INTRODUCTION TO COSMOLOGY

SECOND EXAM

Instructor: Murat Özer 11/01/2021

Student's Name:

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1. a) Write down the relation between flux f, luminosity L, and luminosity distance d_L.

$$f = \frac{L}{4\pi d_L^2}$$

b) What is the relation between L and period P for Cepheids observed by H. S. Leavitt?

$$L = \text{Const.}\, P = f 4\pi d_L^2$$

c) If the pulsational periods ~~of~~ P_A and P_B of Cepheids A and B are equal and if $\bar{f}_A / \bar{f}_B = 100$, find $d_L(A)$ in terms of $d_L(B)$.

$$P_A = P_B \Rightarrow \bar{f}_A d_L^2(A) = \bar{f}_B d_L^2(B)$$

$$\frac{\bar{f}_A}{\bar{f}_B} = 100 = \frac{d_L^2(B)}{d_L^2(A)}$$

$$\Rightarrow \boxed{d_L(A) = \frac{d_L(B)}{10}}$$

2. a) For an object of luminosity L at a distance d_L from us, define the apparent magnitude m and absolute magnitude M.

$$m = -2.5 \log\left(\frac{L}{4\pi d_L^2}\right); \quad M = -2.5 \log\left(\frac{L}{4\pi (10\,pc)^2}\right)$$

b) Obtain the expression for "distance modulus" $m-M$ when d_L is measured in parsecs.

$$m-M = -2.5\left[\cancel{\log L} - \cancel{\log 4\pi} - \log d_L^2 - \cancel{\log L} + \cancel{\log 4\pi} + \log (10)^2\right]$$

$$= 5 \log d_L^{(pc)} - 5$$

c) From your answer in (b), obtain $m-M$ when distances are measured in Megaparsecs.

$$m-M = 5\log\left(\frac{d_L(Mpc) \times 10^6}{1Mpc}\right) - 5 = 5\log\left(\frac{d_L}{1Mpc}\right) + 30 - 5$$

$$= 5\log\left(\frac{d_L}{1Mpc}\right) + 25$$

d) Find the distance d_L to an object whose distance modulus is 35.

$$\frac{m-M-25}{5} = \log\left(\frac{d_L}{1Mpc}\right)$$

$$\Rightarrow d_L(Mpc) = 10^{(m-M-25)/5} = 10^{(35-25)/5} = 10^2$$

$$= 100\,Mpc.$$

3. a) The scale factor $a(t)$ can be expanded in a Taylor series around the present moment $t=t_0$. Keeping the first two terms $a(t)$ is given as

$$a(t) \approx 1 + H_0(t-t_0), \text{ where } H_0 \equiv \left.\frac{\dot{a}}{a}\right|_{t=t_0}$$

Calculate the proper distance $d_p(t_0)$ of a light signal emitted at $t=t_e$.

$$d_p(t_0) = c\,a(t_0)\int_{t_e}^{t_0}\frac{dt}{a(t)} = c\int_{t_e}^{t_0}\frac{dt}{1+H_0\cdot(t-t_0)} = c\int_{t_e}^{t_0}\left[1-H_0\cdot(t-t_0)\right]dt$$

Let $\xi = t-t_0$; $d\xi = dt$

$$d_p(t_0) = c\int_{t_e-t_0}^{0}(1-H_0\xi)\,d\xi = c\left(\xi - H_0\frac{\xi^2}{2}\right)\Bigg|_{-(t_0-t_e)}^{0}$$

$$\cong c(t_0-t_e) + \frac{cH_0}{2}(t_0-t_e)^2$$

b) Express $d_p(t_0)$ in terms of $t_L \equiv t_0 - t_e$. What is t_L called?

$$d_p(t_0) \approx c\,t_L + \frac{cH_0}{2}t_L^2$$

t_L is called the lookback time.

c) What are the physical meanings of the two terms in $d_p(t_0)$?

