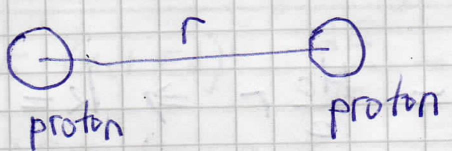
Ch. 3 Newton vs Einstein

On cosmological scales, (scales greater than 100 Mpc) the dominant force determining the evolution of the universe is gravity.

* The weak force is effective on scales of $l_w \sim 10^{-18}$ m or less
  the strong force " " $l_s \sim 10^{-15}$ or less

* gravity & electromagnetism are long range forces. Gravity is negligible on small scales compared to electromagnetism.



$$F_G = G\frac{m_p m_p}{r^2}, \quad F_e = k_e \frac{e \cdot e}{r^2}$$

$$\frac{F_e}{F_G} = \frac{k_e e^2}{G\, m_p^2} = \frac{9\times 10^9 \left(1.6\times 10^{-19}\right)^2}{6.67\times 10^{-11}\left(1.67\times 10^{-27}\right)^2}$$

$$= 1.24 \times 10^{36}$$

+ On large scales the universe is electrically neutral and so there are no electric forces

* Intergalactic magnetic fields are sufficiently small and can be negligible on cosmological scales.

Ways of looking at gravity:

1) The classical Newtonian viewpoint: Gravity is a force that causes massive bodies to be accelerated.
2) The modern Einsteinian viewpoint: Gravity is a manifestation of the curvature of spacetime.

* In most contexts they yield the same predictions.

In the limit of deep potential minima (to use Newtonian language or strong spatial curvature (to use general relativistic language) general relativity yields the correct results.

The way of Newton:

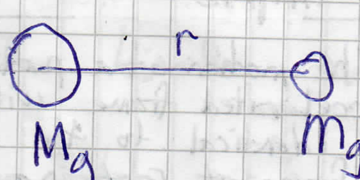
The space is unchanging and Euclidean.

Euclidean space = Flat space.

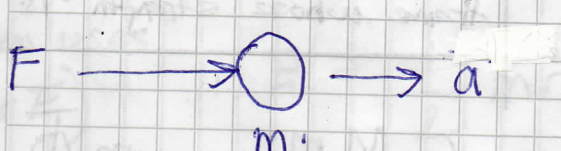
Flat does not mean 1-dimensional. There are 2, 3, ... dimensional flat spaces.

* Shortest distance between two points is a straight line.
* The sum of angles in a triangle add up to 180°.
* Circumference of a circle = $2\pi r$
* If no net force acts on an object it moves on a straight line at constant speed (Newton's 1st law)

Two masses, M_g and M_i:



$$F_G = G\frac{M_g m_g}{r^2}$$; Gravitational mass

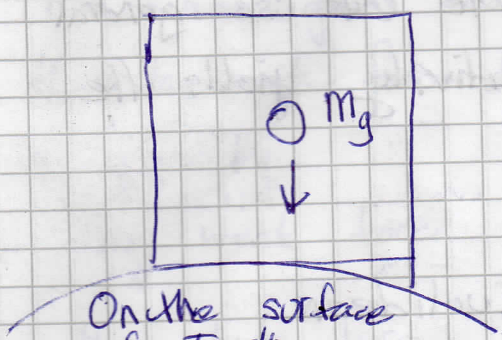


$$F = m_i a$$; Inertial mass

Experimental finding: $m_g = m_i$

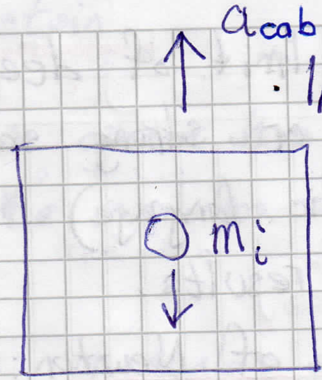
$$\frac{m_i}{m_g} = 1 + 10^{-13}$$

Gravitational Force



On the surface of Earth

$$\vec{F}_G = -G\frac{M_E \cdot m_g}{r^2}\hat{r} = g\, m_g (-\hat{r})$$



· Inertial force (Fictitious force)

