

Lowering an Index:

If A^μ is a vector, then the quantity

$$A_\mu \equiv g_{\mu\nu} A^\nu$$

is a covector.

$$\begin{aligned} A_{\mu'} &= g_{\mu'\nu'} A^{\nu'} = \left[\frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} g_{\alpha\beta} \right] \left[\frac{\partial x^{\nu'}}{\partial x^\sigma} A^\sigma \right] \\ &= \frac{\partial x^\alpha}{\partial x^{\mu'}} \underbrace{\left[\frac{\partial x^\beta}{\partial x^{\nu'}} \frac{\partial x^{\nu'}}{\partial x^\sigma} \right]}_{\delta^\beta_\sigma} g_{\alpha\beta} A^\sigma = \frac{\partial x^\alpha}{\partial x^{\mu'}} \delta^\beta_\sigma g_{\alpha\beta} A^\sigma \\ &= \frac{\partial x^\alpha}{\partial x^{\mu'}} \underbrace{g_{\alpha\beta} A^\beta}_{A_\alpha} = \frac{\partial x^\alpha}{\partial x^{\mu'}} A_\alpha. \end{aligned}$$

This is how a covector transforms.

Raising an Index: $A^\mu = g^{\mu\nu} A_\nu$

Example: Minkowski spacetime

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$[g_{\mu\nu}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{if } A^\mu = \begin{bmatrix} A^t \\ A^x \\ A^y \\ A^z \end{bmatrix} \text{ then } A_\mu = g_{\mu\nu} A^\nu$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A^t \\ A^x \\ A^y \\ A^z \end{bmatrix}$$

$$A_\mu = \begin{bmatrix} -A^t \\ A^x \\ A^y \\ A^z \end{bmatrix}$$

$$A_\mu A^\mu = A^2 = -(A^t)^2 + (A^x)^2 + (A^y)^2 + (A^z)^2$$

The scalar product of a Vector & covector

$A_\mu B^\mu = A^\mu B_\mu$ is independent of coordinate system

$$A^{\mu'} B_{\mu'} = \left(\frac{\partial x^{\mu'}}{\partial x^\alpha} A^\alpha \right) \left(\frac{\partial x^\beta}{\partial x^{\mu'}} B_\beta \right) = \underbrace{\frac{\partial x^\beta}{\partial x^{\mu'}} \frac{\partial x^{\mu'}}{\partial x^\alpha}}_{\delta_\alpha^\beta} A^\alpha B_\beta \\ = \delta_\alpha^\beta A^\alpha B_\beta = A^\alpha B_\alpha$$

Note that $A^\mu B_\mu = A^\mu g_{\mu\nu} B^\nu = A_\mu g^{\mu\nu} B_\nu$.

The Inverse Metric:

Definition: $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$

$g^{\mu\nu}$ is the inverse metric of $g_{\mu\nu}$. It transforms as

$$g^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^\alpha} \frac{\partial x^{\nu'}}{\partial x^\beta} g^{\alpha\beta}$$

Example:

$$[g_{\mu\nu}] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}$$

Find $g^{\mu\nu}$

Using $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$

$\Rightarrow$

$$g^{tt} g_{tt} = g^{tt} (-1) = 1 \Rightarrow g^{tt} = -1$$

$$g^{rr} g_{rr} = g^{rr} (1) = 1 \Rightarrow g^{rr} = 1$$

$$g^{\theta\theta} g_{\theta\theta} = g^{\theta\theta} (r^2) = 1 \Rightarrow g^{\theta\theta} = \frac{1}{r^2}$$

$$g^{\phi\phi} g_{\phi\phi} = g^{\phi\phi} (r^2 \sin^2 \theta) = 1 \Rightarrow g^{\phi\phi} = \frac{1}{r^2 \sin^2 \theta}$$

$$\Rightarrow g^{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \frac{1}{r^2} & \\ & & & \frac{1}{r^2 \sin^2 \theta} \end{bmatrix}$$

All the off diagonal elements are zero.

Example: Spherical coordinates:

Given: $X^a = \begin{bmatrix} 1 \\ r \\ 0 \end{bmatrix}$, $Y_a = \begin{bmatrix} 0 \\ -r^2 \\ \cos^2 \theta \end{bmatrix}$. Find X_a and Y^a .

