Ch. 11 : Precession of the Perihelion (of Mercury)

perihelion: the point in the path of a celestial body (as a planet) that is nearest to the sun (opposite aphelion)

Calculating the shape of an orbit:

We found for equatorial orbits: $r^2 \frac{d\phi}{d\tau} = l$, and $\tilde{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{c^2 r} + \frac{l^2}{2c^2r^2} - \frac{GMl^2}{c^4r^3}$

Note that $\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} + \cancel{\frac{dr}{d\theta}\frac{d\theta}{d\tau}}^{0}$; $\left(\theta = \frac{\pi}{2}\right)$

$\Rightarrow \frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{l}{r^2}$ substituting this in the equation for $\tilde{E}$ we get

$$\tilde{E} = \frac{1}{2}\left(\frac{dr}{d\phi}\right)^2 \frac{l^2}{c^2 r^4} - \frac{GM}{c^2 r} + \frac{l^2}{2c^2r^2} - \frac{GMl^2}{c^4r^3}$$

Define $u \equiv \frac{1}{r}$, $\frac{du}{d\phi} = \frac{du}{dr}\frac{dr}{d\phi} = -\frac{1}{r^2}\frac{dr}{d\phi} = -u^2\frac{dr}{d\phi}$

$\Rightarrow \frac{dr}{d\phi} = -\frac{1}{u^2}\frac{du}{d\phi}$

$$\tilde{E} = \frac{1}{2}\left(\frac{1}{\cancel{u^4}}\right)\left(\frac{du}{d\phi}\right)^2\frac{l^2}{c^2}\cancel{u^4} - u\frac{GM}{c^2} + \frac{l^2}{2c^2}u^2 - \frac{GMl^2}{c^4}u^3$$

Taking the derivative wrt ϕ:

$$\cancel{\frac{du}{d\phi}}\frac{d^2u}{d\phi^2}\frac{l^2}{c^2} - \cancel{\frac{du}{d\phi}}\frac{GM}{c^2} + \frac{l^2}{c^2}u\cancel{\frac{du}{d\phi}} - 3\frac{GMl^2}{c^4}u^2\cancel{\frac{du}{d\phi}} = 0$$

$$\frac{d^2u}{d\phi^2}\frac{\ell^2}{c^2} - \frac{GM}{c^2} + \frac{\ell^2}{c^2}u - \frac{3GM\ell^2}{c^4}u^2 = 0$$

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{\ell^2} + 3\frac{GM}{c^2}u^2 \quad \text{(Schwarzschild)} \; (*)$$

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{\ell^2} \quad \text{(Newtonian)}$$

Perturbation Calculation:

The Schwarzschild orbit equation (*) cannot be solved analytically. Perturbatively: Consider a circular orbit with a constant radius r_c. $u_c = \frac{1}{r_c}$ must satisfy the orbit equation (*):

$$\frac{d^2u_c}{d\phi^2} + u_c = \frac{GM}{\ell^2} + 3\frac{GM}{c^2}u_c^2$$

Consider a perturbation of this orbit that is slightly non-circular:

$$u(\phi) = u_c + u_c w(\phi), \text{ where } w(\phi) \ll 1$$
$$= u_c(1+w)$$

Inserting $u(\phi)$ into the above equation (*)

$$\underbrace{\frac{d^2u_c}{d\phi^2}}_{0}(1+w) + u_c\frac{d^2w}{d\phi^2} + u_c(1+w) = \frac{GM}{\ell^2} + \frac{3GM}{c^2}u_c^2(1+w)^2$$

$$u_c \cdot \frac{d^2w}{d\phi^2} + u_c \cdot w + \cancel{u_c} = \cancel{\frac{GM}{\ell^2}} + \cancel{\frac{3GM}{c^2}u_c^2} + \frac{3GM}{c^2}u_c^2(2w + w^2)$$

$$\frac{d^2w}{d\phi^2} + w = \frac{6GM}{c^2}u_c w + \cancel{\frac{3GM}{c^2}u_c w^2}$$

negligible because $w^2 \ll 1$

$$\Rightarrow \frac{d^2w}{d\phi^2} + w\left(1 - \frac{6GM}{c^2}u_c\right) \approx 0$$

Solution : $w(\phi) = A\cos(\omega\phi + \phi_0)$, where $\omega \equiv \left(1 - \frac{6GM}{c^2 r_c}\right)^{1/2}$ and A and ϕ_0 are constants.