In deep space where $F_G = 0$

$$\vec{F}_{fic} = m_i \vec{a} = -m_i \vec{a}_{cab} = m_i a_{cab}(-\hat{r})$$

$$F = F_G$$

$$g\, m_g = a_{cab}\, m_i \Rightarrow a_{cab} = g\left(\frac{m_g}{m_i}\right)$$

Principle of Equivalence:

The Gravitational Force and the Inertial Force produce indistinguishable effects if $a = g$ (numerically). This will be true if and only if $m_g = m_i$ #

The motion of a particle in an accelerated frame with $a = g$ is identical to its motion in a frame at rest in a gravitational frame whose strength is g.

$$F_G = G\frac{M_g m_g}{r^2}$$

Gravitational Potential Energy $V_G = -G\dfrac{Mm}{r} = m\Phi_G$

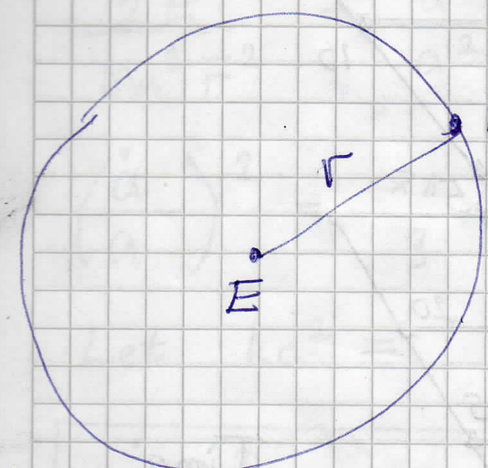
$F_G = -\dfrac{\partial V_G}{\partial r}$, or in vector form

$$\vec{F}_G = -m_g \vec{\nabla}\Phi_G = +m_i \vec{a} \Rightarrow \vec{a} = -\vec{\nabla}\Phi_G$$

because $m_g = m_i$

Expansion in a Newtonian World:

Consider an observer (us) in the expanding medium of the universe with mass density ρ. Since the universe looks the same everywhere, we can consider the Earth as its center. Next consider a "particle" m a distance r away



The total mass in the sphere of radius r is:

$$M = \frac{4\pi}{3}\rho r^3$$

The gravitational force on m is

$$F_G = \frac{GMm}{r^2} = \frac{4\pi}{3}G\rho m r$$

The particle m has a gravitational potential energy

$$V_G = -\frac{GMm}{r} = -\frac{4\pi}{3}G\rho m r^2$$

The kinetic energy of the particle is $T = \frac{1}{2}m\dot{r}^2$

The total mechanical energy $E = T + V_G$ is

(Ryden uses U for E)

