INTRODUCTION TO COSMOLOGY
SECOND EXAM

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June 5, 2017

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1) a) Define the critical energy density $\mathcal{E}_c(t)$ and express the Friedmann equation in terms of $\mathcal{E}_c(t)$.

$$H^2 = \frac{8\pi G}{3c^2}\mathcal{E} - \frac{kc^2}{R_0^2 a^2}$$

$$\mathcal{E}_c(t) = \frac{3c^2 H^2}{8\pi G}$$

$$1 = \frac{\mathcal{E}(t)}{\mathcal{E}_c(t)} - \frac{kc^2}{R_0^2 a^2 H^2}$$

b) Define the parameter $\Omega(t)$ and express the Friedmann equation in terms of $\Omega(t)$.

$$\Omega(t) = \frac{\mathcal{E}(t)}{\mathcal{E}_c(t)} \quad ; \quad 1 = \Omega(t) - \frac{kc^2}{R_0^2 a^2 H^2}$$

c) Express the Friedmann equation in a form which has $H(t)^2/H_0^2$ on the left.

$$\frac{H^2(t)}{H_0^2} = \frac{8\pi G}{3c^2 H_0^2}\mathcal{E}(t) - \frac{kc^2}{R_0^2 a(t)^2 H_0^2} = \frac{\mathcal{E}(t)}{\mathcal{E}_{c,0}} - \frac{kc^2}{R_0^2 a(t)^2 H_0^2}$$

$$\mathcal{E}(t) = \mathcal{E}_r(t) + \mathcal{E}_m(t) + \mathcal{E}_\Lambda = \frac{\mathcal{E}_{r,0}}{a^4} + \frac{\mathcal{E}_{m,0}}{a^3} + \mathcal{E}_\Lambda$$

$$\frac{H^2(t)}{H_0^2} = \frac{\Omega_{r,0}}{a^4(t)} + \frac{\Omega_{m,0}}{a^3(t)} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a(t)^2}$$

2) Consider a closed universe with an energy density given by $\mathcal{E} = \frac{\mathcal{E}_{*,0}}{a^2} + \frac{\mathcal{E}_{m,0}}{a^3}$. The cosmological constant is zero.

a) Find $\mathcal{E}_{*,0}$ ~~in terms of a~~ so that the universe has the critical energy density.

$$H^2 = \frac{8\pi G}{3c^2}\left(\frac{\mathcal{E}_{*,0}}{a^2} + \frac{\mathcal{E}_{m,0}}{a^3}\right) - \frac{kc^2}{R_0^2 a^2} \quad ; \text{if we let } \frac{8\pi G}{3c^2}\mathcal{E}_{*,0} = \frac{kc^2}{R_0^2}$$

($k=1$ for a closed universe)

$$\Rightarrow H^2 = \frac{8\pi G}{3c^2}\frac{\mathcal{E}_{m,0}}{a^3}$$

b) What is the value of the density parameter Ω_m for such a universe ?

$$1 = \frac{8\pi G}{3c^2}\frac{\mathcal{E}_{m,0}}{H^2} = \frac{\mathcal{E}_{m,0}}{\mathcal{E}_c} = \Omega_m . \text{ Thus } \Omega_m = 1$$

c) Find the expression for pressure P for such a universe by using the fluid equation.

Fluid equation $\dot{\mathcal{E}} + 3\frac{\dot{a}}{a}(\mathcal{E} + P) = 0$

$$-2\frac{\mathcal{E}_{*,0}\dot{a}}{a^3} - \cancel{3\frac{\mathcal{E}_{m,0}\dot{a}}{a^4}} + 3\frac{\dot{a}}{a}\left(\frac{\mathcal{E}_{*,0}}{a^2} + \cancel{\frac{\mathcal{E}_{m,0}}{a^3}} + P\right) = 0$$

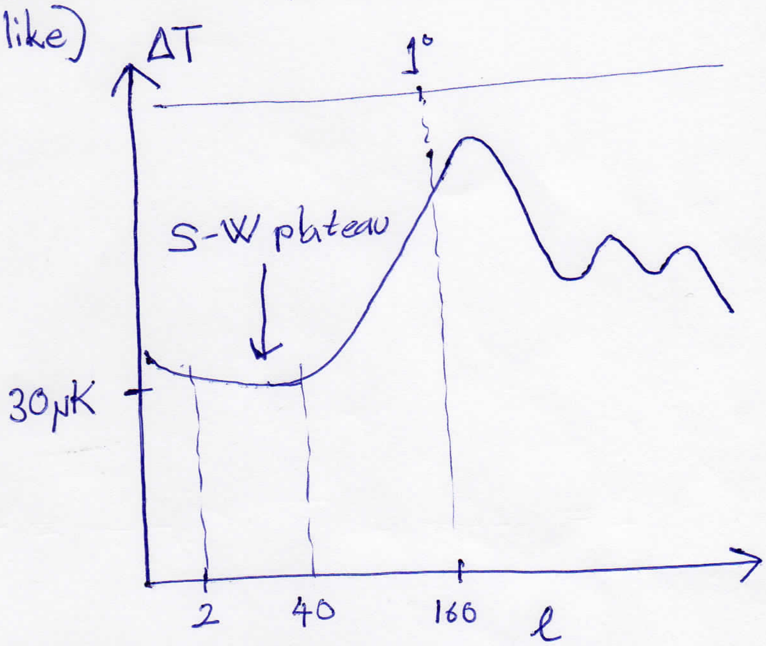
$$\frac{\mathcal{E}_{*,0}}{a^2} + 3P = 0 \Rightarrow P = -\frac{1}{3}\frac{\mathcal{E}_{*,0}}{a^2}$$