$c\,t_L$ is what the proper distance would be in a static universe

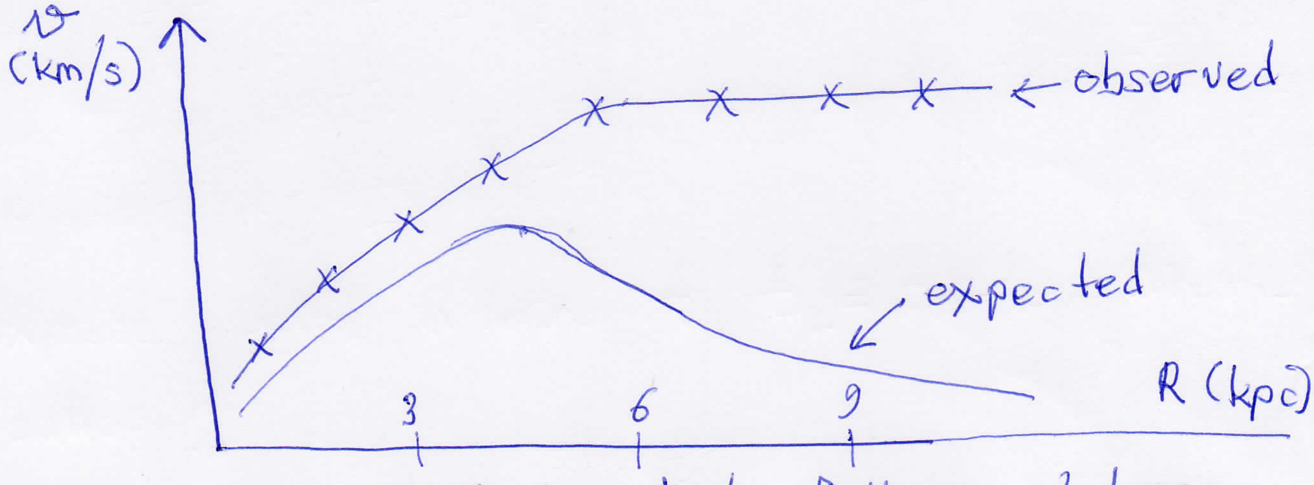
$c\dfrac{H_0}{2}t_L^2$ correction due to the expansion of the universe

4. a) Suppose that a star is in a circular orbit around the center of its galaxy. Obtain the linear velocity ϑ of the star as a function of $M(R)$ and R, where $M(R)$ is the mass contained within a sphere of radius R.

$$F = m_* a = m_* \frac{\vartheta^2}{R} = \frac{G m_* M(R)}{R^2}$$

$$\Rightarrow \quad \vartheta = \sqrt{\frac{GM(R)}{R}}$$

b) Draw a diagram of ϑ vs R for a typical galaxy and show the curves due to observations and visible matter.



c) What is the conclusion that follows at large radii?

We deduce the presence of a dark halo within which the visible stellar disk is embedded.

5. a) Consider a system of N particles (galaxies) interacting gravitationally. Write down the expression for the virial theorem when the moment of inertia I of the system is constant.

The virial theorem: $\ddot{I} = 2\underset{\text{pot. ener.}}{W} + 4\underset{\text{kin. ener.}}{K} = 0 \Rightarrow K = -\frac{1}{2}W$

b) Express the total mass M of the system in terms of the average of the square of the velocities of the particles and the half-mass radius r_h of the system.

$$\frac{1}{2}M\langle v^2\rangle = \frac{1}{2}\alpha\frac{GM^2}{r_h}$$

c) What is the ratio $M_{calculated}/M_{observed}$ equal to for the Coma cluster?

For Coma cluster: $\dfrac{M_{calc.}}{M_{observ.}} \approx 10$

d) What is the conclusion that follows from part c?

There must be invisible mass ~~to~~ necessary to hold the cluster together.

6. a) Write down the expression for the pressure and the energy density for a homogeneous scalar field $\phi(t)$.

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$
$$\mathcal{E} = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad ; \quad \left(\dot{\phi} \equiv \frac{d\phi}{dt}\right)$$

b) Obtain the equation satisfied by $\phi(t)$ using the conservation of energy equation in an adiabatically expanding universe.

Adiabatic expansion $\Rightarrow dQ = 0 \Rightarrow dE + p\,dV = 0$ volume

$$\frac{d}{dt}(\mathcal{E}a^3) + p\frac{d}{dt}(a^3) = 0$$

$$\frac{d}{dt}\left[\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)a^3\right] + \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)\frac{d}{dt}(a^3) = 0$$

$$\left(\ddot{\phi}\dot{\phi} + \frac{dV}{d\phi}\cdot\dot{\phi}\right)a^3 + \left(\frac{1}{2}\dot{\phi}^2 + \cancel{V}\right)3a^2\dot{a} + \left(\frac{1}{2}\dot{\phi}^2 - \cancel{V}\right)3a^2\dot{a} = 0$$

Dividing by $\dot{\phi}a^3 \Rightarrow \ddot{\phi} + \frac{dV}{d\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = 0$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0$$

c) Obtain the equation satisfied by $\phi(t)$ in the steady state model of the universe.