$X_a = g_{ab} X^b$, where $g_{ab} = \begin{bmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2 \theta \end{bmatrix}$

$X_1 = g_{1b} X^b = g_{11} X^1 + g_{12} X^2 + g_{13} X^3$
 $= (1) X^1 = 1 \cdot 1 = 1$

$X_2 = g_{2b} X^b = g_{22} X^2 = r^2 \cdot r = r^3$

$X_3 = g_{3b} X^b = g_{33} X^3 = (r^2 \sin^2 \theta) \cdot 0 = 0$

$\Rightarrow X_a = \begin{bmatrix} 1 \\ r^3 \\ 0 \end{bmatrix} = \begin{bmatrix} X_r \\ X_\theta \\ X_\phi \end{bmatrix}$

$Y^a = g^{ab} X_b$ (Raising an index)

where $g^{ab} = \begin{bmatrix} 1 & & \\ & 1/r^2 & \\ & & 1/(r^2 \sin^2 \theta) \end{bmatrix}$

$Y^1 = g^{1b} Y_b$

$= g^{11} Y_1 = 1 \cdot 0 = 0$

$Y^2 = g^{2b} Y_b = g^{22} Y_2 = \frac{1}{r^2} (-r^2) = -1$

$Y^3 = g^{3b} Y_b = g^{33} Y_3 = \frac{1}{r^2 \sin^2 \theta} (\cos^2 \theta) = \frac{\cot^2 \theta}{r^2}$

$Y^a = \begin{bmatrix} 0 \\ -1 \\ \frac{\cot^2 \theta}{r^2} \end{bmatrix} = \begin{bmatrix} Y^r \\ Y^\theta \\ Y^\phi \end{bmatrix}$

Raising an Index

$$A^N = g^{NV} A_V$$

Tensors: An n th-rank tensor T is a quantity with D^n components $T^{N^1 \dots N^n}_{a^1 \dots a^n}$, where D is the number of dimensions, that transform when one changes coordinate systems according to the rule

$$T^{N^1 \dots N^n}_{a^1 \dots a^n} = \frac{\partial x^{N^1}}{\partial x^{a^1}} \frac{\partial x^{N^2}}{\partial x^{a^2}} \dots \frac{\partial x^{N^n}}{\partial x^{a^n}} T^{b^1 \dots b^n}_{x^1 \dots x^n}$$

In 4D:

A scalar: $\phi' = \phi$ ($4^0 = 1$, no indices
0-rank tensor)

Vectors and covectors are first-rank tensors

$$A^{N'} = \frac{\partial x^{N'}}{\partial x^\alpha} A^\alpha; \quad A_{N'} = \frac{\partial x^\alpha}{\partial x^{N'}} A_\alpha$$

(1 index; $4^1 = 4$ components)

2nd-rank tensors: T^{NV} , T_{NV} , T^N_V

$$T^{N'}_{V'} = \frac{\partial x^{N'}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^{V'}} T^\alpha_\beta; \quad 4^2 = 16 \text{ components}$$

Tensor Equations: If a tensor equation is true in one coordinate system then it is true in all coordinate systems.

Suppose in a coordinate system we have

$$T^{NV}_\alpha = C_0 A^{NV}_\alpha$$

in another coordinate system with primed coordinates

$$T^{N'V'}_{\alpha'} \rightarrow T^{N'V'}_{\alpha'}$$

$G \rightarrow G_0$, and $A^\mu_\alpha \rightarrow A^{\mu'}_{\alpha'}$ so that the equation becomes $T^{\mu'\nu'}_{\alpha'} = G_0 A^{\mu'\nu'}_{\alpha'}$

We say that such a law is manifestly covariant. The form of the equation remains the same. Einstein assumed that the true laws of physics would have a simple form when expressed in tensor equations.