$$E = \frac{1}{2}m\dot{r}^2 - \frac{4\pi}{3}G\rho m r^2$$

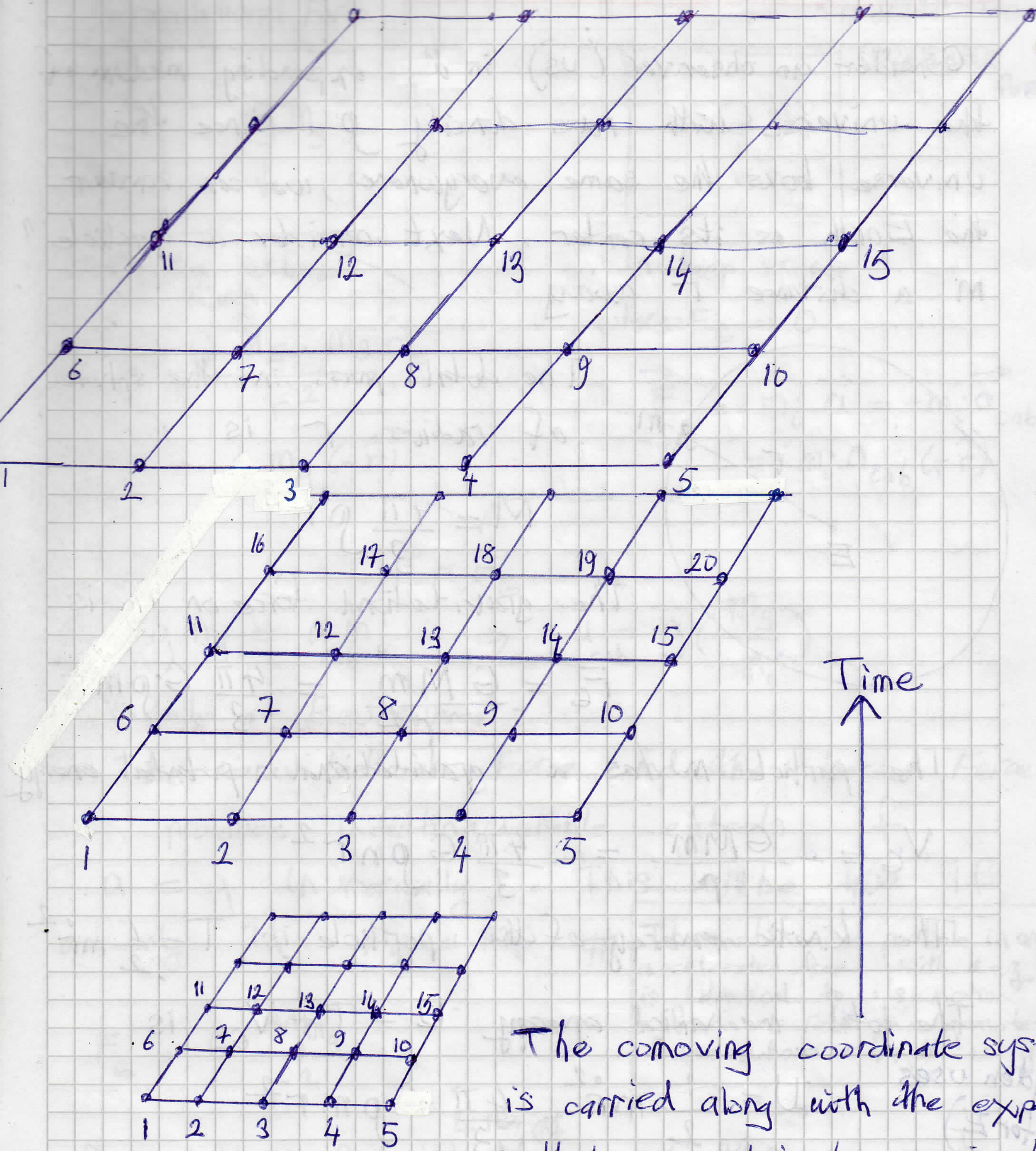
Comoving coordinates:

Coordinates that are carried along with the expansion. The galaxies remain at fixed locations in the $\vec{r}$ coordinate system:

$$\vec{r}(t) = a(t)\,\vec{r}$$

$a(t)$: the scale factor of the Universe.

$\vec{r}$: comoving coordinates, which are time independent



The comoving coordinate system is carried along with the expansion so that any objects remain at a fixed coordinate values

"•" : galaxies

conventions : i) the comoving radial coordinate is unitless and $a(t)$ has distance units.

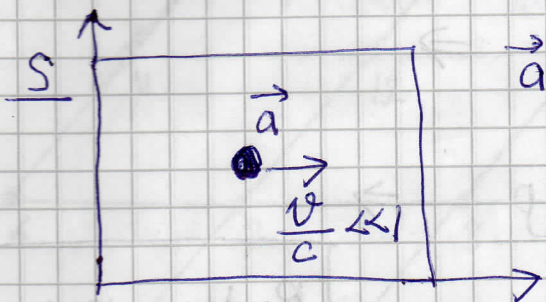
Ryden's convention → ii) $a(t)$ is unitless, $a(t_0) = 1$ and $r(t)$ & r have units of distance.

$r(t_0) = r$ at present.

The special way of Einstein : (special relativity)

* gravity is not present
* space is Euclidean

An inertial reference frame :



