Ch. 10 Inflation & the Very Early Universe

The standard Hot Big Bang scenario has three problems.

The Flatness Problem:

The Friedmann equation

$$H(t)^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{kc^2}{a(t)^2 R_0^2}$$

The critical energy density

$$\varepsilon_c(t) = \frac{3c^2}{8\pi G} H(t)^2 \Rightarrow$$

$$1 = \frac{\varepsilon(t)}{\varepsilon_c(t)} - k\frac{c^2}{a(t)^2 H(t)^2 R_0^2}$$

$$\Omega(t) \equiv \frac{\varepsilon(t)}{\varepsilon_c(t)}$$

$$1 - \Omega(t) = -k\left(\frac{c}{a(t)H(t)R_0}\right)^2$$

The CMB and type Ia supernova results give the constraint

$$|1 - \Omega_0| \leq 0.005$$

$$\Rightarrow \left(\frac{c}{H_0 R_0}\right)^2 \leq 0.005 \Rightarrow \frac{c}{0.0707 H_0} \leq R_0$$

$$R_0 \geq \frac{14c}{H_0}$$

$$1 - \Omega_0 = -k\left(\frac{c}{H_0 R_0}\right)^2 \Rightarrow -k\left(\frac{c}{R_0}\right)^2 = (1-\Omega_0)H_0^2$$

$$\Rightarrow 1 - \Omega(t) = \frac{H_0^2(1-\Omega_0)}{H(t)^2 a(t)^2}$$

When the universe was dominated by radiation & matter, at times $t \ll t_{m\Lambda} \approx 10\,\text{Gyr}$,

$$\frac{H(t)^2}{H_0^2} = \frac{\Omega_{r0}}{a^4} + \frac{\Omega_{m0}}{a^3} \Rightarrow$$

$$1 - \Omega(t) = \frac{(1-\Omega_0)}{\left(\frac{\Omega_{r0}}{a^4} + \frac{\Omega_{m0}}{a^3}\right) a^2} = \frac{(1-\Omega_0)\, a^2(t)}{\Omega_{r0} + a(t)\,\Omega_{m0}}$$

Now, with $\Omega_{r0} \simeq 9\times10^{-5}$; $\Omega_{m0} \approx 0.31$, then at

$$a(t) = a_{rm} = \frac{\Omega_{r0}}{\Omega_{m0}} \approx 2.9\times10^{-4}$$

$$|1-\Omega_{rm}| \leq \frac{(0.005)(2.9\times10^{-4})^2}{9\times10^{-5} + 2.9\times10^{-4}(0.31)} = \frac{4.2\times10^{-10}}{18\times10^{-5}}$$

$$\leq 2.3\times10^{-6} \cancel{\Rightarrow \Omega_{rm} \approx 1.0000023}$$

$$\Rightarrow 0.9999977 \leq \Omega_{rm} \leq 1.0000023$$

When the universe was dominated by radiation only

$$1 - \Omega(t) \approx \frac{(1-\Omega_0)}{\frac{\Omega_{r0}}{a^4}\cdot a^2} = \frac{(1-\Omega_0)\, a(t)^2}{\Omega_{r0}} = \frac{(1-\Omega_0)\frac{t}{t_0}}{\Omega_{r0}}$$

$\left(a(t) = \left(\frac{t}{t_0}\right)^{1/2} \text{ has been used}\right)$

At $t \approx 1$ sec (when $T \approx 10^{10}$ K, nucleosynthesis)

$$t_0 = 13.7 \text{ Gyr} = 13.7\times10^{9} \times 3.16\times10^{7} \text{ sec}$$

$$= 4.3\times10^{17} \text{ sec}$$

$$|1-\Omega_{nucl}| \approx \frac{0.005\,(1)}{9\times10^{-5}\,(4.3\times10^{17})} = \frac{5}{9\times4.3}\times10^{-15}$$

$$\lesssim 1.3\times10^{-16} \cancel{\Rightarrow \Omega_{nucl} \approx 1.00000000000000013}$$

$0.99999999999999987 \leq \Omega_{nucl} \leq 1.00000000000000013$

This is called fine tuning

At the Planck time $t = t_{Pl} \approx 5.4\times10^{-44}$ sec

$$|1-\Omega(t)|_{Pl} \leq \frac{(0.005)\; 5.4\times10^{-44}}{9\times10^{-5}\,(4.3\times10^{17})} = \frac{5\times5.4}{9\times4.3}\times10^{-59}$$

$$\lesssim 6.9\times10^{-60}$$

So, we see that the deviation of $\Omega(t)$ from one was incredibly small; or $\Omega(t)$ was very close to one by 7 parts (~ 6.9) in 10^{60} at $t = t_{Planck}$

* The easiest way out of this dilemma is to suppose that the universe must have precisely the critical density. But there seems no reason to prefer this choice over any other. An explanation of such a value is required.

$$|1 - \Omega(a_{rm})| \leq 10^{-6}$$

$$|1 - \Omega(a_{\text{nuclear synth}})| \leq 10^{-16}$$

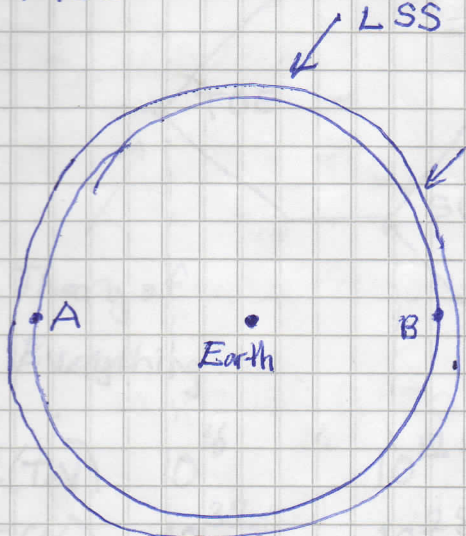
$$|1 - \Omega(a_{pl})| \leq 10^{-60}$$

The near-flatness observed today, $\Omega(a_0) \sim 1$, requires an extreme fine-tuning of Ω close to 1 in the early universe.