Tensor Operations:

i) Tensor sum: $P_1^\mu + P_2^\mu = P_{tot}^\mu$

$P_1^\mu + P_{2\mu} = P_{tot}^\mu$: wrong!

ii) Tensor product: $A^\mu B_\nu = C^\mu_\nu$

iii) contraction: $R^\alpha_{\mu\alpha\nu} = R_{\mu\nu}$

iv) Raising an index: $g^{\mu\alpha} R_{\alpha\nu} = R^\mu_\nu$

v) Lowering an index: $g_{\mu\alpha} T^{\alpha\nu} = T_\mu^\nu$

vi) Multiplying by a scalar $C T_{\mu\nu}$

The Gradient of a Tensor:

ϕ : a scalar, $\partial_\mu \phi$ a covector

Consider a vector A^μ . The components of the gradient ∇A^μ of A^μ transforms when we change coordinate systems as:

$$\partial_{\nu'} A^{\mu'} = \frac{\partial A^{\mu'}}{\partial x^{\nu'}} = \frac{\partial}{\partial x^{\nu'}} \left(\frac{\partial x^{\mu'}}{\partial x^\alpha} A^\alpha \right)$$

$$= \frac{\partial}{\partial x^{\nu'}} \frac{\partial x^\beta}{\partial x^\alpha} \left(\frac{\partial x^{\mu'}}{\partial x^\beta} A^\alpha \right) = \frac{\partial x^\beta}{\partial x^{\nu'}} \frac{\partial}{\partial x^\beta} \left(\frac{\partial x^{\mu'}}{\partial x^\alpha} A^\alpha \right)$$

$$= \frac{\partial x^\beta}{\partial x^{\nu'}} \left(\frac{\partial^2 x^{\mu'}}{\partial x^\beta \partial x^\alpha} A^\alpha + \frac{\partial x^{\mu'}}{\partial x^\alpha} \frac{\partial A^\alpha}{\partial x^\beta} \right)$$

$$= \frac{\partial x^\beta}{\partial x^{\nu'}} \frac{\partial^2 x^{\mu'}}{\partial x^\beta \partial x^\alpha} A^\alpha + \frac{\partial x^\beta}{\partial x^{\nu'}} \frac{\partial x^{\mu'}}{\partial x^\alpha} (\partial_\beta A^\alpha)$$

This term violates the transformation law for a tensor

Note: In special relativity
the gradient of a vector
transforms like a tensor.

$$\frac{\partial^2 x^{\mu'}}{\partial x^{\rho} \partial x^{\sigma}} = 0, \text{ so}$$

The only way to have a tensor is if the transformation is linear. If the transformation is non-linear, the gradient of the vector will not transform like a tensor. This is why in special relativity, the transformation between frames is linear.

$$g_{\mu\nu} = g_{\mu'\nu'} \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}}$$

$$A^{\mu} B^{\nu} = C^{\rho}$$

$$R^{\mu\nu} = R^{\mu\rho} R^{\rho\nu}$$

$$\partial_{\mu} R^{\nu\rho} = R^{\nu\rho} \partial_{\mu}$$

$$\partial_{\mu} T^{\nu\rho} = T^{\nu\rho} \partial_{\mu}$$

$$C^{\mu\nu} = C^{\mu\rho} C^{\rho\nu}$$

The Gradient of a Tensor:

Consider a vector A^{μ} . The components of the gradient $\partial_{\nu} A^{\mu}$ transform like a tensor. This is because the transformation is linear in special relativity.

$$\partial_{\nu} A^{\mu} = \frac{\partial A^{\mu}}{\partial x^{\nu}} = \frac{\partial A^{\mu'}}{\partial x^{\nu}} \frac{\partial x^{\mu'}}{\partial x^{\mu}} = \frac{\partial A^{\mu'}}{\partial x^{\nu'}} \frac{\partial x^{\mu'}}{\partial x^{\nu}} \frac{\partial x^{\nu'}}{\partial x^{\mu}} = \frac{\partial A^{\mu'}}{\partial x^{\nu'}} \frac{\partial x^{\mu'}}{\partial x^{\nu}} \frac{\partial x^{\nu'}}{\partial x^{\mu}}$$

$$\partial_{\nu} A^{\mu} = \frac{\partial A^{\mu}}{\partial x^{\nu}} = \frac{\partial A^{\mu'}}{\partial x^{\nu}} \frac{\partial x^{\mu'}}{\partial x^{\mu}} = \frac{\partial A^{\mu'}}{\partial x^{\nu'}} \frac{\partial x^{\mu'}}{\partial x^{\nu}} \frac{\partial x^{\nu'}}{\partial x^{\mu}} = \frac{\partial A^{\mu'}}{\partial x^{\nu'}} \frac{\partial x^{\mu'}}{\partial x^{\nu}} \frac{\partial x^{\nu'}}{\partial x^{\mu}}$$

$$\partial_{\nu} A^{\mu} = \frac{\partial A^{\mu}}{\partial x^{\nu}} = \frac{\partial A^{\mu'}}{\partial x^{\nu}} \frac{\partial x^{\mu'}}{\partial x^{\mu}} = \frac{\partial A^{\mu'}}{\partial x^{\nu'}} \frac{\partial x^{\mu'}}{\partial x^{\nu}} \frac{\partial x^{\nu'}}{\partial x^{\mu}} = \frac{\partial A^{\mu'}}{\partial x^{\nu'}} \frac{\partial x^{\mu'}}{\partial x^{\nu}} \frac{\partial x^{\nu'}}{\partial x^{\mu}}$$