3) a) What is the Sachs-Wolfe effect?

The creation of temperature fluctuations by variations in the gravitational potential is known as the Sachs-Wolfe effect.

b) What is the Sachs-Wolfe plateau?
(Draw a figure if you like)

The region of the ΔT curve where the temperature fluctuations are nearly constant ($2 < \ell < 40$) is known as the Sachs-Wolfe plateau.



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4) The rate at which neutrons and protons interact with neutrinos is given by $\Gamma \approx G_F^2 T^5$, where G_F is the Fermi constant of weak interactions. In the early radiation-dominated universe the Hubble parameter is given by $H = 1.66\sqrt{g_*}\sqrt{G}\,T^2$, where g_* is the number of spin states.

a) Find the temperature T_f at which the ratio of the number of neutrons to the number of protons is frozen.

At freeze-out $\Gamma = H$

$G_F^2 T_f^5 = \text{const.}\, T_f^2$

$$T_f = \left(\frac{\text{const.}}{G_F^2}\right)^{1/3} = \left(\frac{1.66\sqrt{g_*}\sqrt{G}}{G_F^2}\right)^{1/3}$$

b) In kinetic equilibrium the number of x-particles is given by $n_x = g_x \left(\frac{m_x k_B T}{2\pi\hbar^2} \right)^{3/2} \exp\left(-\frac{m_x c^2}{k_B T} \right)$,

where g_x is the number of spin states for the x-particle.

Write the expression for the ratio of the number of neutrons to the number of protons at $T = T_f$

$$g_n = g_p = 2 \Rightarrow \frac{n_n}{n_p} = \left(\frac{m_n}{m_p} \right)^{3/2} \exp\left(-\frac{c^2}{k_B T_f} (m_n - m_p) \right)$$

c) What is the (approximate) numerical value of the ratio in part b) ?

At $T = T_f$ $\frac{n_n}{n_p} \approx \frac{1}{5}$

d) What are the interactions between neutrons, protons, electrons, and neutrinos ?

$$n + \nu_e \rightleftarrows p + e^-$$

$$n + e^+ \rightleftarrows p + \overline{\nu}_e$$

5) a) Consider a system of N particles (galaxies) interacting gravitationally. Write down the expression for the virial theorem when the moment of inertia I of the system is constant.

$$\ddot{I} = 2U + 4K$$

when $I =$ constant $\ddot{I} = 0$

$$2K + U = 0 \quad \text{or} \quad K = -\frac{1}{2}U$$

b) Express the total mass M of the system in terms of the average of the square of the velocities of the particles and the half-mass radius r_h of the system.

$$\left.\begin{aligned} K &= \frac{1}{2}M\langle v^2\rangle \\ U &= -\alpha\frac{GM^2}{r_h} \end{aligned}\right\} \quad \frac{1}{2}M\langle v^2\rangle = \frac{1}{2}\alpha\frac{GM^2}{r_h} \quad \Rightarrow M = \frac{\langle v^2\rangle r_h}{\alpha G}$$

c) Is the mass for the Coma cluster obtained this way more or less than the observed mass?

More. $\left(M_{calculated}/M_{observed} \approx 10 \right)$

d) What is the conclusion one obtains from part c)?

There must be invisible mass (dark matter) to hold the cluster together.

6) a) How does the energy density of the cosmic microwave background depend on the temperature?

$$\varepsilon_r = \alpha_{SB} T^4$$

b) How does the energy density in radiation depend on the scale factor?

$$T = \frac{\text{const.}}{a(t)} \; ; \; T_0 = \frac{\text{const.}}{a_0} = \frac{\text{const.}}{1}$$

$$\Rightarrow \quad T = T_0 / a(t)$$

c) From parts a) & b), express the cosmic microwave background temperature as a function of the scale factor.

$$\varepsilon_r = \frac{\varepsilon_{r,0}}{a(t)^4} = \alpha_{SB} T^4 \qquad \Bigg| \qquad T = \left(\frac{\varepsilon_{r,0}}{\alpha_{SB}}\right)^{1/4} \frac{1}{a}$$

$$T^4 = \frac{\varepsilon_{r,0}}{\alpha_{SB}} \frac{1}{a^4}$$

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7) a) What was the temperature of radiation-dominated universe at the Planck time?