Why is $\Omega(a_0)$ not much smaller or much larger than 1?

The Horizon Problem:

Consider two antipodal points on the last scattering surface



The current proper distance to the last scattering surface is

$$d_p(t_0) = c\,a(t_0)\int_{t_{es}}^{t_0} \frac{dt}{a(t)}.$$

The current horizon distance

$$d_{hor}(t_0) = c\,a(t_0)\int_{0}^{t_0} \frac{dt}{a(t)}$$

Comparing the two $\Rightarrow$ $d_p(t_0) < d_{hor}(t_0)$

since $t_{es} \ll t_0 \Rightarrow d_p(t_0) \approx d_{hor}(t_0)$

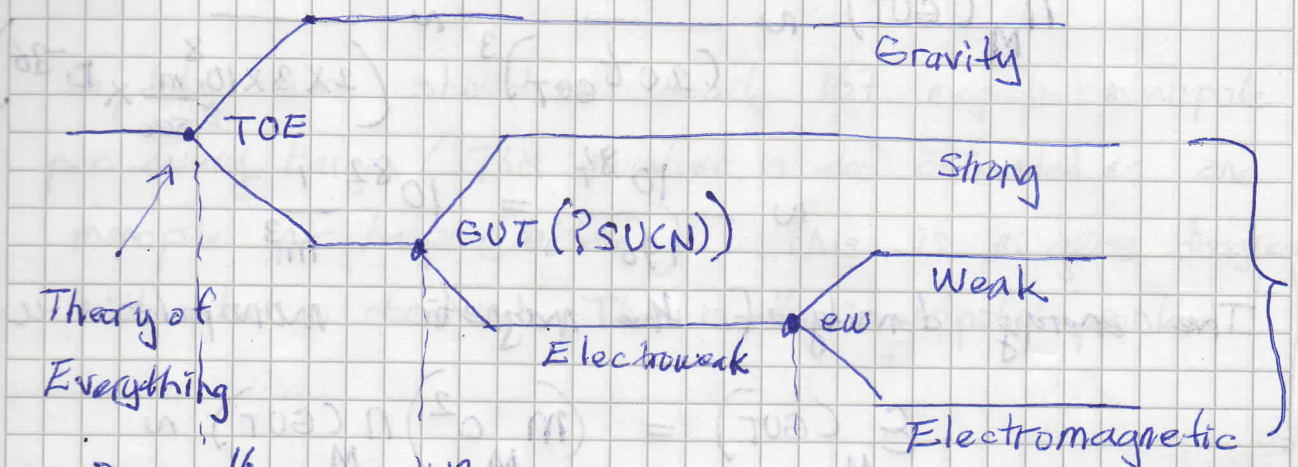
Points A and B are separated by $d(t_0) \approx 2\,d_{hor}(t_0)$

Since $d(t_0) > d_{hor}(t_0)$ points A & B are causally disconnected. A light signal sent from point A has not had time to reach point B, or vice versa. Or points A & B have not had time to come into thermal equilibrium with each other. On the other hand, the two points have the same temperature to within one part in 10^5. Why?

* How can two points that haven't had time to exchange information be so nearly identical in their properties?

The Monopole Problem:

Side Remark: Unification of Forces / Interactions



E(TeV)	10^{16}	10^{12}	1
T(K)	10^{32}	10^{28}	10^{16}
t(sec)	10^{-43}	$10^{-36} = t_{GUT}$	10^{-12}

$SU(3) \times SU(2) \times U(1)$

* One of the predictions of Grand Unified Theories is that the universe underwent a phase transition as the temperature dropped below the GUT temperature of 10^{28} K.

* GUTs predict that the GUT phase transition creates point-like topological defects that act as magnetic monopoles.

→ Rest energy of magnetic monopoles: $M_M c^2 \sim E_{GUT} \sim 10^{12}$ TeV

$$d_{hor}(t) = c\,a(t)\int_0^t \frac{dt}{a(t)} = c\left(\frac{t}{t_0}\right)^{1/2}\int_0^t t^{-1/2}\, t_0^{1/2}\, dt$$

$$= 2ct$$

Horizon volume $\approx (2ct)^3$

* We expect roughly one topological defect per horizon volume, due to the mismatch of fields that are not causally linked.

The number density of magnetic monopoles at the time of their creation, would be

$$n_M(GUT) \sim \frac{1}{(2ct_{GUT})^3} \sim \frac{1}{\left(2\times 3\times 10^8 \frac{m}{sec} \times 10^{-36}\right)^3}$$

$$\sim \frac{10^{84}}{196} = 10^{82} \frac{1}{m^3}$$

The energy density of the magnetic monopoles would be

$$\mathcal{E}_M(GUT) = (m_M c^2)\, n_M(GUT) \sim$$

$$\sim 10^{12} \times 10^{82}\ TeV\, m^{-3} = 10^{94}\ TeV\, m^{-3}$$

This is smaller by ten orders of magnitude than the energy density of radiation

$$\mathcal{E}_\gamma(GUT) \approx \alpha T_{GUT}^4$$

$$\approx 4.72\times 10^{-3}\ MeV.m^{-3}.K^{-4}\,(10^{28}K)^4$$

$$\approx 5\times 10^{103}\ TeV\, m^{-3}$$

$$\frac{\mathcal{E}_\gamma(GUT)}{\mathcal{E}_M(GUT)} \approx 5\times 10^{9} \sim 10^{10}$$

