INTRODUCTION TO COSMOLOGY
FINAL EXAM

05/06/2018

Name :

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1. a) Write down the Friedmann equation with the cosmological constant Λ.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda}{3}$$

b) Assume that Λ is not a true constant, but varies with time. What must be $\Lambda(a)$ so that the universe mimics a flat universe with no cosmological constant and the universe is curved?

We want $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon$

$$\Rightarrow -\frac{kc^2}{R_0^2 a^2} + \frac{\Lambda}{3} = 0 \Rightarrow \Lambda(a) = \frac{3kc^2}{R_0^2 a^2}$$

c) Recalling that Λ is related the vacuum energy density, what must be the value of k, the curvature scalar?

Since free energy and free energy density cannot be negative $\Lambda(a)$ must be positive. $\Rightarrow$ $\boxed{k = +1}$

2. a) Write down the Friedmann equation for a flat universe with no cosmological constant.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\mathcal{E} \quad ; \ (k=0)$$

b) Assume that the energy density $\mathcal{E}$ is proportional to a^{-n}, where n is a positive number.

Find the value of n such that the universe expands acceleratingly and the time dependence of the acceleration is constant.

Assume $\mathcal{E} = \mathcal{E}_0 a^{-n}$, $n>0$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\frac{\mathcal{E}_0}{a^n}$$

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\mathcal{E}_0 a^{(2-n)}$$

$$\dot{a} = \sqrt{\frac{8\pi G \mathcal{E}_0}{3c^2}}\, a^{(1-n/2)}$$

$$\int_0^a \frac{da}{a^{1-n/2}} = \int_0^t C_0\, dt$$

$$a^{-1+\frac{n}{2}+1} = C_0 t \ ; \ C_0 = \sqrt{\frac{8\pi G\mathcal{E}}{3c^2}}$$

$$a = C_0^{2/n} t^{2/n}$$

$$\dot{a} = \frac{2}{n} C_0^{2/n} t^{\frac{2}{n}-1}$$

$$\ddot{a} = \frac{2}{n}\left(\frac{2}{n}-1\right) C_0^{2/n} t^{\frac{2}{n}-2}$$

For $\ddot{a} = \text{const.} \Rightarrow \frac{2}{n} - 2 = 0$

$$\Rightarrow \boxed{n=1}$$

c) Can the value of n be negative? Explain why?

If n is negative, let $n=-|n|$ then $\mathcal{E} = \mathcal{E}_0 a^{+|n|}$

This would mean that as the universe expands the energy density increases. This cannot be so. Hence n cannot be negative.

3. Assume that the scale factor of the universe is given by $a(t) = \frac{t}{t_0}$, where t_0 is the present time.

a) The light emitted by a galaxy at $t = t_e$ reaches us today. Calculate the proper distance of that galaxy.

$$d_{prop}(t_0) = c\, a(t_0) \int_{t_e}^{t_0} \frac{dt'}{a(t')} = c\, t_0 \int_{t_e}^{t_0} t'^{-1} dt' = c\, t_0 \ln\left(\frac{t_0}{t_e}\right)$$

b) Express the proper distance in terms of the redshift z.

$$1+z = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)} = \frac{t_0}{t_e}$$

$$\Rightarrow\ d_{prop}(t_0) = c\, t_0 \ln(1+z)$$

c) Express the proper distance of that galaxy at the time of light emission t_e.