From $T(t) = 0.997 \times 10^{10} K \left(t/1s\right)^{-1/2}$; $t_{Pl} \approx 10^{-44} s$

$$T_{Pl} \approx 10^{10} K \left(10^{44/2}\right) = 10^{32} K.$$

b) Show that in the radiation-dominated era the mean photon energy is given by $E_\gamma = \frac{\alpha_{SB} T_0^3}{n_{\gamma,0} k_B} (k_B T)$.

$$E_\gamma = \frac{\varepsilon_r}{n_\gamma} = \frac{\alpha_{SB} T^4}{n_\gamma} \quad ; \text{ but } \quad n_\gamma(t)\, a^3(t) = n_{\gamma,0} a_0^3 = n_{\gamma,0}$$

$$= \frac{\alpha_{SB} T^4}{n_{\gamma,0}/a^3(t)} = \frac{\alpha_{SB} T (a^3 T^3)}{n_{\gamma,0}} = \frac{\alpha_{SB} (a_0^3 T_0^3)(k_B T)}{n_{\gamma,0} \cdot k_B} = \frac{\alpha_{SB} T_0^3 (k_B T)}{n_{\gamma,0} k_B}$$

8) a) What is the ratio $\frac{\Delta T}{T}$ for the CMB (Cosmic Microwave Background) radiation determined by experimental groups?

$$\frac{\Delta T}{T} \approx 10^{-5}$$

b) i) What is the current value of T_0?

$$T_0 = 2.7255\ K$$

ii) What is the implication of $\Delta T_0 \approx 30\ \mu K$?

The CMB is very closely isotropic.

c) What is the epoch of recombination?

The time at which the baryonic component of the universe goes from being ionized to being neutral. (The time at which H atoms and He atoms are formed.)

d) What is the epoch of photon decoupling?

The time at which photons cease to interact with the electrons (as a result of which the universe becomes transparent.)

e) Write down the Saha equation for recombination.

$$H + \gamma \rightleftharpoons p + e^-$$

$$\frac{n_H}{n_p n_e} = \frac{g_H}{g_p g_e}\left(\frac{m_H}{m_p m_e}\right)^{3/2}\left(\frac{k_B T}{2\pi\hbar^2}\right)^{-3/2} \exp\left[\frac{(m_p + m_e - m_H)c^2}{k_B T}\right]$$

$$g_p = g_e = 2 \ ; \ g_H = 1 + 3 = 4$$

9) a) Obtain the relation $H(z) = -\frac{1}{(1+z)}\frac{dz}{dt}$, where z is the redshift.

$$H = \frac{\dot{a}}{a} = \frac{da}{dt}\frac{1}{a} = \frac{1}{a}\frac{da}{dz}\frac{dz}{dt}$$

$$1+z = \frac{a_0}{a(t)} = \frac{1}{a(t)} \; ; \; a(t) = \frac{1}{1+z} \; ; \; \frac{da}{dz} = -\frac{1}{(1+z)^2}$$

$$\Rightarrow H(z) = (1+z)\left[-\frac{1}{(1+z)^2}\right]\frac{dz}{dt} = -\frac{1}{(1+z)}\frac{dz}{dt}$$

b) For an object of luminosity L at a distance d_L from us, define the apparent and absolute magnitudes m and M.

$$m = -2.5 \log\left(\frac{L}{4\pi d_L^2}\right)$$

$$M = -2.5 \log\left(\frac{L}{4\pi (10pc)^2}\right) \leftarrow d_L = 10pc$$

c) Obtain the expression for "distance modulus" $m-M$ when d_L is measured in parsecs.

$$m-M = -2.5\left[\cancel{\log L} - \cancel{\log 4\pi} - \log d_L^2 - \cancel{\log L} + \cancel{\log 4\pi} + \log 10^2\right]$$

$$m-M = 5\log d_L - 5$$

d) Find the distance d_L to an object whose distance modulus is 25.

$$\log d_L = \frac{m-M+5}{5}$$

$$d_L = 10^{\left(\frac{m-M+5}{5}\right)} = 10^{30/5} = 10^6 \text{ parsecs} = 1\,\text{Mpc}$$

10) Consider the reactions i) $p+n \rightleftarrows D+\gamma$ and

ii) $p+p \rightleftarrows {}^2He$

a) Which reaction occurs most often and why?

Reaction i) occurs most often because reaction ii) occurs through electromagnetism and strong interactions, and there is a Coulomb repulsion (barrier) between the two protons, whereas reaction i) has no such barrier and it occurs through strong interactions.

b) Before nucleosynthesis begins in stars, the primordial helium fraction of the mass of the universe is defined by Y_p = density of helium atoms / density of baryons.

Find Y_p at the time of neutron-proton freeze out.

$$Y_p = \frac{\rho_{He}}{\rho_{baryons}} = \frac{2N_n}{N_n+N_p} = \frac{2}{1+\frac{N_p}{N_n}} \approx \frac{2}{1+5}$$

$$Y_p \approx \frac{1}{3}$$