X

From $n_\gamma(t)a(t)^3 = n_{\gamma 0} a_0^3 = n_{\gamma 0}$; $a(GUT) = \left(\frac{t_{GUT}}{t_0}\right)^{1/2}$

$$n_\gamma(GUT)\left(\frac{t_{GUT}}{t_0}\right)^{3/2} = n_{\gamma 0}$$

$$n_\gamma(GUT) \approx n_{\gamma 0}\left(\frac{t_0}{t_{GUT}}\right)^{3/2} \approx 4.1\times 10^8\, m^{-3}\left(\frac{4.3\times 10^{17} sec}{10^{-36} sec}\right)^{3/2}$$

$$\approx 1.15\times 10^{89}\ m^{-3}$$

$$\frac{n_M(GUT)}{n_\gamma(GUT)} \approx \frac{10^{82}}{10^{89}} = 10^{-7} \approx \frac{n_M(t_0)}{n_\gamma(t_0)}$$

with $\frac{n_\gamma(t_0)}{n_{bary}(t_0)} = \frac{1}{6.1\times 10^{-10}}$

$$\Rightarrow \frac{n_M(t_0)}{n_{bary}(t_0)} = \frac{10^{-7}}{6.1 \times 10^{-10}} = \frac{1}{6} \times 10^{3} \approx 167$$

So, today there should be about 167 magnetic monopole per every baryon (This number is also estimated as one monopole per baryon (nucleon)). This is in gross disagreement with what is observed. This is the monopole problem!

The Inflation Solution:

Inflation is the hypothesis that there was a period, early in the history of our universe, when the universe was expanding with acceleration. $(\ddot{a} > 0)$.

The acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)$$

tells us that

$$\ddot{a} > 0 \ , \ \varepsilon + 3P < 0 \Rightarrow P < -\frac{\varepsilon}{3}$$

Recall that a positive cosmological constant Λ has $\omega = -1 < -\frac{1}{3}$.

So, the simplest implementation of inflation is to assume that the universe was temporarily dominated by a positive cosmological constant.

$$\varepsilon_\Lambda + 3P_\Lambda = \varepsilon_\Lambda - 3\varepsilon_\Lambda = -2\varepsilon_\Lambda$$

$$\frac{\ddot{a}}{a} = +\frac{8\pi G}{3c^2}\varepsilon_\Lambda = \frac{\Lambda}{3} > 0 \ ; \text{ where } \Lambda \equiv \frac{8\pi G}{c^2}\varepsilon_\Lambda = 8\pi G \rho_\Lambda$$

The Friedman equation was

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\varepsilon_\Lambda = \frac{\Lambda}{3} = \text{const.}$$

$$H = \left(\frac{\Lambda}{3}\right)^{1/2} = \text{constant}$$

$$\frac{da}{dt}\frac{1}{a} = H \quad ; \quad \frac{da}{a} = H\,dt$$

t_i = time exponential inflation began, t_f = time exponential inflation ended

$$\ln a(t)\Big|_{a_i}^{a_{\cancel{f}}} = H(t_{\cancel{f}} - t_i)$$

$$\Rightarrow \frac{a_{\cancel{f}}(t)}{a_i} = \exp\left(H(t-t_i)\right)$$

$$a(t) = a_i e^{H(t-t_i)} \quad ; \quad \overbrace{t_i < t < t_f}^{\text{during inflation}}$$

So, we can write

$$a(t) = \begin{cases} a_i \left(\dfrac{t}{t_i}\right)^{1/2} & ; \quad t < t_i \\ a_i e^{H(t-t_i)} & ; \quad t_i < t < t_f \\ a_f \left(\dfrac{t}{t_f}\right)^{1/2} & ; \quad t > t_f \end{cases}$$

, where $a_f = a_i e^{H(t_f - t_i)}$

$$\frac{a(t_f)}{a(t_i)} = \frac{a_f}{a_i} = \frac{a_i e^{H(t_f-t_i)}}{a_i} = e^{H(t_f-t_i)} = e^N$$

where $N \equiv H(t_f - t_i)$ is the # of e-foldings of inflation.

If $(t_f - t_i) \gg H^{-1}$, then $N \gg 1$.

A possible model for inflation:

Assume $t_i \approx t_{GUT} \approx 10^{-36}$ sec, with $H \approx t_{GUT}^{-1} = 10^{36} s^{-1}$

$$\boxed{t_f = (N+1) t_i = (N+1) t_{GUT}}$$

The energy density of Λ during inflation was

$$\varepsilon_\Lambda = \frac{3c^2}{8\pi G} H^2 = \frac{3(3\times10^8 m/s)^2 (10^{36} s^{-1})^2}{8\pi \times 6.67\times 10^{-11} m^3 kg^{-1} s^{-2}}$$

$$= 1.61 \times 10^{98} \, J \, m^{-3}$$

$$= \frac{1.61\times10^{98}}{1.60\times10^{-19}} eV \cdot m^{-3} = 10^{117} eV \cdot m^{-3}$$

$$\varepsilon_\Lambda = 10^{105} \, TeV \cdot m^{-3}$$

(compare this to $\varepsilon_{\Lambda,0} = 0.69 \, \varepsilon_{c,0} \approx 0.0034 \, TeV m^{-3}$

$$\frac{\varepsilon_\Lambda}{\varepsilon_{\Lambda,0}} = \frac{10^{105+3}}{3.4} = 2.9 \times 10^{107}$$ ← this huge ratio is known as the cosmological constant problem.)

