INTRODUCTION TO COSMOLOGY
COMPLETION EXAM

Instructor: Murat Özer

July 5, 2017

Student :

---

1) a) Define the Omega Parameter Ω.

$\Omega = \dfrac{\rho}{\rho_c} = \dfrac{\varepsilon}{\varepsilon_c}$, where ε_c is the critical energy density

b) Using the Friedmann equation find $\dfrac{\Omega - 1}{\Omega}$

$$H^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{kc^2}{R_0^2 a^2}$$

$$1 = \frac{8\pi G}{3c^2}\frac{\varepsilon}{H^2} - \frac{kc^2}{R_0^2 a^2 H^2} = \underbrace{\frac{\varepsilon}{\varepsilon_c}}_{\Omega} - \frac{kc^2}{R_0^2 a^2}\frac{1}{H^2}$$

$$\Omega - 1 = \frac{kc^2}{R_0^2 a^2}\frac{1}{H^2} \; ; \; \frac{\Omega - 1}{\Omega} = \frac{kc^2}{R_0^2 a^2}\frac{1}{H^2}\frac{\varepsilon_c}{\varepsilon} = \frac{kc^2}{R_0^2 a^2 \cancel{H^2}}\frac{3c^2 \cancel{H^2}}{8\pi G}\frac{1}{\varepsilon}$$

$$\frac{\Omega - 1}{\Omega} = \frac{3kc^4}{R_0^2 a^2 8\pi G}\frac{1}{\varepsilon}$$

c) i) How does ε (or ρ) vary with the scale factor a in the standard model of cosmology when universe is radiation-dominated?

$$\varepsilon = \varepsilon_{r,0}\, a^{-4}$$

ii) Thus find how $(\Omega - 1)/\Omega$ varies with a in the radiation dominated era.

$$\frac{\Omega - 1}{\Omega} = \text{const} \cdot a^2$$

2. a) What is Olber's paradox?

Why is the night sky dark if the universe is infinitely large, eternal and static. Instead, it should be bright.

b) What is the primary resolution of it?

The volume of the presently observable universe is not infinite, it is in fact too small to contain sufficiently many visible stars.

c) What is the Cosmological Principle?

On large scales ($\gtrsim 100$ Mpc), the universe is homogeneous and isotropic.

d) In the standard Model of the universe, when the universe cooled to about 3×10^{a} K, it became transparent to photons and today we observe these as the Cosmic Microwave Background (CMB) at a temperature of about 3×10^{b} K. What are the integers a and b?

$$a = 3, \quad b = 0.$$

e) What did the universe primarily consist of at about 1/100 th of a second after the Big Bang?

Electrons, positrons, neutrinos, antineutrinos, photons.

---

3. Consider a flat ($k=0$) universe with scale factor given by

$$a(t) = a_0 e^{\chi t}$$

a) Find the Hubble parameter H

$$H = \frac{\dot{a}}{a} = \frac{a_0 \chi e^{\chi t}}{a_0 e^{\chi t}} = \chi = \text{const.}$$

b) What does the Robertson-Walker metric reduce to for a light ray propagating in the r direction radially?

$$ds^2 = -c^2 dt^2 + a(t)^2 dr^2 = 0$$

c) Obtain the expression $r(t)$ for the light ray.

$$dr^2 = \frac{c^2 dt^2}{a^2} \; ; \; dr = \frac{c\,dt}{a} = c\frac{1}{a_0} e^{-\chi t} dt$$

$$\int_0^r dr = r(t) = \frac{c}{a_0}\int_0^t e^{-\chi t} dt = \frac{c}{a_0} \frac{e^{-\chi t}}{(-\chi)}\Bigg|_0^t$$

$$r(t) = \frac{c}{a_0 \chi}\left(1 - e^{-\chi t}\right)$$

---

4) a) Consider a sphere of radius $R_s(t)$ of the universe as it expands adiabatically $(dQ = 0)$. From the conservation of energy equation $dE + PdV = 0$, obtain the Fluid Equation involving $\mathcal{E}$ (energy density), P (pressure) and a (scale factor).