Steady state model: $\dot{a} = 0$, $V'(\phi) = 0$

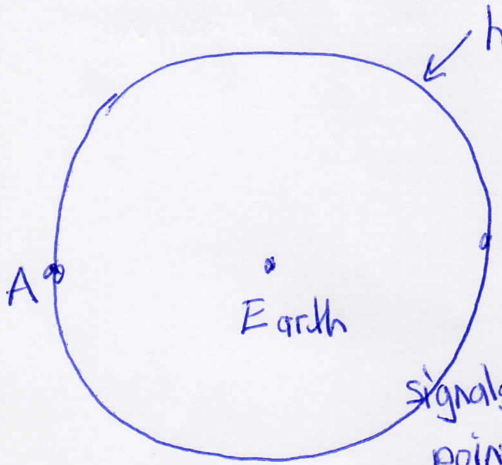
$$\Rightarrow \ddot{\phi} = 0 \Rightarrow \dot{\phi} = \text{const.}$$

7. a) What is the flatness problem of the expanding universe?

$$|1 - \Omega(a_{early})| \leq 10^{-N}, \text{ where } N \gg 1$$

The near-flatness ~~of~~ observed today, $\Omega(a_0) \approx 1$ requires an extreme fine tuning of Ω close to 1 in the early universe.

b) What is the horizon problem?



Points A and B are separated by a distance $d_{AB}(t_0) \approx 2 d_{hor}(t_0)$.

They are causally disconnected by light signals according to the standard model. But the two points have the same temperature to within one percent in 10^5. How can this be so?

c) What is the monopole problem?

$$\frac{\text{Density of monopoles now}}{\text{Density of baryons now}} \approx 167 \text{ (according to calculations)}$$

But we do not see any monopoles! How is this explained?

d) What is the theory that solves these problems called?

The Inflationary Universe Model

8. a) Suppose that the universe starts to inflate exponentially at $t=t_i$ and inflation ends at $t=t_f$. Write down the expressions for the scale factor $a(t)$ for the periods $t<t_i$, $t_i<t<t_f$, and $t>t_f$.

$$a(t)=\begin{cases} a_i\left(\frac{t}{t_i}\right)^{1/2} ; & t<t_i \\ a_i e^{H(t-t_i)} ; & t_i<t<t_f \\ a_f\left(\frac{t}{t_f}\right)^{1/2} ; & t>t_f \,,\; a_f=a_i e^{H(t_f-t_i)} \end{cases}$$

b) Find the ratio $a(t_f)/a(t_i)$ in terms of the constant H and the times t_f and t_i.

$$\frac{a(t_f)}{a(t_i)}=\frac{a_f}{a_i}=\frac{a_i e^{H(t_f-t_i)}}{a_i}=e^{H(t_f-t_i)}$$

c) The number of e-foldings of inflation is defined as $N\equiv H(t_f-t_i)$. What is the value of N that solves the cosmological problems?

$$e^N=e^{H(t_f-t_i)} \;;\; N\approx 60$$

9. a) Write down the (approximate) expression between the time in seconds and the temperature in Kelvins in the very early universe.

$$t(\text{sec}) \approx \frac{10^{20}}{[T(K)]^2}$$

b) What was the temperature of the (radiation-dominated) universe when $t = 1$ sec. ?

$$T \approx \frac{10^{10} K}{\sqrt{t}} = \frac{10^{10}}{1} K = 10^{10} K$$

c) Write down the expressions for the number densities of neutrons and protons when all the existing particles were in thermal equlibrium.

$$n_n = g_n \left(\frac{m_n k_B T}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{m_n c^2}{k_B T}\right)$$

$$n_p = g_p \left(\frac{m_p k_B T}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{m_p c^2}{k_B T}\right)$$

$$g_n = g_p = 2$$

d) Find the (approximate) value of n_n/n_p when $k_B T \gg (m_n - m_p)c^2 = Q_n = 1.29$ MeV.

$$\frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left(-\frac{Q_n}{k_B T}\right) \approx \exp\left(-\frac{Q_n}{k_B T}\right) \approx \exp(-$$

$$\approx \exp(-0.000\#) \approx 1.$$

10. a) What is the ratio $\frac{\Delta T}{T}$ for the CMB (Cosmic Microwave Background) radiation determined by the experimental groups?

$$\frac{\Delta T}{T} \approx 10^{-5}$$

b) What is the current value of T_0?

$$T_0 = 2.7255\ K$$

c) What is the implication of $\Delta T_0 \approx 30\,\mu K$?

The universe is isotropic

d) What is the epoch of recombination?

The time at which the baryonic component of the universe goes from being ionized to being neutral.

e) What is the epoch of photon decoupling?

The time at which photons cease to react with the electrons. When photons decouple the universe becomes transparent.