How inflation resolves the Flatness Problem:

$1 - \Omega(t)$ can be written as

$$|1-\Omega(t)| = \left| -k \frac{c^2}{R_0^2 a(t)^2 H(t)^2} \right| = \frac{c^2}{R_0^2 a(t)^2 H^2}$$

for any universe not perfectly flat.

$$|1-\Omega(t)| = \frac{c^2}{R_0^2} \frac{1}{a_i^2 H^2} e^{-2H(t-t_i)} = \underbrace{\frac{c^2 e^{2Ht_i}}{R_0^2 a_i^2}}_{\text{Const.}} e^{-2Ht}$$

$$|1-\Omega(t_f)| = \text{const.}\, e^{-2Ht_f} = \text{const.}\, e^{-2H(N+1)t_i}$$

$$|1-\Omega(t_i)| = \text{const.}\, e^{-2Ht_i} \; ; \; \text{const} = |1-\Omega(t_i)| e^{2Ht_i}$$

$$\Rightarrow |1-\Omega(t_f)| = |1-\Omega(t_i)| e^{-2H(t_f-t_i)} = |1-\Omega(t_i)| e^{-2N}$$

Suppose that prior to inflation, the universe was actually strongly curved, with

$$|1-\Omega(t_i)| \approx 1$$

so, after N e-foldings of inflation, the deviation of Ω from 1 would be

$$|1-\Omega(t_f)| \approx e^{-2N}$$

Now, $a(t_f) = \left(\frac{t_f}{t_0}\right)^{1/2}$ ← radiation dominated

$$= \left(\frac{(N+1)\,t_i}{t_0}\right)^{1/2} = \left[\frac{(N+1)\,10^{-36}}{4.3\times 10^{17}}\right]^{1/2}$$

$$a(t_f) = 1.52\times 10^{-27}\sqrt{N+1} \rightarrow ?$$

From $|1-\Omega(t_f)| = \dfrac{(1-\Omega_0)\,a^2(t_f)}{\Omega_{r0} + \underbrace{\Omega_{m,0}\,a(t_f)}_{\text{negligible}}}$

$$e^{-2N} \approx \frac{(0.005)\,(1.52)^2\,10^{-54}\,(N+1)}{9\times 10^{-5}}$$

$$\approx \underbrace{\frac{5\times(1.52)^2}{9}}_{1.28} \times 10^{-52}\,(N+1)$$

Taking ln:

$$\ln e^{-2N} = \ln\left(1.28\times 10^{-52}\,(N+1)\right)$$

$$-2N = -52\ln 10 + \ln 1.28 + \ln(N+1)$$

$$= -52(2.30) + 0.588 + 4.11 \rightarrow N = 60$$

$$2N = 114.9 \Rightarrow N = 57.45 \approx 60$$

minimum value required

N may have been much larger than 60.
So, the flatness problem is solved.

How inflation solves the horizon problem:

The horizon distance is given by

$$d_{hor}(t) = c\,a(t)\int_0^t \frac{dt}{a(t)}$$

Prior to inflation, the universe was radiation-dominated. The horizon distance at the beginning of inflation was

$$d_{hor}(t_i) = c\,a(t_i)\int_0^{t_i} \frac{dt}{a(t)}$$

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2} \Rightarrow$$

$$d_{hor}(t_i) = c\,\frac{t_i^{1/2}}{t_0^{1/2}} \cdot t_0^{1/2}\int_0^{t_i} t^{-1/2}\,dt = 2ct_i$$

The horizon distance at the end of inflation was

$$d_{hor}(t_f) = c\,a(t_f)\int_0^{t_f}\frac{dt}{a(t)} = c\,a(t_f)\left[\int_0^{t_i}\frac{dt}{a(t)} + \int_{t_i}^{t_f}\frac{dt}{a(t)}\right]$$

$$= c\,a_i e^N\left[\int_0^{t_i}\frac{dt}{a_i\left(\frac{t}{t_i}\right)^{1/2}} + \int_{t_i}^{t_f}\frac{dt}{a_i e^{H(t-t_i)}}\right]$$

$$= c\,a_i e^N\left[\frac{t_i^{+1/2}}{a_i}2t_i^{1/2} + \frac{1}{a_i}e^{Ht_i}\frac{1}{(-H)}\left(e^{-Ht_f} - e^{-Ht_i}\right)\right]$$

$$= c\,e^N\left[2t_i + \frac{1}{H} - \frac{1}{H}\underbrace{e^{-H(t_f-t_i)}}_{=e^{-N}}\right]$$

$$d_{hor}(t_f) = c\,e^N\left(2t_i + \frac{1}{H}\right)$$

$$\boxed{\begin{aligned}&\frac{c}{H} = ct_i \Rightarrow\\ &d_{hor}(t_f) = e^N\,3ct_i\\ &= \frac{3}{2}d_{hor}(t_i)e^N\\ &d_{hor}(t_f)/d_{hor}(t_i) \approx \frac{3}{2}e^N\end{aligned}}$$