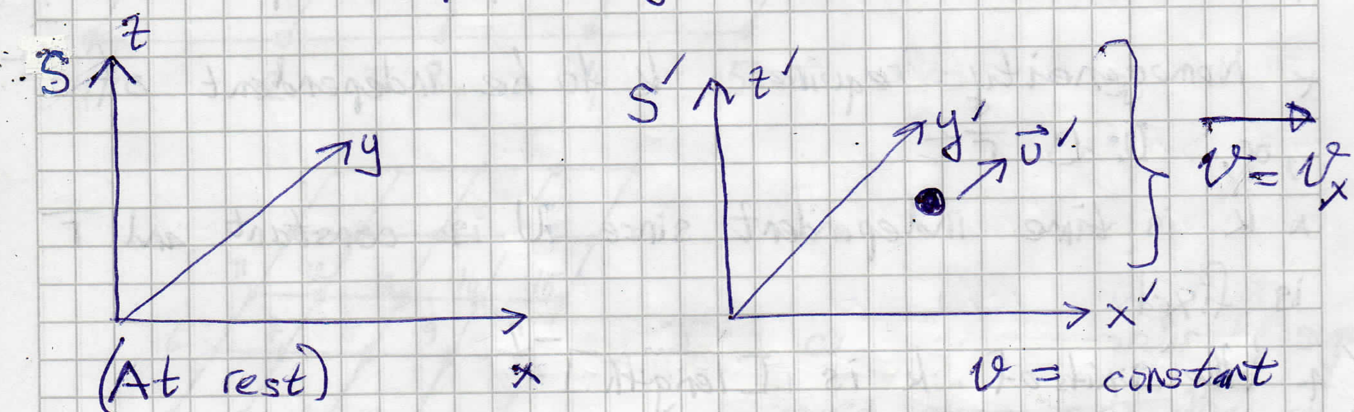
$\vec{a}$: acceleration of a particle relative to the frame S

If the particle obeys

$$\frac{d^2\vec{r}}{dt^2} = \frac{1}{m}\vec{F}$$

then the frame S is inertial.

* A rotating reference frame is not inertial due to Coriolis and centrifugal forces.
* Earth is approximately inertial



Both S and S' are inertial frames

* In Newtonian physics time is absolute

The coordinates are related by the Galilean transformations.

$$x' = x - vt$$

$$y' = y , \quad z' = z$$

$$t' = t$$

$\vec{u}'$: velocity of a particle relative to the S' frame

$$\vec{u}' = \vec{u} - v\,\hat{e}_x$$

velocity of the particle relative to S.

Einstein's first postulate of special relativity : The equations describing the basic laws of physics are the same in all inertial reference frames.

Second postulate : The speed of light in a vacuum has the same value c in all inertial reference frames.

(Michelson Morley experiment (1887)

The frames S and S' coincide at $t = t' = 0$.
A light signal emitted at the origin at $t = t' = 0$

the radius of the sphere of light satisfies

$$R^2 = x^2 + y^2 + z^2 = c^2 t^2$$

$$R'^2 = x'^2 + y'^2 + z'^2 = c^2 t'^2$$

$$\Rightarrow \quad 0 = -c^2 t^2 + x^2 + y^2 + z^2 = -c^2 t'^2 + x'^2 + y'^2 + z'^2$$

This cannot be satisfied by Galilean transformations.
New transformations :

$$\left.\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}\right\} \text{The Lorentz transformations}$$

Time is not absolute!

where $\gamma \equiv \dfrac{1}{\sqrt{1 - v^2/c^2}}$

The spacetime interval $(\Delta s)^2$:

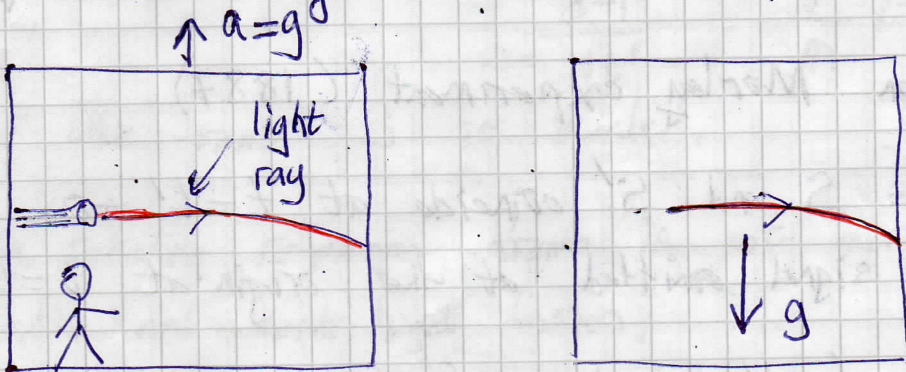
$$(\Delta s)^2 = -c^2(t_1-t_2)^2 + \overbrace{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}^{(\Delta l)^2}$$
$$= -c^2(\Delta t)^2 + (\Delta l)^2$$

In S': $(\Delta s')^2 = -c^2(\Delta t')^2 + (\Delta l')^2$

It can be shown by using the Lorentz transformations that

$$(\Delta s')^2 = (\Delta s)^2$$

The General way of Einstein (General Relativity):



The path of a light ray in a frame accelerated up with $a=g$. The path is not straight but it is curved.

