INTRODUCTION TO COSMOLOGY
FIRST EXAM

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1. a) Define the critical energy density $\mathcal{E}_c(t)$ and express the Friedmann equation in terms of $\mathcal{E}_c(t)$.

$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\mathcal{E}_c$; $\mathcal{E}_c(t) = \frac{3c^2}{8\pi G} H(t)^2$, where $H(t) = \frac{\dot{a}(t)}{a(t)}$

b) What is the value of the curvature constant k for a critical-density-universe?

$k = 0$

c) Now, write down the friedman equation when there is a cosmological constant Λ.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\mathcal{E} - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3}$$

d) Assume that $\Lambda = \Lambda(a)$, where a is the scale factor. Obtain the expression for $\Lambda(a)$ so that the energy density of the universe is critical.

$$\mathcal{E} = \mathcal{E}_c \Rightarrow -\frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3} = 0$$

$$\Rightarrow \Lambda(a) = \frac{3k}{R_0^2 a^2}$$

e) What is the value of k for a universe in part d)?

Assuming $\Lambda(a) = +\text{ive} \Rightarrow k = 1$.

2, a) Assume a flat universe with no cosmological constant. Assume that the energy density $\mathcal{E}$ is proportional to a^{-n}, where n is a positive number and a is the scale factor.
Find the value of n such that the universe expands acceleratingly and the time dependence of the acceleration is constant.

$$\mathcal{E}(a) = \mathcal{E}_0 a^{-n}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2} \mathcal{E}_0 a^{-n}$$

$$\dot{a}^2 = \frac{8\pi G}{3c^2} \mathcal{E}_0 a^{-n+2}$$

$$\dot{a} = \left(\frac{8\pi G \mathcal{E}_0}{3c^2}\right)^{1/2} a^{-n/2+1}$$

$$\ddot{a} = \left(\quad\right)^{1/2}\left(-\frac{n}{2}+1\right) a^{-n/2} \cdot \dot{a} > 0$$

$$\Rightarrow \quad -\frac{n}{2}+1 > 0 \Rightarrow -n > -2$$

$$\Rightarrow \quad n < 2 \Rightarrow \boxed{n = 1} \text{ because } n \text{ is an integer}$$

b) Can the value of n be negative? Explain why?

* n cannot be negative because for a negative n the energy density $\mathcal{E}$ would increase as the universe expands.

** If n is negative, then $\ddot{a} = \frac{4\pi G}{3c^2} \mathcal{E}_0 (2-n) a^{1-n}$ would not be constant for negative n.

3. a) Assume that the scale factor of the universe is given by $a(t) = C_0 . t^2$, where C_0 is a constant. Find C_0.

$a(t_0) = 1 = C_0 t_0^2 \Rightarrow C_0 = \frac{1}{t_0^2}$

b) The light emitted by a galaxy at $t = t_e$ reaches us today. Calculate the proper distance of that galaxy.

$$d_{prop}(t) = c a(t) \int_{t_e}^{t_0} \frac{dt}{a(t)} = c a(t) \frac{1}{t_0^{-2}} \int t^{-2} dt = \frac{c a(t)}{t_0^{-2}} \left. \frac{t^{-1}}{-1} \right|_{t_e}^{t_0}$$

$$d_{prop}(t_0) = c \frac{1}{t_0^{-2}} \left(\frac{1}{t_e} - \frac{1}{t_0} \right) = c t_0 \left(\frac{t_0}{t_e} - 1 \right)$$

c) Express the proper distance in terms of the redshift z and t_0.

$$1 + z = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)} = \frac{t_0^2}{t_e^2} \Rightarrow \frac{t_0}{t_e} = \sqrt{1+z}$$

$$\Rightarrow d_{prop}(t_0) = c t_0 \left(\sqrt{1+z} - 1 \right)$$

d) Calculate the horizon distance at $t = t_0$ (today)

$$d_{Hor}(t_0) = c \int_0^{t_0} \frac{dt}{a(t)} = c t_0^2 \int_0^{t_0} \frac{dt}{t^2} = c t_0^2 \left. \frac{t^{-1}}{-1} \right|_0^{t_0}$$

$$= c t_0^2 \left(\frac{1}{0} - \frac{1}{t_0} \right) = \infty$$

4. a) Write down the line element $d\ell^2$ for the surface of a sphere of radius R.

$$d\ell^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2$$

b) Find the circumference of a circle of radius r.

$$C = \int d\ell = \int \sqrt{\underset{\overset{\shortparallel}{0}}{dr^2} + R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2} = R \sin\left(\frac{r}{R}\right) \int_0^{2\pi} d\theta$$

$$C = 2\pi R \sin\left(\frac{r}{R}\right)$$

5, a) Write down the acceleration equation

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

b) Define the equation of state parameter ω and rewrite the acceleration equation in terms of ω.

$$P = \omega \rho c^2 \Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \rho (1 + 3\omega)$$

c) If the universe is accelerating as it expands, what must be the value of ω?

$$\ddot{a} > 0 \Rightarrow (1 + 3\omega) < 0 \Rightarrow \omega < -\frac{1}{3}$$

d) What kind of a component of the universe can have such an ω value?

A cosmological constant with $\omega = -1$

6. a) Write down the spacetime metric for a homogeneous and isotropic universe with scale factor $a(t)$.

$$ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + S_k(r)^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

$$S_k(r) = \begin{cases} R\sin(r/R), & k=+1 \\ r, & k=0 \\ R\sinh(r/R), & k=-1 \end{cases}$$

b) What is the name of this metric?

The Robertson-Walker metric

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7. a) What is the homogeneity of the universe?

On large scales, there are no preferred locations

b) What is the isotropy of the universe?