If inflation started at $t_i \approx 10^{-36}$ s, then immediately before inflation $d_{hor}(t_i) \approx 2ct_i = 6\times10^8 \times 10^{-36}\,m = 6\times10^{-28}\,m$

$$\Rightarrow d_{hor}(t_f) \approx e^N \times 6 \times 10^{-28}\ m + e^N \frac{c}{H}$$

$$\frac{c}{H} = ct_i = 3\times10^8 \times 10^{-36}\ m = 3\times10^{-28}\ m \Rightarrow$$

$$d_{hor}(t_f) \approx e^N \times 9\times10^{-28}\ m$$

Let this be equal to $n_0\, m \xrightarrow{} \text{meter}$

$$e^N \times 9 \times 10^{-28}\ m = n_0 m$$

$$e^N \approx \frac{n_0}{9} \times 10^{28}$$

$$N \approx \ln\left(\frac{n_0}{9}\times10^{28}\right) = \overbrace{\ln\left(\frac{n_0}{9}\right)}^{0.51 \text{ for } n_0 = 15} + \underbrace{28 \ln 10}_{64.47}$$

Rydon takes $n_0 = 15$, then

$$N \approx 65$$

How inflation solves the Monopole Problem:

From $n_M(t)\,a(t)^3 = n_M(t_{GUT})\,a(t_{GUT})^3$

$$n_M(t) = n_M(t_{GUT})\frac{a(t_{GUT})^3}{a(t)^3}$$

$$n_M(t_f) = n_M(t_i)\left(\frac{a(t_i)}{a(t_f)}\right)^3 \; ; \quad t_{GUT} = t_i$$

$$= 10^{82}\,m^{-3}\left(\frac{a(t_i)}{e^N a(t_i)}\right)^3 = 10^{82}\,m^{-3}\,e^{-3N}$$

$$n_M(t_f) = 10^{82}\,e^{-3(65)}\,m^{-3}$$

at the end of inflation

$$= 10^{82}\,\underbrace{10^{-195/\ln 10}}_{10^{-84.687}} = 10^{-2.687}\,m^{-3}$$

$$= 0.002056\,m^{-3}$$

Then

$$n_M(t_0)\,\underset{1}{a(t_0)^3} = n_M(t_f)\,a(t_f)^3 \rightarrow (2\times10^{-27})^3$$

$$n_M(t_0) = 2\times10^{-3}\times 8\times10^{-81}\,m^{-3}$$

$$= 1.6\times10^{-83}\,m^{-3}$$

density of monopoles today

incredibly small !!!

The horizon distance at the time of last scattering is

$$d_{hor}(t_{ls}) = c\,a(t_{ls})\int_{t_i}^{t_{ls}} \frac{dt}{a(t)} \approx c\,a(t_{ls})\int_{t_i}^{t_f}\frac{dt}{a(t)}$$

t_i = beginning of inflation

t_f = end of inflation

$$a(t) = a_i e^{H(t-t_i)} = a_i e^{-Ht_i} e^{Ht}$$

$$\Rightarrow d_{hor}(t_{ls}) \approx \frac{c\,a(t_{ls})}{a_i e^{-Ht_i}}\int_{t_i}^{t_f} e^{-Ht}\,dt = \frac{c\,a(t_{ls})}{a_i} e^{Ht_i}\left(\frac{e^{-Ht_f}-e^{-Ht_i}}{(-H)}\right)$$

$$\approx c\frac{a(t_{ls})}{a_i H}\left(-e^{-H(t_f-t_i)}+1\right) \quad ; \quad a_f = a_i e^{H(t_f-t_i)} = a_i e^{N}$$

$$\approx c\frac{a_{ls}}{H a_f} e^{-N}\left(1-e^{-N}\right) = c\frac{a_{ls}}{a_f H}\left(e^{N}-1\right) \approx c\frac{a_{ls}}{a_f H}e^{N}$$

The angular diameter distance at $t = t_{ls}$ is

$$d_A(t_{ls}) = \frac{d_p(t_0)}{(1+z_{ls})} = \frac{d_p(t_0)}{a_0/a_{ls}} = \frac{d_p(t_0)}{a_0}a_{ls}$$

since $d_p(t_0) \sim \frac{c}{H_0} \Rightarrow d_A(t_{ls}) \approx \frac{c\,a_{ls}}{a_0 H_0}$

Requiring $d_{hor}(t_{ls}) > d_A(t_{ls}) \Rightarrow$ during inflation

$$c\frac{a_{ls}}{a_f H}e^{N} > c\frac{a_{ls}}{a_0 H_0} \Rightarrow e^{N} > \frac{a_f H}{a_0 H_0} \approx \frac{a_r H_r}{a_0 H_0}$$

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) = \frac{8\pi G}{3c^2}\left(\varepsilon_r(t)+\varepsilon_m(t)\right) = \frac{8\pi G}{3c^2}\left(\frac{\varepsilon_{r,0}}{a(t)^4}+\frac{\varepsilon_{m,0}}{a(t)^3}\right)$$

At radiation-matter equality

$$H_{rm}^2 = \frac{8\pi G}{3c^2}\left(\frac{\varepsilon_{r0}}{a_{rm}^4}+\frac{\varepsilon_{m0}}{a_{rm}^3}\right) ; \quad \frac{H(t)^2}{H_{rm}^2} = \frac{\varepsilon_{r0}/a(t)^4+\varepsilon_{m0}/a(t)^3}{\varepsilon_{r0}/a_{rm}^4+\varepsilon_{m0}/a_{rm}^3} = \frac{\varepsilon_{r0}\left(\frac{a_{rm}}{a}\right)^4+\left(\frac{a_{rm}}{a}\right)^3 \times \varepsilon_{m0}\cdot a_{rm}}{\varepsilon_{r0}+\varepsilon_{m0}\cdot a_{rm}}$$

From $\varepsilon_{rm} = \varepsilon_{r0}/a_{rm}^4 = \frac{\varepsilon_{m0}}{a_{rm}^3} \Rightarrow \varepsilon_{r0} = a_{rm}\varepsilon_{m0} \Rightarrow H(t)^2/H_{rm}^2 = \frac{(a_{rm}/a)^4+(a_{rm}/a)^3}{2}$