Equivalence principle requires that the path of a light should be curved in a frame at rest in a gravitational field with strength g.

* In Euclidean space the shortest path between two points is a straight line. (Gravity absent)

* When gravity is present the path taken by light is not a straight line. → Space is not Euclidean in the presence of gravity.

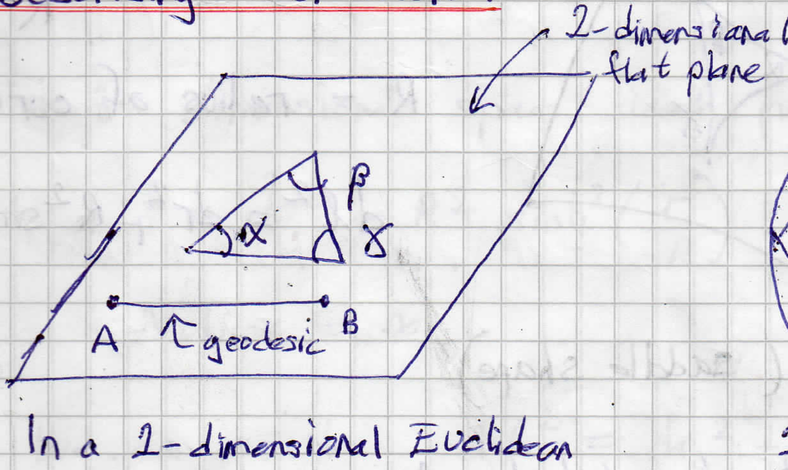
* The presence of mass (that creates the gravitational field)

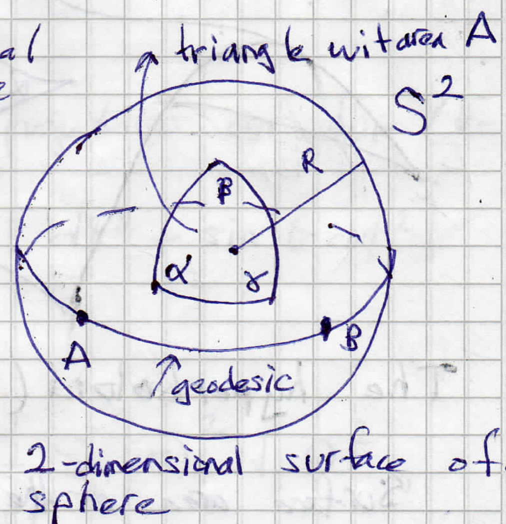
causes space to be curved.

* since $E = mc^2$, the presence of mass & energy causes space to be curved.

* A falling object in a gravitational field follows a geodesic in curved space-time.

Describing Curvature:





In a 2-dimensional Euclidean or flat space $\alpha + \beta + \gamma = \pi$

2-dimensional surface of a sphere

$$\alpha + \beta + \gamma = \pi + \frac{A}{R^2}$$

Distance between points (r, θ) and $(r+dr, \theta + d\theta)$ is

$$dl^2 = dr^2 + r^2 d\theta^2$$

Area of the plane $= \infty$

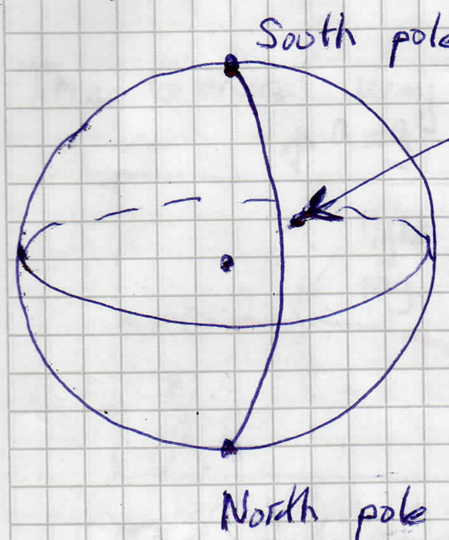
* All spaces in which $\alpha + \beta + \gamma > \pi$ are called positively curved spaces

* The curvature of a sphere is homogeneous and isotropic (uniform curvature)