$$E = \mathcal{E} V$$

$$V = \text{const.}\, a^3$$

$$d(\mathcal{E} a^3) + P\,da^3 = 0$$

$$a^3 d\mathcal{E} + 3a^2 da . \mathcal{E} + 3Pa^2 da = 0$$

dividing by a^3: $d\mathcal{E} + \frac{3\mathcal{E}}{a} da + \frac{3P}{a} da = 0$

Taking the time derivative

$$\frac{d\mathcal{E}}{dt} + 3(\mathcal{E}+P)\frac{1}{a}\frac{da}{dt} = 0$$

$$\boxed{\dot{\mathcal{E}} + 3\frac{\dot{a}}{a}(\mathcal{E}+P) = 0}$$

This is the fluid equation.

b) Obtain the Acceleration Equation from the Friedmann equation and the fluid equation.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\mathcal{E} - \frac{kc^2}{R_0^2 a^2}$$

$\times$ by a^2

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\mathcal{E}a^2 - \frac{kc^2}{R_0^2}$$

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\left(\dot{\mathcal{E}}a^2 + 2\mathcal{E}a\dot{a}\right) - 0$$

dividing by $2a\dot{a}$ $\Rightarrow$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(\dot{\mathcal{E}}\frac{a}{\dot{a}} + 2\mathcal{E}\right)$$

using the fluid equation

$$\dot{\mathcal{E}}\frac{a}{\dot{a}} = -3(\mathcal{E}+P) \Rightarrow$$

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}\left(-3\mathcal{E} - 3P + 2\mathcal{E}\right)$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\mathcal{E} + 3P\right)}$$

This is the acceleration equation.

c) If the universe is accelerating as it expands, what must be the value of ω defined by $P = \omega\mathcal{E}$, where P is the pressure and $\mathcal{E}$ is the energy density?

$$\frac{\ddot{a}}{a} > 0 \Rightarrow (\mathcal{E} + 3P) < 0 \Rightarrow \mathcal{E}(1+3\omega) < 0 \Rightarrow \boxed{\omega < -\frac{1}{3}}$$

d) What kind of a component of the universe can have such an ω value?

A cosmological constant with $\omega = -1$.

5) Consider a universe with $a(t) = bt^{1/3}$

a) Find b with the convention that $a(t_0) = 1$, where t_0 is the present time.

$$a(t_0) = b\, t_0^{1/3} = 1 \Rightarrow \boxed{b = t_0^{-1/3}}$$

b) Calculate the horizon distance, $d_{hor}(t)$ at time t.

$$d_{hor}(t) = c\, a(t) \int_0^t \frac{dt'}{a(t')} = c\, a(t) \frac{1}{b} \int_0^t t'^{-1/3} dt' = \frac{c\, a(t)}{b\, 2/3} t^{2/3}$$

$$= \frac{3}{2} c \cdot t^{1/3} \cdot t^{2/3} = \frac{3}{2} c t$$

c) The light emitted by a galaxy at $t = t_e$ reaches us today. Calculate the proper distance of that galaxy.

$$d_{prop}(t_0) = c \overset{=1}{a(t_0)} \int_{t_e}^{t_0} \frac{dt'}{a(t')} = c \frac{1}{b} \frac{3}{2} \left(t_0^{2/3} - t_e^{2/3} \right)$$

$$= \frac{3c}{2} \left(t_0 - t_0^{1/3} t_e^{2/3} \right) = \frac{3c}{2} t_0 \left(1 - \left(\frac{t_e}{t_0} \right)^{2/3} \right).$$

d) Express the proper distance in terms of the redshift z,

$$1 + z = \frac{a(t_0)}{a(t_e)} = \frac{1}{a(t_e)} = \frac{1}{b\, t_e^{1/3}} = \frac{t_0^{1/3}}{t_e^{1/3}} =$$

$$\Rightarrow d_{prop}(t_0) = \frac{3 c t_0}{2} \left[1 - \frac{1}{(1+z)^2} \right]$$

e) Find the Hubble distance $d_H(t_0)$.