Let $H_r = H(t_r)$ at $t = t_r$, the beginning of radiation dominance after inflation

$$H_r = \frac{H_{rm}}{\sqrt{2}}\left[\left(\frac{a_{rm}}{a_r}\right)^4+\left(\frac{a_{rm}}{a_r}\right)^3\right]^{1/2} \approx \frac{H_{rm}}{\sqrt{2}}\left(\frac{a_{rm}}{a_r}\right)^2 \quad \text{because } a_{rm}/a_r \gg 1$$

$$\Rightarrow a_r^2 \approx (H_{rm}/H_r)\cdot a_{rm}^2/\sqrt{2} \Rightarrow a_r = \sqrt{H_{rm}/(\sqrt{2}H_r)}\cdot a_{rm}$$

From $H(t)^2/H_0^2 = \frac{\Omega_{r0}}{a^4}+\frac{\Omega_{m0}}{a^3} \Rightarrow H_{rm}^2 = H_0^2\left(\frac{\Omega_{r0}}{a_{rm}^4}+\frac{\Omega_{m0}}{a_{rm}^3}\right) = 2H_0^2\,\Omega_{m0}/a_{rm}^3$

$$H_{rm} = H_0\sqrt{2\Omega_{m0}}\cdot a_{rm}^{-3/2} ; \quad a_r H_r/a_0 H_0 = \frac{1}{H_0}\left[H_0\sqrt{2\Omega_{m0}}\cdot a_{rm}^{-3/2}/(\sqrt{2}H_r)\right]^{1/2} a_{rm} H_r$$

$$= \left(\Omega_{m0}\,a_{rm}\right)^{1/4}\left(\frac{H_r}{H_0}\right)^{1/2} = \Omega_{r0}^{1/4}\left(\frac{H_r}{H_0}\right)^{1/2}$$

So, $\dfrac{a_r H_r}{a_0 H_0} = \Omega_{r0}^{1/4} \left(\dfrac{H_r}{H_0}\right)^{1/2}$; $(a_0 = 1)$

From $H(t) = \sqrt{\dfrac{8\pi G}{3c^2}\, \varepsilon(t)}$

$$\left(\frac{H_r}{H_0}\right)^{1/2} = \left(\frac{\varepsilon_r(t)}{\varepsilon_{c,0}}\right)^{1/4}, \text{ where } \varepsilon_{c,0} = \frac{3c^2}{8\pi G} H_0$$

using $\varepsilon_r(t) = \varepsilon_{r,0}/a(t)^4 = \varepsilon_{r0}\left[\dfrac{}{(t/t_0)^{1/2}}\right]^4 = \varepsilon_{r0}\left(\dfrac{t_0}{t}\right)^2$

$$\varepsilon_r(t_r) = \varepsilon_{r0}\left(t_0/t_r\right)^2$$

$$\Rightarrow \frac{a_r H_r}{a_0 H_0} = \left[\Omega_{r0} \frac{\varepsilon_{r0}}{\varepsilon_{c0}} \left(\frac{t_0}{t_r}\right)^2\right]^{1/4}$$

$$\varepsilon_{r0} = \alpha T_0^4 = 0.26 \times 10^{-6}\ eV\,m^{-3}$$

$$\Omega_{r0} = 9 \times 10^{-5} \ ; \ \varepsilon_{c0} = 5 \times 10^{9}\ eV\,m^{-3}$$

Taking $t_0 = 4.3 \times 10^{17}$ sec

$t_r \approx t_{pl} \approx 10^{-44}$ sec

$$e^N > \frac{a_r H_r}{a_0 H_0} = 5.424 \times 10^{25}$$

$$N > \ln\left(5.424 \times 10^{25}\right) = 59 \approx 60$$

The energy-momentum (stress-energy) tensor for a scalar field ϕ

$$T_{\alpha\beta} = \frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta} - g_{\alpha\beta}\left[\frac{1}{2} g^{\rho\sigma}\frac{\partial\phi}{\partial x^\rho}\frac{\partial\phi}{\partial x^\sigma} + V(\phi)\right]$$

Comparing this to the energy-momentum tensor of a perfect fluid

$$T_{\alpha\beta} = (\varepsilon + p)\, U_\alpha U_\beta + p\, g_{\alpha\beta}$$

$$U_\alpha = -\left[- g^{\rho\sigma}\frac{\partial\phi}{\partial x^\rho}\frac{\partial\phi}{\partial x^\sigma}\right]^{-1/2}\frac{\partial\phi}{\partial x^\alpha}$$

$$T_{\alpha\beta} = \left(-\frac{1}{2} g^{\rho\sigma}\frac{\partial\phi}{\partial x^\rho}\frac{\partial\phi}{\partial x^\sigma} + V - \frac{1}{2} g^{\rho\sigma}\frac{\partial\phi}{\partial x^\rho}\frac{\partial\phi}{\partial x^\sigma} - V\right)$$

$$\times \underbrace{\frac{1}{\left(-\frac{1}{2} g^{\rho\sigma}\frac{\partial\phi}{\partial x^\rho}\frac{\partial\phi}{\partial x^\sigma}\right)}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta}}_{= U_\alpha U_\beta} - g_{\alpha\beta}[\,--\,]$$

$$\Rightarrow\quad p = -\frac{1}{2} g^{\rho\sigma}\frac{\partial\phi}{\partial x^\rho}\frac{\partial\phi}{\partial x^\sigma} - V(\phi)$$

$$\varepsilon = -\frac{1}{2} g^{\rho\sigma}\frac{\partial\phi}{\partial x^\rho}\frac{\partial\phi}{\partial x^\sigma} + V(\phi)$$