+ distance between points (r, θ) and $(r+dr, \theta+d\theta)$ is

$$dl^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2$$

Area of the surface of a sphere $= 4\pi R^2$

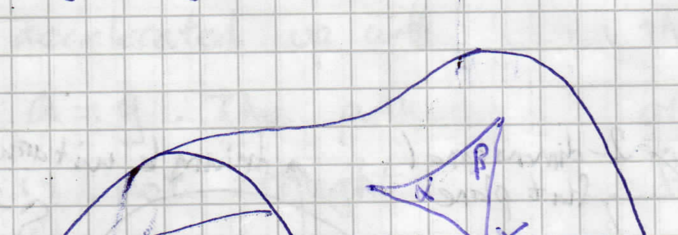


$l_{max} = \pi R$ ← max distance possible when points are the S & N poles.

(For a plane $l_{max} = \infty$)

$R =$ radius of sphere
$=$ radius of curvature

A Negatively curved two-dimensional space



$$\alpha + \beta + \gamma = \pi - \frac{A}{R^2}$$

R = radius of curvature

$$dl^2 = dr^2 + R^2 \sinh^2\left(\frac{r}{R}\right) d\theta^2$$

The hyperboloid (saddle shape)

Surface area of the hyperboloid $= \infty$

Maximum distance between two points $= \infty$

* Two dimensional homogeneous and isotropic spaces can be specified by two quantities, k and R.

k = curvature constant

R = radius of curvature

$$k = \begin{cases} 0, & \text{flat space} \\ +1, & \text{+ively curved space (surface of a sphere)} \\ -1, & \text{−ively " " (hyperboloid)} \end{cases}$$

Metrics for 3-dimensional spaces

* $dl^2 = dx^2 + dy^2 + dz^2$ (3-dim. Euclidean space)

$= dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$; in (r, θ, ϕ) coordinates

Volume = ∞

* a 3-dimensional space with positive curvature ($k = 1$)

$$dl^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

Volume = finite = $\frac{4\pi}{3} R^3$

* a 3-dimensional space with negative curvature ($k = -1$)

$$dl^2 = dr^2 + R^2 \sinh^2\left(\frac{r}{R}\right)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

Volume = ∞

* We can write $dl^2 = dr^2 + S_k(r)^2\, d\Omega^2$,

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$,

and

$$S_k(r) = \begin{cases} R\sin(r/R), & k = +1 \\ r, & k = 0 \\ R\sinh(r/R), & k = -1 \end{cases}$$

* in the limit $r \ll R$, $S_k \approx r$

* For $k = 0$ and -1, $S_k \to \infty$ as $r \to \infty$

* For $k = +1$, $S_{max} = R$ at $\frac{r}{R} = \frac{\pi}{2}$

  $S_{min} = 0$ at $\frac{r}{R} = \pi$

* If we switch the radial coordinate from r to $x \equiv S_k(r)$ the metric for a homogeneous and isotropic 3-dimensional space can be written in the form

$$d\ell^2 = \frac{dx^2}{1 - k\frac{x^2}{R^2}} + x^2 d\Omega^2$$

* Four-dimensional homogeneous and isotropic spacetimes:

Consider two spacetime points (t, r, θ, ϕ) and $(t+dt, r+dr, \theta+d\theta, \phi+d\phi)$

The spacetime separation between these two points (events) is

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\Omega^2$$ (the Minkowski metric)

A photon's trajectory obeys

$$ds^2 = 0 = -c^2dt^2 + dr^2 + r^2d\Omega^2$$

For a radially moving photon $d\Omega^2 = 0 \Rightarrow$

$$c^2dt^2 = dr^2$$

$$\frac{dr}{dt} = \pm c$$

* Minkowski metric applies when there is no gravity

If the universe is spatially homogeneous and isotropic, and if distances expand or contract as a function of time, then the metric of the universe is

