GENERAL RELATIVITY
EXAM 1

April 9, 2021

Name:

Number:

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1. Write down how the following objects transform under a coordinate transformation from $x^\alpha \to x'^\alpha$. (Make sure that the primed objects are on the left hand side. F is a scalar function)

a) B'_μ

b) B'^μ

c) $R'^\mu{}_\nu$

d) $T'^{\mu\nu}{}_{\sigma\gamma}$

e) $\partial'_\mu F'$

f) $A'^\mu B'^\nu$

g) $A'_\mu B'_\nu$

h) $A'_\mu B'^\mu$

i) F'

j) $\partial'_\mu \partial'^\mu F'$

2. Calculate the following $(i, j = 1, 2, 3\,;\ \alpha, \beta = 0, 1, 2, 3)$:

a) i) $g_{\mu 1}\, g^{\mu 1}$

(ii) $\delta^{\alpha}_{\ 2}\, \delta^{2}_{\ \alpha}$

(iii) $\delta^{1}_{\ 3}\, \delta^{3}_{\ 1}$

(iv) $\eta_{ij}\, \eta^{ij}$

(v) $g^{\mu\nu}\, g_{\mu\nu}$

b) Circle the equations that violate the rules for indices in the following:

(i) $p^{\mu} = m\, g^{\mu\nu}\, U_{\mu}$

(ii) $\eta_{\mu\nu}\, p^{\mu} p^{\nu} = -m^2 c^2$

(iii) $\dfrac{dp^{\mu}}{d\tau} = q F^{\mu\nu}\, U^{\nu}$

(iv) $B_{\mu\nu} = \Lambda^{\mu\nu} A^{\alpha} A_{\alpha}$

(v) $R^{\mu}_{\ \nu}\, R^{\nu}_{\ \mu} = T^{\mu\nu}$

3. Consider the following spacetime metric

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{r_s}{r}\right), & 0, & 0, & 0 \\ 0 & \left(1-\frac{r_s}{r}\right)^{-1}, & 0, & 0 \\ 0 & 0 & r^2, & 0 \\ 0 & 0 & 0, & r^2\sin^2\theta \end{pmatrix}$$

a) Write down the inverse metric $g^{\mu\nu}$

b) Given $X_\mu = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ and $Y^\mu = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Find X^μ and Y_μ.

4) The following covariant and contravariant vectors are given

$$X_\mu = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad Y^\mu = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

Calculate $X_\mu Y^\mu$ in 4-dimensional

a) Euclidean spacetime

b) Minkowski spacetime

c) Schwarzschild spacetime whose metric is given in question 3.

5. Consider a metric given by

$$ds^2 = g_{tt} c^2 dt^2 + g_{ij} dx^i dx^j$$

a) Find g_{tt} such that the proper time and the coordinate time become identical.

b) By using the Minkowski metric find the speed of light.

6. The electromagnetic field tensor $F^{\alpha\beta}$ is defined as

$$F^{\alpha\beta} = \partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha},$$

where A^{α} is the electromagnetic four-potential.

a) Show that $F^{\alpha\beta}$ is antisymmetric

b) Find the value of $\partial_{\alpha}\partial_{\beta}F^{\alpha\beta}$

c) Find the value of $\partial^{\alpha}F^{\mu\nu} + \partial^{\nu}F^{\alpha\mu} + \partial^{\mu}F^{\nu\alpha}$

7. Consider a hypothetical two-dimensional spacetime metric given by

$$ds^2 = -\left(1-\frac{R}{r}\right)c^2 dt^2 + 2c\,dt\,dr + \left(1-\frac{R}{r}\right)^{-1} dr^2$$

a) Write down the matrix for the metric $g_{\mu\nu}$ $(\mu = 0, 1)$

b) Find the matrix for $g^{\mu\nu}$.

8. In a hypothetical multi-dimensional spacetime the metric tensor satisfies $g_{\alpha\beta}\, g^{\alpha\beta} = 11$

a) What are the numerical values that α and β can take on?

b) What is the dimension of this spacetime?

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9. a) Given the tensor $A_{\alpha\beta}$ in a spacetime whose metric tensor is some $g_{\alpha\beta}$. Raise the index to find $A^{\alpha}{}_{\beta}$.

b) Given the tensor $B^{\alpha\beta}$. Lower the index to find $B^{\alpha}{}_{\beta}$.

c) Say true or false: $A^{\alpha\beta}{}_{\mu} + B^{\alpha\beta}{}_{\nu} = C^{\alpha\beta}{}_{\mu\nu}$

d) Say true or false: $A^{\alpha\beta}\, B^{\gamma}{}_{\alpha\beta} = C^{\gamma}$

e) Say true or false: $A_{\alpha} B^{\alpha} + A^{\beta} B_{\beta} + C^{\gamma}{}_{\gamma} = F$
where F is a scalar.

10. a) Write down the 4-velocity $\underset{\sim}{U}$ as a column vector.

b) Find the value of the scalar product $\underset{\sim}{U} \cdot \underset{\sim}{U}$

c) Write down the 4-momentum $\underset{\sim}{P}$ as a column vector.

d) Find the value of the scalar product $\underset{\sim}{P} \cdot \underset{\sim}{P}$