Assuming that the field ϕ is homogeneous, namely $\frac{\partial\phi}{\partial x^i} = 0 \quad (i = 1,2,3)$

$$p = -\frac{1}{2} g^{00}\frac{\partial\phi}{\partial x^0}\frac{\partial\phi}{\partial x^0} - V(\phi)$$

$$ds^2 = -d(ct)^2 + a(t)^2[\quad]$$

$$g_{00} = g^{00} = -1 \Rightarrow p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$$\mathcal{E} = \frac{1}{2}\dot{\phi}^2 + V(\phi) \qquad (c = \hbar = 1)$$

$$[\hbar c^3] = E \cdot T \cdot (L T^{-1})^3 \qquad ; [E] = M L^2 T^{-2}$$

$$[\dot{\phi}^2] = [\phi^2] T^{-2}$$

$$\left[\frac{\dot{\phi}^2}{\hbar c^3}\right] = \frac{[\phi^2]\cancel{T^{-2}}}{E \cancel{T^{-2}} L^3} = \frac{E}{L^3} \Rightarrow [\phi] = E = \text{energy}$$

$$[V(\phi)] = \frac{E}{L^3} \qquad \left[\mathcal{E} = \frac{1}{2}\frac{1}{\hbar c^3}\dot{\phi}^2 + V(\phi)\right]$$

The equation of state is

$$\omega_\phi = \frac{P_\phi}{\mathcal{E}_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}$$

if the kinetic energy (density) is much less than the potential energy (density), we have.

$$\dot{\phi}^2 \ll V(\phi) \Rightarrow \omega \approx -1 < -\frac{1}{3}$$

The conservation of energy equation

$$dE + p\,dV = 0 \quad \text{gives}$$

$$\frac{d}{dt}(\mathcal{E} a^3) + p\frac{d}{dt}(a^3) = 0$$

$$\frac{d}{dt}\left[\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)a^3\right] + \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)\frac{d}{dt}a^3 = 0$$

$$\left(\dot{\phi}\ddot{\phi} + \frac{dV}{d\phi}\right)a^3 + \left(\frac{1}{2}\dot{\phi}^2 + \cancel{V(\phi)}\right)3a^2\dot{a} + \left(\frac{1}{2}\dot{\phi}^2 - \cancel{V(\phi)}\right)3a^2\dot{a} = 0$$

Dividing by $\dot{\phi} a^3 \Rightarrow$

$$\ddot{\phi} + \frac{dV}{d\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = 0 \quad \text{or} \quad \boxed{\ddot{\phi} + 3H(t)\dot{\phi} + V'(\phi) = 0}$$

Note: The scalar field which gives rise to inflation is called the inflaton.

$$H(t) = \sqrt{\frac{8\pi G \mathcal{E}}{3c^2}} = \left[\frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\right]^{1/2} \quad (*)$$

$$\boxed{H \approx \sqrt{\frac{8\pi G}{3c^2} V(\phi)}}$$

From $(*)$ $\Rightarrow$ squaring $(*)$ and differentiating

$$2H\dot{H} = \frac{8\pi G}{3c^2}\underbrace{\left(\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi}\right)}_{= -3H\dot{\phi}^2} = -\frac{8\pi G H \dot{\phi}^2}{c^2}$$

$$\Rightarrow \dot{H} = -\frac{4\pi G}{c^2}\dot{\phi}^2$$

So, $$\frac{\dot{H}}{H^2} \approx -\frac{4\pi G}{c^2}\frac{\dot{\phi}^2}{\frac{8\pi G}{3c^2}V(\phi)} = -\frac{3}{2}\frac{\dot{\phi}^2}{V(\phi)}$$

$$\Rightarrow \frac{|\dot{H}|}{H^2} \approx \frac{3}{2}\frac{\dot{\phi}^2}{V(\phi)} \ll 1 \quad \left(\text{or } -\frac{\dot{H}}{H^2} \ll 1\right)$$

From $\dot{\phi}^2 \ll V(\phi)$

$$\Rightarrow 2\dot{\phi}\ddot{\phi} \ll V'(\phi)\dot{\phi}$$

$$2\ddot{\phi} \ll V'(\phi) = -\ddot{\phi} - 3H\dot{\phi}$$

$$3\ddot{\phi} \ll -3H\dot{\phi} \Rightarrow \left|\frac{\ddot{\phi}}{H\dot{\phi}}\right| \ll 1$$

The $\ddot{\phi}$ term may be dropped in the equation of ϕ:

$$\Rightarrow \dot{\phi} \approx -\frac{V'(\phi)}{3H} \quad ; \text{ putting } H \approx \sqrt{8\pi G V}$$

$$\dot{\phi} \approx -\frac{V'(\phi)}{\sqrt{24\pi G V(\phi)}}$$

$$\Rightarrow \frac{3}{2}\frac{\dot{\phi}^2}{V(\phi)} = \frac{3}{2}\frac{V'^2}{24\pi G V^2} = \boxed{\frac{1}{16\pi G}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \ll 1}$$

Suppose that during some time interval the field ϕ changes from ϕ_i to ϕ_f , with $0 < V(\phi_f) < V(\phi_i)$

