i)}$$

where $H_\phi = \sqrt{\frac{4\pi G}{3c^2} m^2 \phi^2}$

b) Find the condition on ϕ imposed by the flatness condition.

$$V(\phi) = \frac{1}{2} m^2 \phi^2$$

$$V'(\phi) = m^2 \phi$$

$$\frac{V'}{V} = \frac{2}{\phi} \;; \quad \left(\frac{V'}{V}\right)^2 = \frac{4}{\phi^2} \ll 16\pi G$$

$$\Rightarrow \quad |\phi| \gg \frac{1}{2\sqrt{\pi G}}$$

8) a) How many space dimensions are there in the Robertson-Walker metric?

Three (r, θ, ϕ)

b) What is the dimension of the embedding space of the universe?

Four

c) Is the embedding space of the universe flat or curved?

Flat

d) Write down the definition of the redshift if the observed wavelength of light is λ_o and the emitted wavelength is λ_e.

$$z = (\lambda_o - \lambda_e)/\lambda_e$$

e) Relativistic Doppler effect yields $\dfrac{\lambda_o}{\lambda_e} = \sqrt{\dfrac{1+\frac{v}{c}}{1-\frac{v}{c}}}$.

Find the approximate value of the redshift for $(v/c) \ll 1$.

$$z = \lambda_o/\lambda_e - 1 \approx \left(1+\tfrac{1}{2}\tfrac{v}{c}\right)/\left(1-\tfrac{1}{2}\tfrac{v}{c}\right) - 1 \approx \left(1+\tfrac{1}{2}\tfrac{v}{c}\right)\left(1+\tfrac{1}{2}\tfrac{v}{c}\right) - 1 \approx 1 + \tfrac{v}{c} - 1 + \tfrac{1}{4}\left(\tfrac{v}{c}\right)^2 \approx v/c$$

f) Write down the Hubble law for a galaxy at a distance d from us and with a velocity v.

$$v = H_o d$$

g) Write down the Robertson-Walker metric at a fixed time t and find the horizon distance from the observer and a galaxy.

$$ds^2 = 0 + a(t)^2 \left[dr^2 + S_k^2 d\Omega^2\right]$$

h) State True or False: The ratio of the photon energy density to the neutrino energy density today is less than one.

False

i) State True or False: In a single component Lambda only universe the scale factor $a(t)$ increases with time exponentially.

True

j) State True or False: In a $k=1$ universe the present value of the density parameter Ω_o is equal to one.

False

9) a) What was the temperature of the radiation-dominated universe at the Planck time?

$$T(K) = 10^{10}\left(t(\text{sec})\right)^{-1/2} \;;\; t_{Pl} \approx 10^{-44}\,\text{sec}$$

$$\Rightarrow T_{Pl} \approx 10^{32}\,K.$$

b) Show from the energy density in radiation that aT = constant, where a and T are the scale factor and the temperature, respectively.

$$\varepsilon_r = \alpha_{SB} T^4 = \varepsilon_{r,0}/a^4 \Rightarrow (aT)^4 = \frac{\varepsilon_{r,0}}{\alpha_{SB}} = \text{const.}$$

$$\Rightarrow aT = (\text{const.})^{1/4} = \left(\varepsilon_{r,0}/\alpha_{SB}\right)^{1/4} = \text{const.}$$

c) Given that $T_0 = 2.7255\,K$, $T_{Pl} = 1.42 \times 10^{32}\,K$, find the value of a_{Pl} (assuming that inflation did not occur.)

$$aT = a_0 T_0 = T_0 \;,\; (\text{because } a_0 = 0)$$

$$a_{Pl} = \frac{T_0}{T_{Pl}} = \left(\frac{2.73}{1.42}\right) \times 10^{-32}$$

d) Show that in the radiation-dominated era the mean photon energy is given by $E_\gamma = \dfrac{\alpha_{SB} T_0^3}{n_{\gamma,0} k_B}(k_B T)$.

$$E_\gamma = \frac{\varepsilon_\gamma}{n_\gamma} \;;\; n_\gamma(t)\,a(t)^3 = n_{\gamma,0}\,a_0^3 = n_{\gamma,0}$$

$$E_\gamma = \frac{\alpha_{SB} T^4 a(t)^3}{n_{\gamma,0}} = \frac{\alpha_{SB}}{n_{\gamma,0}} \frac{k_B}{k_B} T (aT)^3 = \frac{\alpha_{SB} (a_0 T_0)^3 T k_B}{n_{\gamma,0} k_B}$$

$$= \frac{\alpha_{SB} T_0^3 (k_B T)}{n_{\gamma,0} k_B}$$

10) a) The rate at which neutrons and protons interact with neutrinos is given by $\Gamma \approx G_F^2 T^5$, where G_F is the Fermi constant of weak interactions and $\hbar = c = 1$. In the early radiation-dominated universe the Hubble parameter is given by $H = 1.66\sqrt{g_*}\sqrt{G}\,T^2$, where g_* is the number of spin states and $\hbar = c = 1$.

a) Find the temperature T_f at which the ratio of the number of neutrons to the number of photons is frozen.

This occurs when $\Gamma \approx H$

$$G_F^2 T_f^5 = (\text{const.}) T_f^2 \Rightarrow T_f = \left[\frac{1.66\sqrt{g_*}\sqrt{G}}{G_F^2}\right]^{1/3}$$

b) For an object of luminosity L at a distance d_L from us, define the apparent magnitude m and the absolute magnitude M.

$$m = -2.5\log\left(\frac{L}{4\pi d_L^2}\right) ; \quad M = -2.5\log\left(\frac{L}{4\pi(10\,psc)^2}\right)$$

c) Obtain the expression for "distance modulus" $m-M$ when d_L is measured in parsecs.

$$m - M = -2.5\left[\cancel{\log\frac{L}{4\pi}} - 2\log d_L - \cancel{\log\frac{L}{4\pi}} + 2\log 10\right]$$

$$= 5(\log_{10} d_L - 1)$$

d) Find the distance d_L to an object whose distance modulus is 30.

$$\log_{10}(d_L) = \frac{m-M}{5} + 1 = (m - M + 5)/5$$

$$d_L = 10^{(m-M+5)/5} = 10^{(30+5)/5} = 10^{7}\ psc.$$

in psc