$$d_H(t) = c H^{-1}(t) = c \frac{a}{\dot{a}} = c \frac{b\, t^{1/3}}{b \frac{1}{3} t^{-2/3}} = 3ct$$

$$d_H(t_0) = 3 c t_0 .$$

6. a) In the early universe write down the expression that relates the temperature to the time.

$$T(t) \approx 10^{10} K \left(\frac{t}{1 sec}\right)^{-1/2}.$$

b) What is the Planck time t_{Pl} equal to?

$$t_{Pl} \approx 10^{-44} sec.$$

c) From a), find the temperature T_{Pl}.

$$T_{Pl} \approx 10^{10} K \left(10^{22}\right) = 10^{32} K.$$

d) Show that in the radiation-dominated era the mean photon energy is given by $E_\gamma = \dfrac{\alpha_{SB} T_0^3}{n_{\gamma,0} k_B} (k_B T)$.

$$\varepsilon_\gamma = \alpha_{SB} T^4$$

$$E_\gamma = \frac{\varepsilon_\gamma}{n_\gamma}$$

$$n_\gamma(t) a^3(t) = n_{\gamma,0} a_0^3 = n_{\gamma,0}$$

$$n_\gamma = n_{\gamma,0} a^{-3}(t)$$

$$E_\gamma = \frac{\alpha_{SB} T^4}{n_{\gamma,0}} a^3 = \frac{\alpha_{SB} T}{n_{\gamma,0}} (aT)^3$$

$$= \frac{\alpha_{SB} (k_B T)}{n_{\gamma,0} k_B} (a_0 T_0)^3$$

$$= \frac{\alpha_{SB} T_0^3}{n_{\gamma,0} k_B} (k_B T)$$

e) Show from the energy density in radiation that $aT = \text{constant}$, where a and T are the scale factor and the temperature, respectively.

$$\varepsilon_\gamma = \alpha_{SB} T^4 = \varepsilon_{r,0} / a^4 \Rightarrow (aT)^4 = \varepsilon_{r,0} / \alpha_{SB} = \text{const.}$$

$$\Rightarrow aT = \left(\varepsilon_{r,0} / \alpha_{SB}\right)^{1/4} = \text{const.}$$

7.) a) Write down the expressions for the pressure and the energy density for a homogeneous scalar field $\phi(t)$.

$$P(\phi) = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$$\mathcal{E}(\phi) = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

b) starting from the conservation of energy equation in an adiabatically expanding universe, obtain the equation satisfied by $\phi(t)$.

$$dE + P\,dV = 0$$

$$\frac{d(\mathcal{E}a^3)}{dt} + P\frac{da^3}{dt} = 0$$

$$\dot{\mathcal{E}}a^3 + \mathcal{E}\frac{da^3}{dt} + P\frac{da^3}{dt} = 0$$

$$\dot{\mathcal{E}}a^3 + (\mathcal{E}+P)\frac{da^3}{dt} = 0$$

$$\dot{\phi}\ddot{\phi}a^3 + \dot{\phi}^2(3a^2\dot{a}) \cancel{=0}$$

$$+ \frac{dV}{d\phi}\frac{d\phi}{dt}.a^3 = 0$$

Dividing by $a^3\dot{\phi}$ $\Rightarrow$

$$\boxed{\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0}$$

$$V'(\phi) = \frac{dV}{d\phi}$$

c) What is the equation satisfied by ϕ in the steady state model of the universe?

Steady state model

$$H^2 = \frac{8\pi G}{3c^2}\mathcal{E} = \text{const},$$

$$= \frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right) = \text{const}.$$

Taking derivative $\Rightarrow$

$$0 = \dot{\phi}\ddot{\phi} + V'\dot{\phi}$$

$$\ddot{\phi} + V' = 0$$

$$\Rightarrow 3\frac{\dot{a}}{a}\dot{\phi} = 0$$

$$\Rightarrow \dot{\phi} = 0$$

8. Suppose the universe inflates exponentially as the inflaton field ϕ rolls down the potential $V(\phi) = g\phi^4$.

a) Assuming that $H_\phi \approx$ constant, obtain from the Friedmann equation the time dependence of the scale factor.