$\uparrow$ initial value $\quad \uparrow$ final value

and with $\left(\frac{V'}{V}\right)^2 \ll 16\pi G$

During this period $\frac{a(t_f)}{a(t_i)} = \exp\left[\int_{t_i}^{t_f} H dt\right] = \exp\left[\int_{\phi_i}^{\phi_f} \frac{H d\phi}{\dot\phi}\right]$

$$\simeq \exp\left[\int_{\phi_i}^{\phi_f} \sqrt{\frac{8\pi G V}{3}} \frac{d\phi}{\left(-\frac{V'}{\sqrt{24\pi G V}}\right)}\right]$$

$$\simeq \exp\left[-\int_{\phi_1}^{\phi_2} \left(\frac{8\pi G V(\phi)}{V'(\phi)}\right) d\phi\right]$$

Example: The power-law potential

$$V(\phi) = g\,\phi^\alpha \quad ; \; g > 0 \, , \alpha > 0$$

The flatness condition: $V' = g\alpha\phi^{\alpha-1}$

$$\frac{V'}{V} = \frac{g\alpha\phi^{\alpha-1}}{g\phi^\alpha} = \frac{\alpha}{\phi} \; ; \; \left(\frac{V'}{V}\right)^2 = \frac{\alpha^2}{\phi^2} \ll 16\pi G$$

$|\phi| \gg \frac{\alpha}{\sqrt{16\pi G}}$

$$\frac{8\pi G V}{V'} = 8\pi G \frac{g\phi^\alpha}{g\alpha\phi^{\alpha-1}} = \frac{8\pi G}{\alpha}\phi$$

~~For $\alpha = 4$~~

$$\frac{a_f}{a_i} \simeq \exp\left[-\int_{\phi_i}^{\phi_f} \frac{8\pi G}{\alpha}\phi\, d\phi\right] = \exp\left[-\frac{4\pi G}{\alpha}\left(\phi_f^2 - \phi_i^2\right)\right]$$

$$= \exp\left[+\frac{4\pi G}{\alpha}\left(\phi_i^2 - \phi_f^2\right)\right] \quad ; \; \phi_i > \phi_f$$

For $\frac{a_f}{a_i} = e^{60} \Rightarrow \frac{4\pi G}{\alpha}\phi_i^2 = 60$

$$|\phi_i| = \sqrt{\frac{60\alpha}{4\pi G}} = \sqrt{\frac{15\alpha}{\pi G}}$$

Another example:

$$V(\phi) = g e^{-\lambda\phi} \quad ; \; g>0, \; \lambda>0$$

$$V'(\phi) = -\lambda g e^{-\lambda\phi}$$

$$\frac{V'}{V} = \frac{-\lambda g e^{-\lambda\phi}}{g e^{-\lambda\phi}} = -\lambda$$

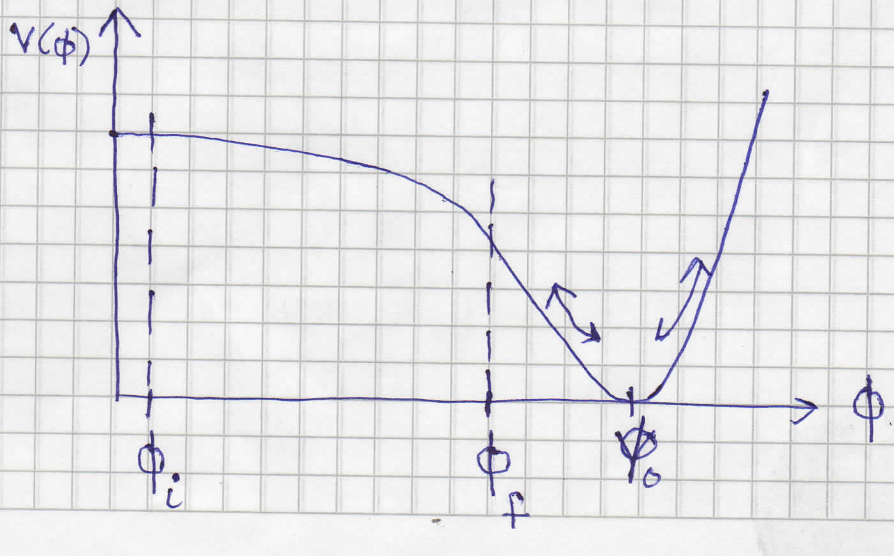
$$\left(\frac{V'}{V}\right)^2 = \lambda^2 \ll 16\pi G \Rightarrow \boxed{\frac{\lambda^2}{16\pi G} \ll 1}$$

$$\frac{a_f}{a_i} = \exp\left[-\int_{\phi_i}^{\phi_f}\left(\frac{8\pi G V(\phi)}{V'(\phi)}\right)d\phi\right]$$

$$\simeq \exp\left[-\int_{\phi_i}^{\phi_f}\frac{8\pi G}{(-\lambda)}d\phi\right] = \exp\left[\frac{8\pi G}{\lambda}(\phi_f - \phi_i)\right]$$

$$\simeq e^N \qquad \phi_f > \phi_i$$

In general, the potential $V(\phi)$ during inflation has the following shape



From conservation of entropy : $S_f = S_i$

$$(a_f T_f)^3 = (a_i T_i)^3$$

or $\quad a_f T_f = a_i T_i$

$$\Rightarrow \frac{T_f}{T_i} = \frac{a_i}{a_f} = e^{-N} = e^{-65} \quad , (N = 65)$$

$$= 5.9 \times 10^{-29}$$

if, when inflation started when $T_i = T_{GUT} \approx 10^{28}$ K

Then, at the end of inflation

$$T_f \approx 6 \times 10^{-29} T_i \approx 0.6 \text{ K},$$

which is pretty low. When the inflaton field ϕ reached ϕ_0, its true vacuum value, it fluctuated and its energy was converted to relativistic particles such as photons, the temperature was restored from ~ 0.6 K to its pre-inflationary value $T_i \approx 10^{28}$ K. This is called "reheating".