From $$\frac{d_{prop}(t_0)}{d_{prop}(t_e)} = \frac{\int a(t_0)\, dr}{\int a(t_e)\, dr} = \frac{a(t_0)\int dr}{a(t_e)\int dr} = 1+z$$

$$\Rightarrow\ d_{prop}(t_e) = \frac{d_{prop}(t_0)}{1+z} = c\, t_0 \frac{\ln(1+z)}{(1+z)}.$$

or: $$\frac{d_{prop}(t_e)}{d_{prop}(t_0)} = a(t_e) \Rightarrow d_{prop}(t_e) = \left(\frac{t_e}{t_0}\right) \cdot c\, t_0 \ln\left(\frac{t_0}{t_e}\right)$$

$$= c\, t_e \ln(t_0/t_e) \quad \#$$

4 . a) Write down the acceleration equation for a universe consisting of N components.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\sum_{i=1}^{N}\left(\mathcal{E}_i + 3P_i\right) \; ; \; \mathcal{E} = \text{energy density}, \; P = \text{pressure}$$

b) Presently, the universe is accelerating and consists of matter, radiation and another component. What must be the value of w_*, where w_* is the value of w for another component.

Presently, $i = m, r, \Lambda$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\,\text{[redacted]}\,\Big(\underbrace{\mathcal{E}_r + w_r\mathcal{E}_r}_{\text{negligible}} + \mathcal{E}_m + \overset{0}{\overset{=}{w_m}}P_m + \mathcal{E}_\Lambda + w_\Lambda\mathcal{E}_\Lambda\Big)$$

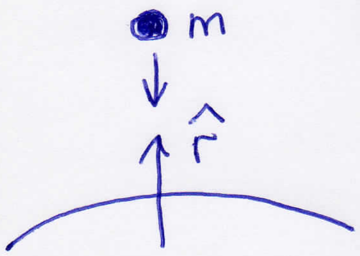
$$= -\frac{4\pi G}{3c^2}\,\text{[redacted]}\,\Big(\mathcal{E}_m + \mathcal{E}_\Lambda(1 + w_\Lambda)\Big) = -\frac{4\pi G}{3c^2}\mathcal{E}_c\cancel{X}\Big(\Omega_m + \Omega_\Lambda(1 + w_*)\Big)$$

$$= -(\quad)\Big(0.3 + 0.7(1 + w_*)\Big) \Rightarrow 0.3 + 0.7(1 + w_*) < 0 \Rightarrow w_* < -\frac{1}{0.7}$$

c) What kind of a component of the universe can have such an w value?

A cosmological constant with $w_\Lambda = -1$.

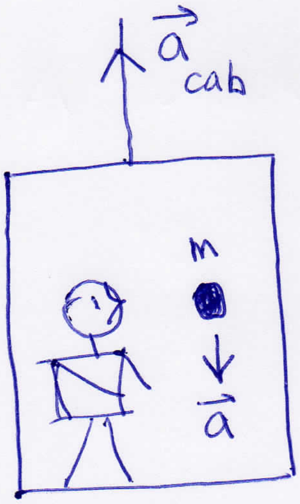
5. Consider a particle of inertial mass m_i and gravitational mass m_g. When the particle is dropped from rest at a height h from the surface of the earth, what is the acceleration of the particle (in vector form)?



$$\vec{F} = m_i \vec{a} = m_g \vec{g} = m_g g(-\hat{r})$$

$$\Rightarrow \vec{a} = \frac{m_g}{m_i} g(-\hat{r})$$

b) Now suppose that the particle is taken to deep space where there are no fields of any kind. If the particle is dropped by an observer in a cabin in deep space and observed that it has the same value of the acceleration in part a. What must be the acceleration of the cabin (in vector form)?



The equivalence of gravity and inertia requires the acceleration of the particle as in part a).

Then $\vec{a}_{cab} = -\vec{a} = \frac{m_g}{m_i} g \hat{r}$

upward

6. a) For an object of luminosity L at a distance d_L from us, define the apparent magnitude m and the absolute magnitude M.

$$m = -2.5 \log\left(\frac{L}{4\pi d_L^2}\right) \; ; \; M = -2.5 \log\left(\frac{L}{4\pi (10pc)^2}\right)$$

$d_L = 10\,pc$

b) Obtain the expression for "distance modulus" $m-M$ when d_L is measured in parsecs.