Chapter 7: Maxwell's Equations

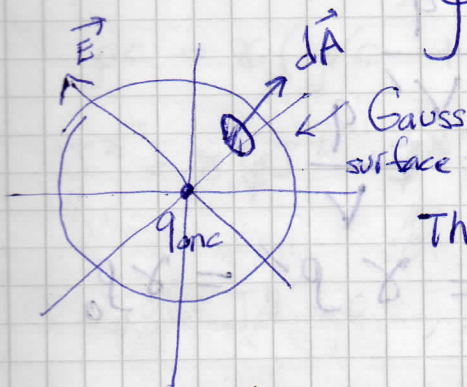
Recall the principle of relativity: The laws of physics are the same in all IRFs.

So, a law of physics written in tensor form will be manifestly covariant and automatically satisfies the principle of relativity.

Basic Electrostatics:

Gauss' Law: $\oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\epsilon_0} = 4\pi k q_{\text{enc}}$

(this is the integral form)

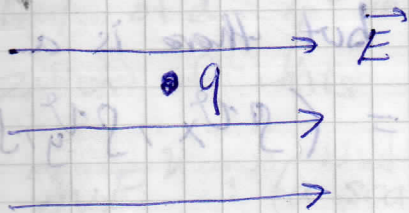


The differential form:

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi k \rho \quad (*)$$

k : Coulomb constant

Electric force on a charge q :

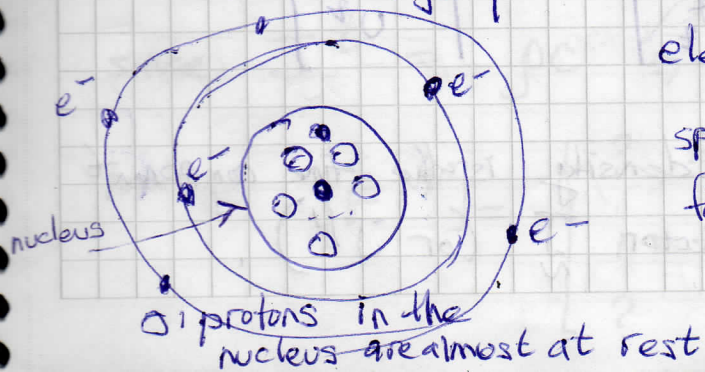


$$\frac{d\vec{p}}{dt} = \vec{F}_e = q\vec{E} \quad (**)$$

$\vec{p}$: momentum of the charge q

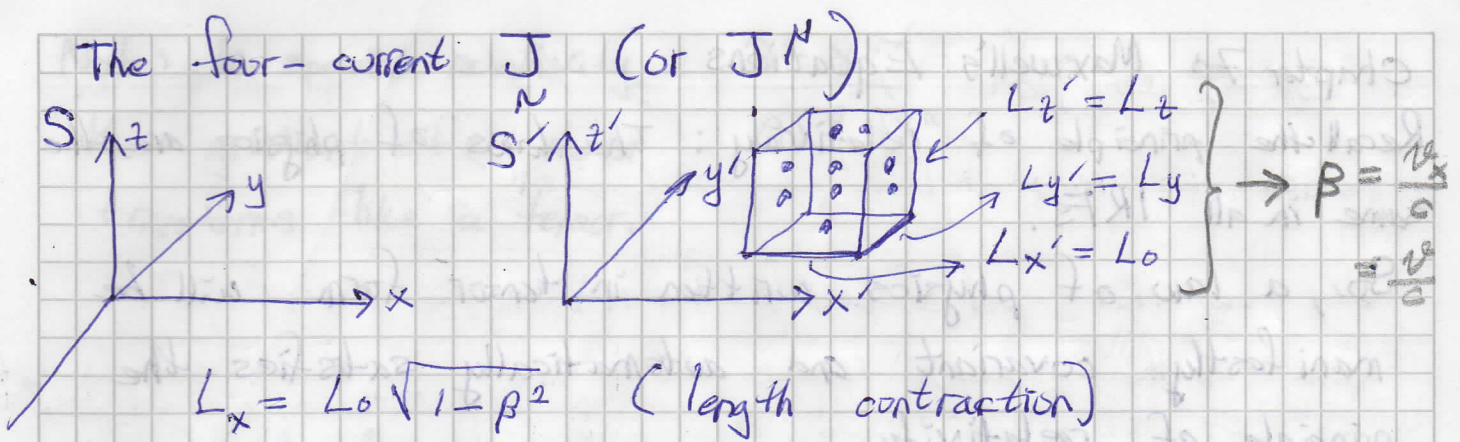
Question: What are the tensor versions of equations (*) and (**) ? Our task is to answer this question.