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_k(r)^2 d\Omega^2 \right]$$

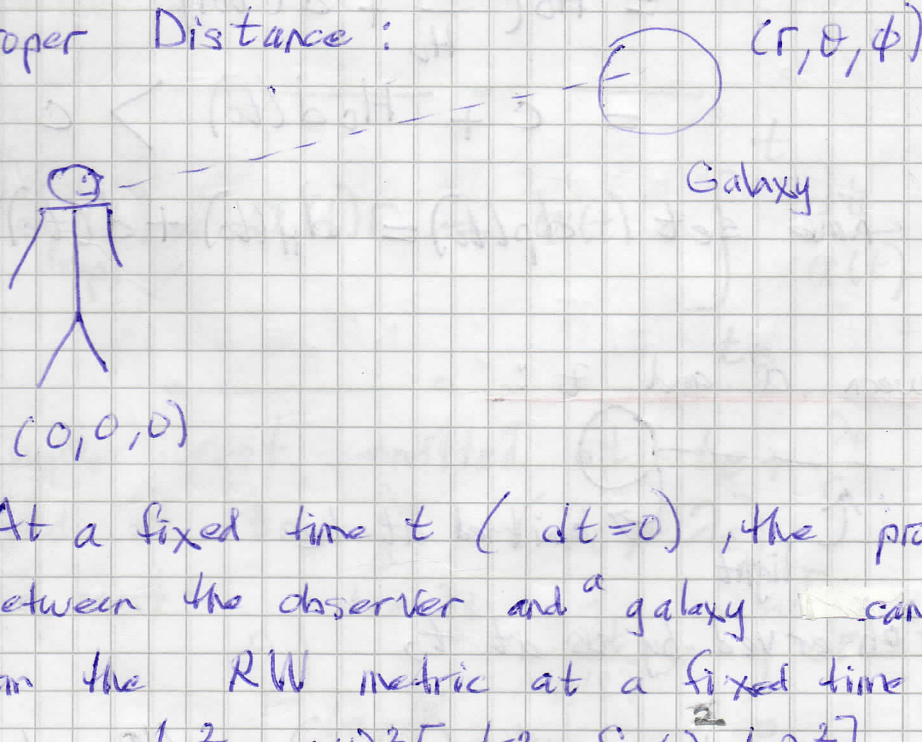
(The Robertson-Walker metric)

t : the cosmological proper time = the cosmic time

(r, θ, ϕ) : the comoving coordinates of a point in space.

* if the expansion of the universe is homogeneous & isotropic then the comoving coordinates of any point remain constant with time.

Proper Distance :



At a fixed time t ($dt = 0$), the proper distance between the observer and a galaxy can be found from the RW metric at a fixed time t

$$ds^2 = a(t)^2 \left[dr^2 + S_k(r)^2 d\Omega^2 \right]$$

The angles θ and ϕ are also constant $\Rightarrow$

$$ds = a(t) dr$$

proper distance $d_p(t) = a(t) \int_0^r dr = a(t) r$

$$\boxed{d_p(t) = a(t) r}$$

$$\dot{d}_p = \dot{a} r = \frac{\dot{a}}{a}(a r) = \frac{\dot{a}}{a} d_p = v_p$$

At the present $t = t_0$, $v_p(t_0) = H_0 d_p(t_0)$

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)}$$

v_p proper velocity

The Hubble distance : $d_H(t_0) = \frac{c}{H_0} = (4380 \pm 130)$ Mpc

Note : Points separated by a proper distance d_p greater than the Hubble distance $d_H(t_0)$ have a proper velocity

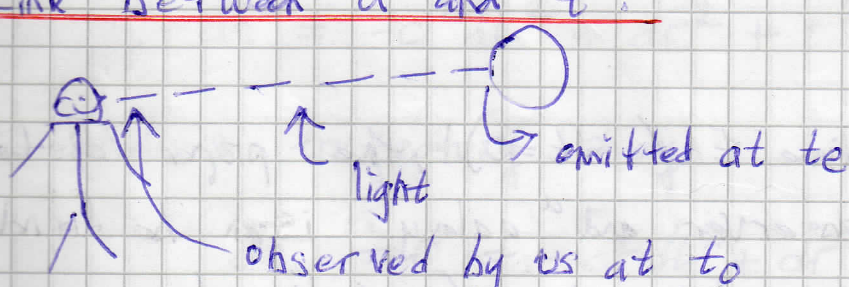
$$v_p = H_0 \left(d_H(t_0) + d(t_0) \right)$$

$$= H_0 \left(\frac{c}{H_0} + d(t_0) \right)$$

$$= c + H_0 d(t_0) > c \quad , (!!!)$$

where we have set $d_p(t_0) = d_H(t_0) + d(t_0) > d_H(t_0)$

Link between a and z :