The orbit's perihelion occurs when $w(\phi)$ is a minimum , that is when $(\omega\phi + \phi_0) = \pi + 2\pi n$, $n = 0, 1, 2, \ldots$

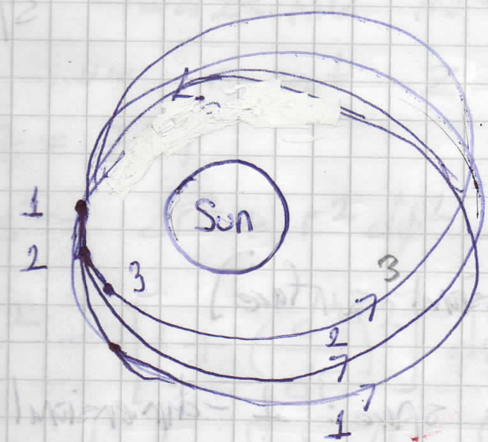
1st perihelion : $n=0$ $\quad \omega\phi_1 + \phi_0 = \pi$

2nd " : $n=1$ $\quad \omega\phi_2 + \phi_0 = 3\pi$ $\quad \Big\}\ \omega\Delta\phi = 2\pi$

3rd " : $n=2$ $\quad \omega\phi_3 + \phi_0 = 5\pi$ $\quad \Big\}\ \omega\Delta\phi = 2\pi$

$$\Rightarrow \Delta\phi = \frac{2\pi}{\omega} = \left(1 - \frac{6GM}{c^2 r_c}\right)^{-1/2} 2\pi \approx \left(1 + \frac{3GM}{c^2 r_c}\right) 2\pi$$

$$\approx 2\pi + \frac{6\pi GM}{c^2 r_c} \quad , \left(\frac{6GM}{c^2 r_c} \ll 1 \text{ has been assumed} \right)$$



$1 \text{ rad} = 206265$ arc-sec

For Mercury; $r_c \approx 57.9 \times 10^9$ m

For Sun : $\frac{GM}{c^2} = 1477$ m

$$\delta = 6\pi \frac{GM}{c^2 r_c} = \frac{6(3.14)(1477)}{5.79 \times 10^{10}} \text{ rad} = 4806 \times 10^{-10} \text{ rad}$$

$T_{Mercury} = 0.241$ y ; $\frac{T_{century}}{T_{Mercury}} = \#$ of turns $= 415$

$\delta = 41.1$ arc-sec/century

The Effect of Spatial Curvature: The precession arises both because $g_{tt} = -\left(1 - \frac{2GM}{c^2 r}\right) \neq 1$ and

$$g_{rr} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \neq 1$$

At a given time t:

$dt = 0$

$$ds^2 = ds^2_{spatial} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2$$

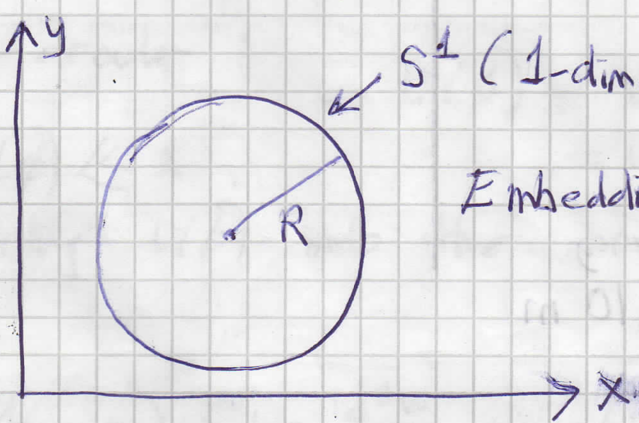
$$+ r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

Since the orbits remain in the equatorial plane, we can take $\theta = \frac{\pi}{2}$ and $d\theta = 0$.