$$m-M = -2.5\left[\cancel{\log L} - \cancel{\log 4\pi} - \log d_L^2 - \cancel{\log L} + \cancel{\log 4\pi} + \log 10^2\right]$$

$$= 5\left[\log d_L - 1\right]$$

$$m-M = 5 \log d_L - 5$$

c) Find the distance d_L to an object whose distance modulus is 40.

$$\log d_L = \frac{m-M+5}{5} = \frac{40+5}{5} = 9$$

$$d_L = 10^9\,pc$$

7. a) Consider a sphere of radius $R_s(t)$ of the universe as it expands adiabatically ($dQ=0$). Obtain the fluid equation involving $\mathcal{E}$ (energy density), P (pressure) and a (scale factor) from the conservation of energy equation.

$$dQ = dE + PdV = 0$$

$$V = \text{const.}\, a^3$$

$$E = \mathcal{E} V = \text{const.}\, \mathcal{E} a^3$$

$$\Rightarrow d(\text{const.}\, \mathcal{E} a^3) + P\, d(\text{const.}\, a^3) = 0$$

$$d(\mathcal{E} a^3) + P\, da^3 = 0$$

$$a^3 d\mathcal{E} + \mathcal{E}\, 3a^2 da + 3Pa^2 da = 0$$

Dividing by a^3:

$$d\mathcal{E} + \mathcal{E} 3 \frac{da}{a} + 3P\frac{da}{a} = 0$$

Taking the time derivative

$$\frac{d\mathcal{E}}{dt} + \frac{3}{a}\frac{da}{dt}(\mathcal{E}+P) = 0$$

$$\boxed{\dot{\mathcal{E}} + 3\frac{\dot{a}}{a}(\mathcal{E}+P) = 0}$$

This is the fluid eqn.

b) Obtain the acceleration equation from the Friedmann equation and the fluid equation.

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\mathcal{E} - \frac{kc^2}{R_0^2 a^2}$$

$\times$ by a^2:

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\mathcal{E} a^2 - \frac{kc^2}{R_0^2}$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\left(\dot{\mathcal{E}} a^2 + 2\mathcal{E} a\dot{a}\right)$$

$\div$ by $2a\dot{a}$:

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(\dot{\mathcal{E}}\frac{a}{\dot{a}} + 2\mathcal{E}\right)$$

From the fluid equation $\Rightarrow$

$$\dot{\mathcal{E}}\frac{a}{\dot{a}} = -3(\mathcal{E}+P) \Rightarrow$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}(-3\mathcal{E} - 3P + 2\mathcal{E})$$

$$\boxed{\frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2}(\mathcal{E} + 3P)}$$

This is the acceleration eqn.

8. a) Show that in the radiation-dominated era the mean photon energy is given by $E_\gamma = \dfrac{\alpha_{SB} T_0^3}{n_{\gamma,0} k_B} (k_B T)$, where α_{SB}, T_0, $n_{\gamma,0}$, and k_B are the Stephan-Bolzman constant, the current background radiation temperature, the photon number density at present, and the Bolzman constant.

We know that $\mathcal{E}_\gamma = \alpha_{SB} T^4$

and $E_\gamma = \dfrac{\mathcal{E}_\gamma}{n_\gamma}$ ← mean photon energy

$n_\gamma(t)\, a^3(t) = n_{\gamma,0}\, a_0^3 = n_{\gamma,0}$

$n_\gamma = n_{\gamma,0}\, a^{-3}(t)$

$$\Rightarrow E_\gamma = \frac{\alpha_{SB} T^4}{n_{\gamma,0}\, a^{-3}} = \frac{\alpha_{SB} T}{n_{\gamma,0}} (aT)^3$$

$$= \frac{\alpha_{SB} (k_B T)}{n_{\gamma,0} k_B} (a_0 T_0)^3 = \frac{\alpha_{SB} T_0^3}{n_{\gamma,0} k_B} (k_B T).$$

b) Show from the energy density in radiation that $aT =$ constant, where a and T are the scale factor and the temperature.