Electric charge q is a frame independent quantity.



electrons are moving at relativistic speeds. However $Q_p + Q_{e^-} = 0$ for neutral atoms.

This proves that q is frame independent.



$$V' = L_x' L_y' L_z'$$

$$V = L_x L_y L_z = \underbrace{L_0 L_y L_z}_{V'} \sqrt{1-\beta^2} = V' \sqrt{1-\beta^2}$$

$$\rho' : \text{charge density in } S' = \frac{q}{V'}$$

$$\rho : \text{charge density in } S = \frac{q}{V}$$

$$\Rightarrow \rho = \frac{q}{V} = \frac{q}{V' \sqrt{1-\beta^2}} = \frac{\rho'}{\sqrt{1-\beta^2}} = \gamma \rho' = \gamma \rho_0$$

In the S' frame the total charge q moves with speed v in the $+x$ direction. There is no current density in S' (the rest frame of the charge) but there is a current density in S $\underline{J} = \rho \underline{v} = (\rho v_x, \rho v_y, \rho v_z)$

Define the four-current $\underline{J}$

$$\underline{J} = \begin{bmatrix} \rho c \\ \rho v_x \\ \rho v_y \\ \rho v_z \end{bmatrix} = \rho_0 \begin{bmatrix} \gamma c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{bmatrix} = \rho_0 \begin{bmatrix} c \\ v_x \\ v_y \\ v_z \end{bmatrix} \equiv \underline{J}^N$$

$$= \begin{bmatrix} J^t \\ J^x \\ J^y \\ J^z \end{bmatrix}$$

The charge density is the time component of a 4-vector $\underline{J}$ (or $\underline{J}^N$).

$$\underline{J} = \Lambda^{-1} \underline{J}' \quad (J^\mu = (\Lambda^{-1})^\mu_\nu J'^\nu)$$

$$\begin{bmatrix} J^t \\ J^x \\ J^y \\ J^z \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} J^{t'} \\ J^{x'} \\ J^{y'} \\ J^{z'} \end{bmatrix}$$

$$J^t = \gamma (J^{t'} + \beta J^{x'}) \quad ; \quad (J^{x'} = 0 \text{ because there is no current density in } S')$$

$$\rho c = \gamma (\rho' c + 0) = \gamma \rho' c \Rightarrow \rho = \gamma \rho'$$

$$J^x = \gamma (\beta J^{t'} + J^{x'}) = \gamma \beta \rho' c \Rightarrow J^x = \gamma \beta \rho c$$

$$J^y = J^{y'} = 0, \quad J^z = J^{z'} = 0$$

Generalization of Gauss's Law:

Since the Right side of Eq'n (*) is the time component of a 4-vector, the left side of the generalization of this eq'n must be also.

$\partial_\nu E^\nu$: (a scalar) does not work

$\partial_\nu F^{\nu\mu}$: a 4-vector, where $F^{\nu\mu}$ is the (2nd rank) ^{field} tensor.