$$\Rightarrow \quad ds^2 = (\qquad)^{-1} dr^2 + r^2 d\phi^2$$ ← curved space

Side Remark

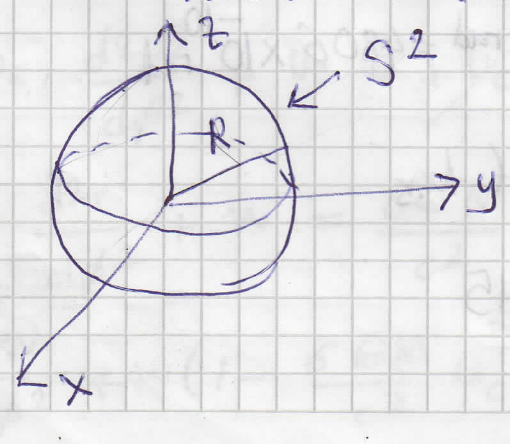
i) $ds^2 = R^2 d\phi^2$



S^1 (1-dimensional surface)

Embedding space: 2-dimensional Euclidean space

ii) $ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2$



S^2

Embedding space: 3-dimensional Euclidean space

ii) Schwarzschild equatorial plane:

$$ds^2 = \frac{dr^2}{\left(1-\frac{2GM}{c^2 r}\right)} + r^2 d\phi^2 \qquad (***)$$

Embedding space: 3-dimensional Euclidean space labeled by polar coordinates (r, ϕ, z)

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2$$

In this 3-dimensional space, a two-dimensional surface can be described by specifying the height z as a function of r and ϕ.

Since Schwarzschild metric is symmetric under rotations in ϕ, we expect z to be a function of r only.

$$z = z(r)$$

$$\Rightarrow ds^2 = dr^2 + r^2 d\phi^2 + \left(\frac{dz}{dr}\right)^2 dr^2$$

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\phi^2$$

We want to find the function $z(r)$ that makes the metric of this surface match the metric in $(***)$.

$$1 + \left(\frac{dz}{dr}\right)^2 = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}$$

$$\frac{dz}{dr} = \pm\sqrt{\frac{1}{(\quad)} - 1}$$

$$\int dz = \pm \int \sqrt{\frac{1}{(\quad)} - 1}\, dr$$

$$z = \pm \int \sqrt{\frac{\frac{2GM}{c^2 r}}{\left(1 - \frac{2GM}{c^2 r}\right)}}\, dr$$