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3c^2}\,\mathcal{E} \approx \frac{8\pi G}{3c^2} V(\phi) = \frac{8\pi G}{3c^2} g\phi^4 = H_\phi^2 = \text{const.}$$

$$\int_{a_i}^{a} \frac{da}{a} = \int_{t_i}^{t} H_\phi \, dt$$

$$\ln\frac{a}{a_i} = H_\phi(t - t_i) + \text{const.}$$

$$a(t) = a_i e^{H_\phi(t-t_i)}$$

b) Find the condition on ϕ imposed by the flatness condition on the potential $V(\phi)$.

$$\frac{V'}{V} = \frac{4g\phi^3}{g\phi^4} = \frac{4}{\phi}$$

$$\left(\frac{V'}{V}\right)^2 = \frac{16}{\phi^2} \ll 16\pi G$$

$$\Rightarrow \phi^2 \gg \frac{1}{\pi G}$$

9) a) In a region of space of radius R full of gas, what is meant by hydrostatic equilibrium?

The inward gravitational force is balanced by an outward pressure gradient, as a result no gravitational collapse occurs.

b) What is the time it takes for the pressure gradient to build up?

$$t_{pre} \sim \frac{R}{c_s},$$

where c_s is the local speed of sound.

c) The dynamical time for collapse is $t_{dyn} = \frac{1}{\sqrt{4\pi G \bar{\rho}}}$.
What is the Jeans length equal to?

$$\lambda_J = 2\pi c_s t_{dyn} = 2\pi c_s \frac{\sqrt{c^2}}{2\sqrt{\pi G \mathcal{E}}} = c_s \left(\frac{\pi c^2}{G \mathcal{E}}\right)^{1/2}$$

d) What is the physical significance of Jeans length for collapse

Overdense regions larger than λ_J collapse under their own gravity. Overdense regions smaller than λ_J oscillate in density, they constitute stable standing sound waves.

e) Why do not density fluctuations in the air collapse?

The average height of the atmosphere is ~ 12 km which is much lass than λ_J.

10) a) Write down the definition of the redshift z if the observed and emitted wavelengths of the light are λ_0 and λ_e, respectively.

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e}$$

b) Relativistic Doppler effect yields $\frac{\lambda_0}{\lambda_e} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$.

Find the approximate value of the redshift for $\frac{v}{c} \ll 1$

$$1+z = \left(\frac{1+v/c}{1-v/c}\right)^{1/2} \approx \left(1+\frac{1}{2}\frac{v}{c}\right)\left(1+\frac{1}{2}\frac{v}{c}\right) = 1+\frac{v}{c}+\frac{1}{4}\left(\frac{v}{c}\right)^2 \approx 1+\frac{v}{c}$$

$$\Rightarrow z \approx \frac{v}{c}$$

c) If the rate at which neutrons and protons interact with neutrinos is given by $\Gamma = G_F^2 T^5$ and the Hubble parameter is given by $H = C_0 T^2$, where G_F and C_0 are known constants, find the temperature T_f at which the ratio of the number of neutrons to the number of photons is frozen.

At freeze out $\Gamma = H \Rightarrow G_F^2 T^5 = C_0 T^2$

$$\Rightarrow T = \left(C_0/G_F^2\right)^{1/3}$$

d) In a $k=0$ universe what is the condition that produces an exponentially expanding universe?

$$H = \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3c^2}\varepsilon} = \text{constant} \Rightarrow \int_{a_i}^{a}\frac{da}{a} = \int H dt = H(t-t_i)$$

$$\Rightarrow a = a_i e^{H(t-t_i)}$$, namely ε must be constant so that H is constant.

e) What is the approximate value of the present value of the density parameter Ω_0? $\Omega_0 \approx 1$