Light follows a null geodesic ($ds = 0$) $\quad (d\theta = d\phi = 0)$

$$c^2 dt^2 = a(t)^2 dr^2$$

$$c \frac{dt}{a(t)} = dr$$

1st wave crest emitted

Integrating $c\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^r dr = r$ $\circledast$

Another form for RW metric

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

Note : $k = (+1, 0, -1) \leftarrow$ unitless

$(r, \theta, \phi) \leftarrow$ unitless

$a(t) \rightarrow$ with units of length

$$d_{P_{rop.}}(t) = a(t) r = a(t)\, c \int_{t_e}^{t} \frac{dt}{a(t)}$$

2nd wave crest emitted at $t_e + \frac{\lambda_e}{c}$ and is observed at $t_0 + \frac{\lambda_0}{c}$ $(\lambda_0 \neq \lambda_e)$

For the 2nd crest: $c\int_{t_e+\frac{\lambda_e}{c}}^{t_0+\frac{\lambda_0}{c}} \frac{dt}{a(t)} = \int_0^r dr = r$ $\circledast\circledast$

Comparing $\circledast$ and $\circledast\circledast$ $\Rightarrow \int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e+\frac{\lambda_e}{c}}^{t_0+\lambda_0/c} dt/a(t)$ $\circledast\circledast\circledast$

Subtracting $\int_{t_e+\frac{\lambda_e}{c}}^{t_0} dt/a(t)$ from each side of Eq. $\circledast\circledast\circledast$

$\Rightarrow \int_{t_e}^{t_e+\lambda_e/c} dt/a(t) = \int_{t_0}^{t_0+\lambda_0/c} dt/a(t)$. Since $a(t)$ is effectively constant between emission and observation times $\Rightarrow$

$\Rightarrow$ we may write

$$\underbrace{\frac{1}{a(t_e)}\int_{t_e}^{t_e+\lambda_e/c} dt}_{= \frac{\lambda_e}{c}} = \underbrace{\frac{1}{a(t_0)}\int_{t_0}^{t_0+\frac{\lambda_0}{c}} dt}_{\frac{\lambda_0}{c}}$$

$$\Rightarrow \quad \frac{\lambda_e}{a(t_e)} = \frac{\lambda_0}{a(t_0)}$$

since $\quad z = \dfrac{\lambda_0 - \lambda_e}{\lambda_e} = \dfrac{\lambda_0}{\lambda_e} - 1$

$$\Rightarrow \quad 1 + z = \frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)}$$

So, if we observe a galaxy with a redshift $z=2$, we are observing it as it had a scale factor $a(t_e) = \frac{1}{3}$

The three dimensional curved surface, S^3, the three-sphere, embedded in four dimensional Euclidean space is given by

$$x^2+y^2+z^2+w^2 = R^2 \text{, where}$$

$$x = R\sin\theta\sin\phi \,.\, \sin\chi$$
$$y = R\sin\theta\cos\phi \,.\, \sin\chi$$
$$z = R\cos\theta \,.\, \sin\chi$$
$$w = R\cos\chi$$

$$dl^2 = dx^2 + dy^2 + dz^2 + dw^2$$
$$= R^2\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$
$$0 \le \theta \le \pi, \quad 0 \le \chi \le \pi, \quad 0 \le \phi \le 2\pi$$

Let $\chi = \frac{r}{R}$ Eq. (3.31) Ryden

$$\Rightarrow dl^2 = dr^2 + R^2\sin^2\left(\frac{r}{R}\right)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

Let $r(t) = a(t)\bar{r}$; $R(t) = a(t)\bar{R}$

$$dl^2 = a(t)^2\left[d\bar{r}^2 + \bar{R}^2\sin^2\left(\frac{\bar{r}}{\bar{R}}\right)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Let $r' = \bar{R}\sin\left(\frac{\bar{r}}{\bar{R}}\right)$

$$dr'^2 = \bar{R}\cos\left(\frac{\bar{r}}{\bar{R}}\right) \cdot \frac{1}{\bar{R}}\, d\bar{r}$$
$$dr'^2 = \cos^2\left(\frac{\bar{r}}{\bar{R}}\right) d\bar{r}^2 = \left[1 - \sin^2\left(\frac{\bar{r}}{\bar{R}}\right)\right] d\bar{r}^2$$

$$dl^2 = a(t)^2\left[\frac{dr'^2}{1 - \frac{r'^2}{\bar{R}^2}} + r'^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$