$$= \pm \sqrt{\frac{2GM}{c^2}} \int \sqrt{\frac{1}{r - \frac{2GM}{c^2}}}\, dr$$

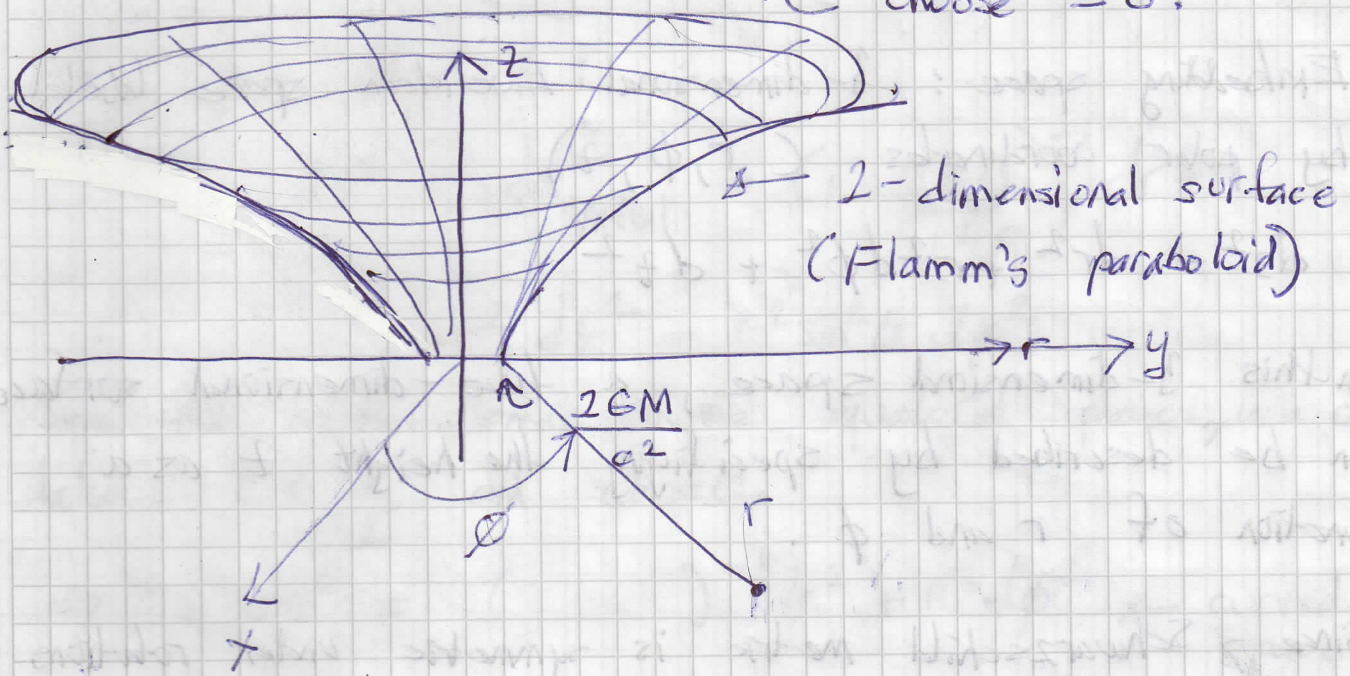
$$x \equiv r - \frac{2GM}{c^2}$$

$$dx = dr$$

$$z = \pm\sqrt{\frac{2GM}{c^2}} \int x^{-1/2}\,dx = \pm\sqrt{}\;\frac{x^{1/2}}{1/2} + \text{const},$$

$$z(r) = \pm\sqrt{\frac{8GM}{c^2}}\sqrt{r - \frac{2GM}{c^2}} + \text{const},$$

$\uparrow$ choose $= 0$.



Ch. 12 Photon Orbits

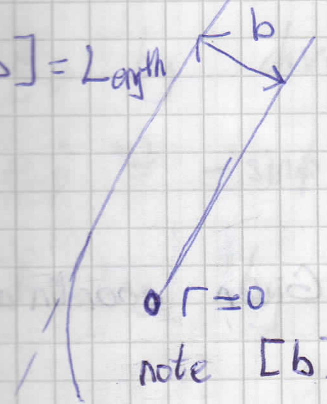
The proper time along a photon worldline is zero.

To overcome this problem we let $m \to 0$ in the equations for massive particles.

The impact parameter b:

$$b = \frac{\ell}{ec} = \frac{r^2\left(\frac{d\phi}{d\tau}\right)}{c\left(1-\frac{2GM}{c^2 r}\right)\frac{dt}{d\tau}} = r^2\left(1-\frac{2GM}{c^2 r}\right)^{-1} \times \frac{d\phi}{cdt}$$

$[b] = \text{Length}$



b remains well defined as $m \to 0$

In flat space $b = r^2 \frac{d\phi}{cdt}$

note $[b] = L$

$$0 = -c^2 d\tau^2 = ds^2 = -\left(1-\frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1-\frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

for photons only

At $\theta = \frac{\pi}{2}$:, Divide by $\left(1-2\frac{GM}{c^2 r}\right)c^2 dt^2$

$$0 = -1 + \left(1-\frac{2GM}{c^2 r}\right)^{-2}\left(\frac{dr}{cdt}\right)^2 + r^2 \frac{1}{(\ )}\left(\frac{d\phi}{cdt}\right)^2$$

$$-1 + (\ )^{-2}\left(\frac{dr}{cdt}\right)^2 + r^2 \frac{1}{(\ )} \frac{b^2}{r^4}\left(1-\frac{2GM}{c^2 r}\right)^2$$

$$1 = (\ )^{-2}\left(\frac{dr}{cdt}\right)^2 + \frac{b^2}{r^2}(\ )$$

Dividing by b^2:

$$\frac{1}{b^2} = \left[\frac{1}{b}\left(1-\frac{2GM}{c^2 r}\right)^{-1}\frac{dr}{cdt}\right]^2 + \frac{1}{r^2}\left(1-\frac{2GM}{c^2 r}\right)$$