$$\mathcal{E}_\gamma = \alpha_{SB} T^4 = \frac{\mathcal{E}_{r,0}}{a^4} \Rightarrow (aT)^4 = \frac{\mathcal{E}_{r,0}}{\alpha_{SB}}$$

$$\Rightarrow aT = \left(\frac{\mathcal{E}_{r,0}}{\alpha_{SB}}\right)^{1/4} = \text{const.}$$

9. a) In the early universe when photons, electrons, neutrinos, protons, and neutrons are in equilibrium at a temperature T write down the expressions for the number densities n_n and n_p for neutrons and protons.

$$n_n = g_n \left(\frac{m_n k_B T}{2\pi \hbar^2}\right)^{3/2} \exp\left(-\frac{m_n c^2}{k_B T}\right) \quad ; \quad g_n = 2$$

$$n_p = g_p \left(\frac{m_p k_B T}{2\pi \hbar^2}\right)^{3/2} \exp\left(-\frac{m_p c^2}{k_B T}\right) \quad ; \quad g_p = 2$$

b) Hence find the ratio $\frac{n_n}{n_p}$.

$$\frac{n_n}{n_p} = \overbrace{\left(\frac{m_n}{m_p}\right)^{3/2}}^{\approx 1} \exp\left(-\frac{(m_n - m_p)c^2}{k_B T}\right)$$

$$= \exp\left(-\frac{Q_n}{k_B T}\right) \quad ; \quad Q_n = (m_n - m_p)c^2$$

c) When is the ratio $\frac{n_n}{n_p}$ is frozen?

When Γ, the rate at which neutrons and protons interact with neutrinos, equals H, the expansion rate of the universe, the ratio n_n/n_p is frozen

d) What is the ratio $\left(\frac{n_n}{n_p}\right)_{freeze}$ equal to?

$$\left(\frac{n_n}{n_p}\right)_{freeze} \approx 0.2 = \frac{1}{5}$$

10. a) Define the Omega Parameter Ω.

$$\Omega = \frac{\rho}{\rho_c} = \frac{\mathcal{E}}{\mathcal{E}_c}$$, where $\mathcal{E}_c$ is the critical energy density.

b) Using the Friedmann equation find $\frac{1-\Omega}{\Omega}$.

$$H^2 = \frac{8\pi G}{3c^2}\mathcal{E} - \frac{kc^2}{R_0^2 a^2}$$

$$1 = \frac{8\pi G}{3c^2}\frac{\mathcal{E}}{H^2} - \frac{kc^2}{R_0^2 a^2 H^2}$$

$$= \frac{\mathcal{E}}{\mathcal{E}_c} - \frac{kc^2}{R_0^2 a^2 H^2}$$

$$= \Omega - kc^2/(R_0^2 a^2 H^2)$$

$$1 - \Omega = -\frac{kc^2}{R_0^2 a^2 H^2}$$

$$\frac{1-\Omega}{\Omega} = -\frac{kc^2}{R_0^2 a^2 H^2}\frac{\mathcal{E}_c}{\mathcal{E}} = \frac{-kc^2}{R_0^2 a^2 H^2}\frac{3c^2 H^2}{8\pi G\mathcal{E}}$$

$$= -\frac{3kc^4}{R_0^2 a^2 8\pi G}\frac{1}{\mathcal{E}}.$$

c)(i) How does $\mathcal{E}$ (or ρ) vary with the scale factor a in the standard model of cosmology when the universe is matter-dominated?

$$\mathcal{E} = \mathcal{E}_{m,0}\, a^{-3}$$

(ii) Thus find how $\left|\frac{1-\Omega}{\Omega}\right|$ varies with a in the matter-dominated era.

$$\frac{1-\Omega}{\Omega} = (\quad)\frac{1}{\mathcal{E}} = (\quad)\frac{a^3}{\mathcal{E}_{m,0}}$$

$$= \frac{\text{const.}}{a^2}\frac{a^3}{\mathcal{E}_{m,0}} = \text{const.}\, a$$