GENERAL RELATIVITY

COMPLETION EXAM 08/July/2021

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1. For a binary star system with masses m_1 and m_2 seperated by a distance D rotating around their center of mass with angular frequency ω, the metric perturbation is given by

$$h^{jk}_{TT} = -\frac{4GM\eta D^2\omega^2}{c^4 R_0}\begin{bmatrix} \cos[2\omega(t-\frac{R_0}{c})], & \sin[2\omega(t-\frac{R_0}{c})], & 0 \\ \sin[2\omega(t-\frac{R_0}{c})], & -\cos[2\omega(t-\frac{R_0}{c})], & 0 \\ 0, & 0, & 0 \end{bmatrix}$$

where $M \equiv m_1 + m_2$, $\eta \equiv m_1 m_2/M^2$, and R_0 is the distance from the center of mass of the system.

a) Consider a system in the xz-plane rotating around the y-axis. Write down h^{jk}_{TT} for this system.

$$h^{jk}_{TT} = -\frac{4GM\eta D^2\omega^2}{c^4 R_0}\begin{bmatrix} \cos[2\omega(t-R_0/c)], & 0, & \sin[2\omega(t-R_0/c)] \\ 0, & 0, & 0 \\ \sin[2\omega(\ \ )], & 0, & -\cos[2\omega(\ \ )] \end{bmatrix}$$

b) write h^{jk}_{TT} in part (a) as a sum of "plus" and "cross" polarization waves.

$$h^{jk}_{TT} = -\frac{4GM\eta D^2\omega^2}{c^4R_0}\cos[2\omega(\;)]\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}$$

$$-\frac{4GM\eta D^2\omega^2}{c^4R_0}\sin[2\omega(\;)]\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}$$

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2. The Reissner - Nördström metric describes the spacetime geometry outside a spherical object with mass M and electric charge Q. It is given by (in GR units)

$$ds^2 = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 + \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2$$
$$+r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

a) Calculate the radii of the infinite redshift surfaces.

Multiply $\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)$ by r^2 and the set to zero:

$$r^2 - 2Mr + Q^2 = 0$$

$$r_{1,2} = \frac{2M \pm \sqrt{4M^2-4Q^2}}{2}$$

$$r_+ = M + \sqrt{M^2-Q^2}$$

$$r_- = M - \sqrt{M^2-Q^2}$$

b) What are the radii of the event horizons?

Same as r_+ and r_- in part a)

c) What is the condition for a naked singularity?

For a naked singularity $r_\pm$ will be imaginary.

$\Rightarrow$ $M^2 < Q^2$ is the condition.

d) For what value of Q the radius of the event horizon is equal to the radius of the Schwarzschild event horizon?

For $Q = 0$

e) What does cosmic censorship require Q^2 to be?

$Q^2 < M^2$ so that $r_\pm$ are real.

3. In an alternative theory, the metric outside a spherical object with mass M and electric charge Q>0 is given by

$$ds^2 = -\left(1 - 2\frac{GM}{c^2 r} + 2\frac{e}{m_e}\frac{k_e Q}{c^2 r}\right)c^2 dt^2$$
$$+\left(1 - 2\frac{GM}{c^2 r} + 2\frac{e}{m_e}\frac{k_e Q}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

a) Find the position r of the test particle where the proper time measured by the test particle is zero.

$$d\tau = \left[1 - \left(\frac{2GM}{c^2} - \frac{2e}{m_e}\frac{k_e Q}{c^2}\right)\frac{1}{r}\right]^{1/2} dt$$

$$d\tau = 0 \Rightarrow r - \left(\frac{2GM}{c^2} - \frac{2e}{m_e}\frac{k_e Q}{c^2}\right) = 0$$

$$r = \frac{2GM}{c^2} - \frac{2e}{m_e}\frac{k_e Q}{c^2}$$

b) What is the value of r such that the proper time interval and coordinate time interval of the test particle are equal?

For $d\tau = dt \Rightarrow r = \infty$

4. Consider a hypothetical spherical object with a negative mass $-M$.

a) Write down the line element ds^2 outside this object when there is no matter or energy outside it.

$$ds^2 = -\left(1+\frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1+\frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

b) Is there an infinite redshift surface for this object? Explain.

For an infinite redshift surface $r + \frac{2GM}{c^2} = 0$

$\Rightarrow r = -\frac{2GM}{c^2} < 0$. So no infinite redshift surface.

c) Is there an event horizon for this object? Explain.

No event horizon because $r_{EH} = -\frac{2GM}{c^2}$

(r_{EH} cannot be negative)

d) What is the value of $r \neq 0$ for which there is a singularity?

For $r = -\frac{2GM}{c^2}$, $g_{rr} = \frac{1}{\left(1+\frac{2GM}{c^2 r}\right)} \rightarrow \frac{1}{0} = \infty$

e) What kind of singularity is this?

A naked singularity

5. a) Write down the line element ds^2 outside a spherical body of mass M when there is no mass, no charge and no non-gravitational energy outside it.

$$ds^2 = -\left(1-\frac{2GM}{c^2r}\right)c^2dt^2 - \left(1-\frac{2GM}{c^2r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

b) Write down the Einstein field equation to which this ds^2 is a solution.

$$R_{\mu\nu} = 0$$

c) Whas is the name of this solution?