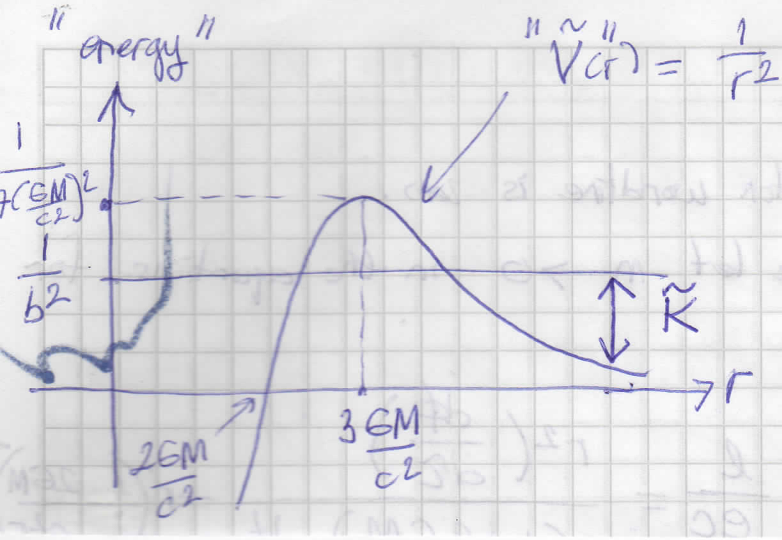
equation of radial motion for photons

(In flat space: $\frac{1}{b^2} = \left[\frac{1}{b}\left(\frac{dr}{cdt}\right)\right]^2 + \frac{1}{r^2}$)

This equation has the form of a conservation (of energy) equation

$$\text{"}\tilde{K}\text{"} = \left[\frac{1}{b}(\ )^{-1}\frac{dr}{cdt}\right]^2 ; \quad \text{"}\tilde{V}\text{"} = \frac{1}{r^2}\left(1-\frac{2GM}{c^2 r}\right)$$

$$\text{"}\tilde{V}(r)\text{"} = \frac{1}{r^2}\left(1 - \frac{2GM}{c^2 r}\right)$$



Ch. 13 Deflection of Light

In the last chapter, we obtained the impact parameter b

$$b = \frac{\ell}{ec} = \frac{r^2(d\phi/d\tau)}{c\left(1-\frac{2GM}{c^2 r}\right)(dt/d\tau)} = r^2\left(1-\frac{2GM}{c^2 r}\right)^{-1}\frac{d\phi}{c\,dt}$$

$$\Rightarrow \frac{d\phi}{c\,dt} = \frac{b}{r^2}\left(1-\frac{2GM}{c^2 r}\right) \quad , (*)$$

Also, $$1 = \left(1-\frac{2GM}{c^2 r}\right)^{-2}\left(\frac{dr}{c\,dt}\right)^2 + \frac{b^2}{r^2}\left(1-\frac{2GM}{c^2 r}\right) \quad (**)$$

(**) is valid for photons only!

We obtain an equation for $u(\phi) = \frac{1}{r}$ from (*):

$$\frac{dr}{c\,dt} = \frac{dr}{du}\frac{du}{d\phi}\frac{d\phi}{c\,dt}$$

$$\frac{dr}{du} = \left(\frac{du}{dr}\right)^{-1} = \left(-\frac{1}{r^2}\right)^{-1} = -r^2$$

$$\Rightarrow \frac{dr}{c\,dt} = -\cancel{r^2}\,\frac{b}{\cancel{r^2}}\left(1-\frac{2GM}{c^2 r}\right)\frac{du}{d\phi}$$

substituting this into (**) we get

$$1 = \cancel{\left(1-\frac{2GM}{c^2 r}\right)^{-2}}\; b^2 \cancel{\left(1-\frac{2GM}{c^2 r}\right)^2}\left(\frac{du}{d\phi}\right)^2 + b^2 u^2\left(1-\frac{2GM}{c^2}u\right)$$

$$1 = b^2\left(\frac{du}{d\phi}\right)^2 + b^2 u^2 - \frac{2GM}{c^2}u^3 b^2$$

Multiplying by $\frac{1}{b^2}$:

$$\frac{1}{b^2} = \left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2GM}{c^2}u^3$$

or, $$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{1}{b^2} + \frac{2GM}{c^2}u^3$$

Taking the derivative of both sides wrt ϕ yields:

$$2\cancel{\frac{du}{d\phi}}\frac{d^2u}{d\phi^2} + 2u\cancel{\frac{du}{d\phi}} = 6\frac{GM}{c^2}u^2\cancel{\frac{du}{d\phi}}$$

$$\Rightarrow \frac{d^2u}{d\phi^2} + u = 3\frac{GM}{c^2}u^2$$; for photons only $(***)$

Perturbation Solution:

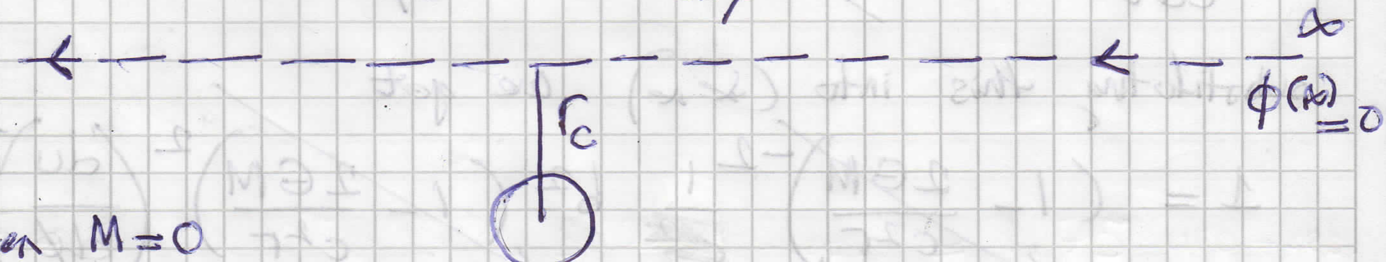
If there were no gravitating object at the origin then $M=0$:

$$\frac{d^2u}{d\phi^2} + u = 0$$

$$\Rightarrow u_{zg}(\phi) = u_c \sin\phi = \frac{\sin\phi}{r_c} = \frac{1}{r_{zg}}$$

zero gravity

$$\Rightarrow r_{zg} = \frac{r_c}{\sin\phi}$$



When $M=0$

Photon comes in from ∞ $(\phi_\infty = 0)$

For $r_c \gg \frac{2GM}{c^2}$ the photon's actual path will not differ much from the $M=0$ case.

Define

$$u(\phi) = u_c \sin\phi + u_c w(\phi) \quad , \; w(\phi) \ll 1$$

unitless

$$\frac{du}{d\phi} = u_c\cos\phi + u_c\frac{dw}{d\phi}$$

$$\frac{d^2u}{d\phi^2} = -u_c\sin\phi + u_c\frac{d^2w}{d\phi^2}$$. Then $(***)$ becomes

$$-\cancel{u_c \sin\phi} + u_c \frac{d^2w}{d\phi^2} + \cancel{u_c \sin\phi} + u_c w = \frac{3GM}{c^2}\Big(u_c^2 \sin^2\phi + \underbrace{2u_c^2 \sin\phi\, w + u_c^2 w^2}_{\text{negligible compared to } u_c^2\sin^2\phi}\Big)$$

Dividing by u_c $\Rightarrow$

$$\frac{d^2w}{d\phi^2} + w = \varepsilon \sin^2\phi = \frac{\varepsilon}{2}(1-\cos 2\phi) \text{ , where}$$

$$\varepsilon = \frac{3GM}{c^2} u_c = \frac{3GM}{c^2 r_c}$$

Try a solution of the form $w(\phi) = A + B\cos 2\phi$

$$\frac{dw}{d\phi} = -2B\sin 2\phi \,, \quad \frac{d^2w}{d\phi^2} = -4B\cos 2\phi$$

$$-4B\cos 2\phi + A + B\cos 2\phi = \frac{\varepsilon}{2} - \frac{\varepsilon}{2}\cos 2\phi$$

$$\cos 2\phi\left(-3B + \frac{\varepsilon}{2}\right) + A = \frac{\varepsilon}{2} \Rightarrow A = \frac{\varepsilon}{2} \,; \; \frac{\varepsilon}{2} = 3B$$

$$\cancel{\cos\theta} = \cancel{1 + \theta^2} \qquad = \cancel{1 + f(\theta)} \quad \Rightarrow$$

$$\cancel{\left(-3B + \frac{\varepsilon}{2}\right) + f(\theta)\left(-3B + \frac{\varepsilon}{2}\right) + A = \frac{\varepsilon}{2} = 0}$$

$$\Rightarrow w(\phi) \approx \frac{\varepsilon}{2} + \frac{\varepsilon}{6}\cos 2\phi$$

$$\Rightarrow u(\phi) \approx \frac{1}{r_c}\left[\sin\phi + \frac{3GM}{2c^2 r_c}\left(1 + \frac{1}{3}\cos 2\phi\right)\right]$$