On large scales, there are no preferred directions

c) What is the cosmological principle?

On large scales, the universe is homogeneous and isotropic.

d) What are the scales for which the cosmological principle holds?

The principle holds for scales $\gtrsim 100$ Mpc

8. a) Using the Friedmann equation with a cosmological constant (independent of time) Λ and the fluid equation show that the acceleration equation becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_m + 3\frac{P_m}{c^2}\right) + \frac{\Lambda c^2}{3}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_m - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3}$$

$$\dot{a}^2 = \left(\frac{8\pi G}{3}\rho_m + \frac{\Lambda c^2}{3}\right)a^2 - \frac{kc^2}{R_0^2}$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}\dot{\rho}_m a^2 + \left(\frac{8\pi G}{3}\rho_m + \frac{\Lambda c^2}{3}\right)2a\dot{a}$$

$$\ddot{a} = \frac{4\pi G}{3}\dot{\rho}_m \frac{a^2}{\dot{a}} + \left(\frac{8\pi G}{3}\rho_m + \frac{\Lambda c^2}{3}\right)a$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\dot{\rho}_m \frac{a}{\dot{a}} + \left(\frac{8\pi G}{3}\rho_m + \frac{\Lambda c^2}{3}\right)$$

· Fluid Eqn: $\dot{\rho}_m + \frac{3\dot{a}}{a}\left(\rho_m + \frac{P_m}{c^2}\right) = 0$

$$\Rightarrow \frac{1}{3}\dot{\rho}_m \frac{a}{\dot{a}} = -\rho_m + \frac{P_m}{c^2}$$

$$\Rightarrow \frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(-\rho_m - \frac{P_m}{c^2}\right) + \frac{8\pi G}{3}\rho_m + \frac{\Lambda c^2}{3}$$

$$= -\frac{4\pi G}{3}\left(\rho_m + \frac{3P_m}{c^2}\right) + \frac{\Lambda c^2}{3}$$

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b) Then find the relation between R_0 and Λ for a matter-dominated static universe.

Matter-dominated $\Rightarrow P_m = 0$; static $\Rightarrow \ddot{a} = 0$

$$0 = -\frac{4\pi G}{3}\rho_m + \frac{\Lambda c^2}{3} \Rightarrow \frac{4\pi G}{3}\rho_m = \frac{\Lambda c^2}{3}$$

Friedman Eqn. $\Rightarrow \frac{\dot{a}^2}{a^2} = 0 = \frac{8\pi G}{3}\rho_m - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3}$

$$0 = 2\frac{\Lambda c^2}{3} - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3} \Rightarrow \Lambda = \frac{k}{R_0^2 a} = \frac{1}{R_0^2 a}$$

($k=1$ because ρ_m = +ive $\Rightarrow \Lambda$ = +ive)

9. a) Consider a universe with scale factor $a(t) = \left(\frac{t}{t_0}\right)^n$, where $n > 0$. Calculate the radial coordinate $r(t)$ for light rays.

For light rays $ds^2 = -c^2 dt^2 + a^2 dr^2 = 0$

$$\Rightarrow dr^2 = \frac{c^2 dt^2}{a^2} \Rightarrow \int_0^r dr = r(t) = c \int \frac{dt^2}{a} = c\, t_0^n \int_0^t t^{-n} dt$$

$$r(t) = c\, t_0^n \left. \frac{t^{-n+1}}{-n+1} \right|_0^t = c\, t_0^n \left(\frac{t^{-n+1}}{-n+1} \right)$$

b) Calculate the horizon distance $d_{hor}(t)$ at time t

Horizon distance $d_{Hor}(t) = a(t)\, r(t) = c \frac{t^n}{\cancel{t_0^n}} \cancel{t_0^n} \left(\frac{t^{-n+1}}{-n+1} \right)$

$$d_{Hor}(t) = \frac{ct}{1-n}$$

c) Find the Hubble distance $d_H(t)$.

$$d_{Hubble}(t) = \frac{c}{H(t)} = \frac{c\, a(t)}{\dot{a}(t)} = \frac{c t^n \cancel{t_0^{-n}}}{\cancel{t_0^{-n}} \cdot n t^{n-1}} = \frac{ct}{n}$$

d) Find the value of n so that $d_{hor}(t_0) = 2 d_H(t_0)$

$$\frac{c\, t_0}{1-n} = \frac{2\, c\, t_0}{n} \Rightarrow n = 2 - 2n$$

$$3n = 2$$

$$\boxed{n = 2/3}$$

e) Find the value of n so that $d_H(t_0) = 2 d_{hor}(t_0)$.

$$\frac{c\, t_0}{n} = 2 \frac{c\, t_0}{1-n}$$

$$\Rightarrow 2n = 1 - n$$

$$3n = 1 \Rightarrow \boxed{n = 1/3}$$

10. a) Write down the definition of the redshift in terms of the wavelengths of the emitted and observed light rays.

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e}$$

b) Assuming that wavelength is proportional to scale factor $a(t)$, express the redshift in terms of $a(t_o)$ and $a(t_e)$.

$$z = \frac{a(t_o) - a(t_e)}{a(t_e)} = \frac{a(t_o)}{a(t_e)} - 1$$

$$1 + z = \frac{a(t_o)}{a(t_e)}$$

c) What is the redshift today?

Putting $t_e = t_o \Rightarrow 1 + z = 1$

$\Rightarrow z = 0$; today

d) When the redshift was 14, what was the observed size of the universe in terms of its observed size today?

$z = 14$

$$1 + 14 = \frac{a(t_o)}{a(t_e)}$$

$$\therefore$$

$$\boxed{a(t_e) = \frac{a(t_o)}{15}}$$