The Schwarzschild solution

6. a) Calculate the following ($\mu, \nu = 0,1,2,3$; $i,j = 1,2,3$)

(i) $g_{\mu 3} g^{\mu 3} = \delta^{\mu}_{3} = \delta^{3}_{3} = 1$

(ii) $\delta^{\mu}{}_{2} \delta^{2}_{\mu} = \delta^{2}_{2} \delta^{2}_{2} = 1$

(iii) $\delta^{1}_{3} \delta^{3}_{1} = 0$

(iv) $\eta_{ij} \eta^{ij} = 3$

(v) $g^{\mu\nu} g_{\mu\alpha} \delta^{\alpha}_{\nu} = g^{\mu\nu} g_{\mu\nu} = 4$

b) Circle the indices that violate the rules for indices:

i) $p^{\mu} = m g^{\mu\nu} U_{\mu}$ ✓ Violates

ii) $\eta_{\mu\nu} p^{\mu} p^{\nu} = -m^2 c^2$

iii) $\dfrac{dp^{\mu}}{d\tau} = q F^{\mu\nu} U^{\nu}$ ✓ Violates

iv) $B^{\mu\nu} = \Lambda^{\mu\nu} A^{\alpha} A_{\beta}$ ✓ Violates

v) $R^{\mu}_{\nu} R^{\nu}_{\mu} = T^{\mu}_{\nu}$ ✓ Violates

7. In a hypothetical multi-dimensional spacetime the metric tensor satisfies $g_{\alpha\beta} g^{\alpha\beta} = 7$

a) What are the numerical values that α and β can take on?

$$\alpha, \beta = 0, 1, 2, 3, 4, 5, 6$$

b) What is the dimension of this spacetime?

Seven

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8. a) Given the tensor $A_{\alpha\beta}$ in a spacetime whose metric tensor is some $g_{\alpha\beta}$. Raise the index to find $A^{\alpha}{}_{\beta}$.

$$A^{\alpha}{}_{\beta} = g^{\alpha\mu} A_{\mu\beta}$$

b) Given the tensor $B^{\alpha\beta}$. Lower the index to find $B^{\alpha}{}_{\beta}$.

$$B^{\alpha}{}_{\beta} = g_{\beta\mu} B^{\alpha\mu}$$

c) Say true or false: $A^{\mu\gamma}{}_{\alpha} + B^{\mu\gamma}{}_{\beta} = C^{\mu\gamma}{}_{\alpha\beta}$: False

d) Say true or false: $A^{\mu\gamma} B^{\nu}{}_{\mu\gamma} = C^{\nu}$: True

e) Say true or false: $A_{\mu} B^{\mu} + A^{\alpha} B_{\alpha} = F$: True

9. Assume that some creatures live in a world whose geometry is described by the metric

$$ds^2 = -\left(1-\frac{r}{R}\right)c^2dt^2 + \left(1-\frac{r}{R}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2,$$

where R = constant and $0 \leq r < \infty$. Suppose the creatures live near $r=0$. If a creature were to cross the radius $r=R$,

a) could the creature return back? Explain why.

If the creature crosses the radius, then $r/R > 1 \Rightarrow$

$$ds^2 = +\left(\frac{r}{R}-1\right)c^2dt^2 - \left(\frac{r}{R}-1\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

So, the time and radial coordinates change function. The creature cannot go back

t becomes radial and r becomes time coord.

b) a creature observes that its clock registers zero time even though the clock is perfect. What is the location of this creature?

For $d\tau = \left(1-\frac{r}{R}\right)^{1/2}dt = 0 \Rightarrow r = R$

c) Write down the matrix for $g^{\alpha\beta}$.

$$g^{\alpha\beta} = \begin{pmatrix} -\left(1-\frac{r}{R}\right)^{-1} & 0 & 0 & 0 \\ 0 & \left(1-\frac{r}{R}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

10. In the weak-field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric and the perturbation $h_{\mu\nu}$ satisfies $|h_{\mu\nu}| \ll 1$. The inverse metric $g^{\mu\nu}$ can be written as $g^{\mu\nu} = \eta^{\mu\nu} + b^{\mu\nu}$, where $|b^{\mu\nu}| \ll 1$.

a) Find $b^{\mu\nu}$ in terms of $h^{\mu\nu}$ to first order in the metric perturbation. Show your steps.

$$g_{\mu\nu} g^{\mu\nu} = (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\mu\nu} + b^{\mu\nu}) = 4$$

$$= \underbrace{\eta_{\mu\nu}\eta^{\mu\nu}}_{4} + \eta_{\mu\nu} b^{\mu\nu} + h_{\mu\nu}\eta^{\mu\nu} + h_{\mu\nu} b^{\mu\nu} = 4$$

$$\Rightarrow \eta_{\mu\nu} b^{\mu\nu} + h_{\mu\nu}\eta^{\mu\nu} = 0$$

$$\eta_{\mu\nu} b^{\mu\nu} + \eta_{\mu\nu} h^{\mu\nu} = 0 \Rightarrow \boxed{b^{\mu\nu} = -h^{\mu\nu}}$$

b) Consider a metric given by

$$ds^2 = g_{00} c^2 dt^2 + g_{ii} dx^i dx^i, \quad (i = 1,2,3)$$

What is g_{00} such that proper time and coordinate time become identical?

$$d\tau = \sqrt{-g_{00}}\, dt = dt \Rightarrow g_{00} = -1$$

c) What are g_{00}, g_{11}, g_{22}, and g_{33} so that ds^2 is the Minkowski metric?

$g_{00} = -1$, $g_{11} = g_{22} = g_{33}$ in Cartesian coordinates

$g_{00} = -1$, $g_{11} = 1$, $g_{22} = r^2$, $g_{33} = r^2 \sin^2\theta$ in spherical coord.