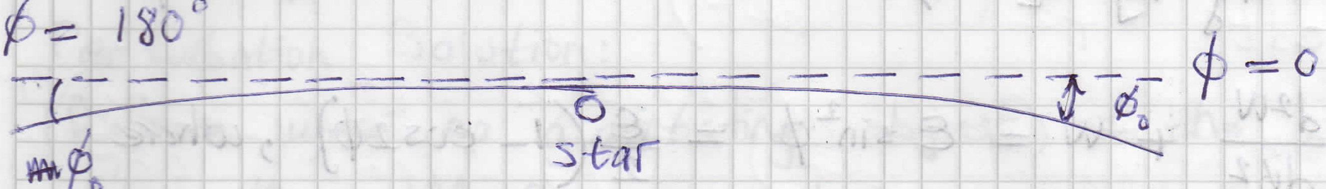
Now r does not become ∞ at $\phi = 0$. Rather r becomes ∞ at some angle ϕ_0 such that

$u(\phi_0) = 0 \Rightarrow$

$$0 = \sin\phi_0 + \frac{3GM}{2c^2 r_c}\left(1 + \frac{1}{3}\cos 2\phi_0\right)$$

small-angle approximation: $\sin\phi_0 \approx \phi_0$, $\cos 2\phi_0 \approx 1$

$$\Rightarrow \quad 0 \approx \phi_0 + \frac{3GM}{2c^2 r_c} \cdot \frac{4}{3} \quad \Rightarrow \quad \phi_0 \approx -\frac{2GM}{c^2 r_c}$$



Total amount of deflection $\delta \approx 2|\phi_0| = \frac{4GM}{c^2 r_c}$

Taking $r_c = R_\odot = 6.96 \times 10^8$ m

$$\frac{GM_\odot}{c^2} = 1477 \text{ m}$$

$$\delta = 8.49 \times 10^{-6} \text{ rad}$$

2π rad $= 180°$, 1 rad $= \frac{180°}{\pi}$

$1° = 3600$ arc-sec.

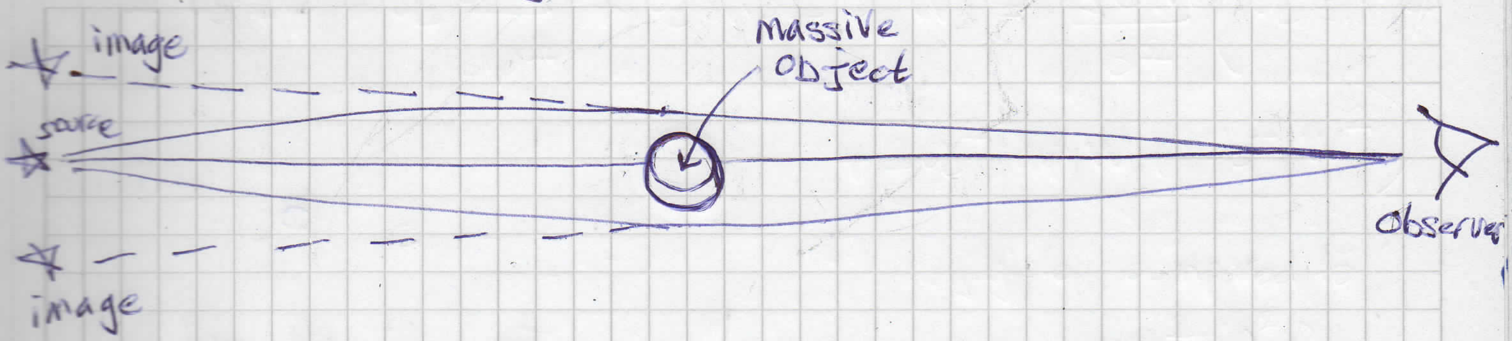
$$\delta = 8.49 \times 10^{-6} \times \frac{180}{3.14} \times 3600 \text{ arc-sec.} = 1.75 \text{ arc-sec.}$$

This was measured by Eddington in 1919

$$\delta = \begin{cases} 1.98'' \pm 0.16 & \text{Sobral data} \\ 1.61'' \pm 0.4 & \text{Principe data} \end{cases}$$

With the announcement of these results Einstein became a world celebrity...

Gravitational Lensing :



* If the source is not directly behind the gravitating body the observer sees two images of the source due to the deflection of light
* If the source is directly behind the gravitating body the observer sees a ring around the lensing body (= gravitating body)