Therefore, we write $\partial_\nu F^{\nu\mu} = \frac{4\pi k}{c^2} J^\mu$

$$\partial_t F^{t\mu} + \partial_x F^{x\mu} + \partial_y F^{y\mu} + \partial_z F^{z\mu} = \frac{4\pi k}{c^2} J^\mu$$

since $J^t = \rho c \xrightarrow{\text{from Eq'n (*) in previous page}} F^{tx} = \frac{E_x}{c}, F^{ty} = \frac{E_y}{c}, F^{tz} = \frac{E_z}{c}$

namely $[F^{\mu\nu}] = \begin{bmatrix} ? & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ ? & - & ? & - \\ ? & - & - & ? \end{bmatrix}$

To obtain the other entities of $F^{\mu\nu}$ we generalize

$$\frac{d\vec{p}}{dt} = q \vec{E} \quad \text{to a 4-vector equation}$$

$$\boxed{\frac{dp^\mu}{d\tau} = q F^{\mu\nu} U_\nu}, \quad \text{where } U_\nu \equiv \eta_{\nu\alpha} U^\alpha$$

is the covector corresponding to U^μ

Now, since $p^2 = \underset{\nu}{\vec{p}} \cdot \underset{\nu}{\vec{p}} = p^\mu p_\mu = -m^2 c^2$

and $\frac{d(p^2)}{d\tau} = 2 p_\mu \frac{dp^\mu}{d\tau} = -\frac{d(m^2 c^2)}{d\tau} = 0$

$$p_\mu \frac{dp^\mu}{d\tau} = q p_\mu U_\nu F^{\mu\nu} = m q U_\mu U_\nu F^{\mu\nu} = 0$$

Also $U_\mu U_\nu (F^{\mu\nu} + F^{\nu\mu}) = 0$ implies that $F^{\mu\nu} = -F^{\nu\mu}$, namely $F^{\mu\nu}$ is antisymmetric. This requires that

$$[F^{\mu\nu}] = \begin{bmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & ? & ? \\ -\frac{E_y}{c} & ? & 0 & ? \\ -\frac{E_z}{c} & ? & ? & 0 \end{bmatrix}, \quad \begin{aligned} F^{tt} &= -F^{tt} \Rightarrow F^{tt} = 0 \\ F^{aa} &= -F^{aa} \Rightarrow F^{aa} = 0 \\ a &= x, y, z. \text{ (or } 1, 2, 3) \end{aligned}$$

Note: This shows that there must be another field, which turns out to be $\vec{B}$.
When there is a magnetic field $\vec{B}$ acting on the charged particle the force is

$$\frac{d\vec{p}}{dt} = q \vec{E} + q \vec{v} \times \vec{B}$$

for $\mu = x$:

$$\begin{aligned} \frac{dp^x}{d\tau} &= q (F^{xt} U_t + F^{xx} U_x + F^{xy} U_y + F^{xz} U_z) \\ &\xrightarrow{\text{non-rel. limit}} q \left(F^{xt} \frac{U_t}{c} + 0 + F^{xy} U_y + F^{xz} U_z \right) \\ &= q E_x + q (F^{xy} U_y + F^{xz} U_z) \end{aligned}$$

$$= q \vec{E}_x + q (\vec{v} \times \vec{B})_x$$

$$= q E_x + q (v_y B_z - v_z B_y)$$

Comparing: $\Rightarrow F^{xy} = B_z, F^{xz} = -B_y$

$$\Rightarrow [F^{\mu\nu}] = \begin{bmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & ? \\ -\frac{E_z}{c} & B_y & ? & 0 \end{bmatrix}$$

Considering $\frac{dp^y}{d\tau} = q (F^{yt} U_t + F^{yx} U_x + F^{yy} U_y + F^{yz} U_z)$

non-rel. limit $\rightarrow q \left(-\frac{E_y}{c} (-c) - B_z v_x + F^{yz} v_t \right)$

$$= q E_y + q (\vec{v} \times \vec{B})_y$$

$$= q E_y + q (v_z B_x - v_x B_z) \Rightarrow F^{yz} = B_x$$

$$\Rightarrow [F^{\mu\nu}] = \begin{bmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix}$$

$$F_{\mu\nu} = \eta_{\mu\alpha} \eta_{\nu\beta} F^{\alpha\beta}$$

$$= \begin{bmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix}$$

Consequences:

Take the gradient of both sides of

$$\partial_\nu F^{\mu\nu} = \frac{4\pi k}{c^2} J^\mu$$

$$\partial_\mu \partial_\nu F^{\mu\nu} = \frac{4\pi k}{c^2} \partial_\mu J^\mu$$

Since $F^{\mu\nu} = -F^{\nu\mu} \Rightarrow \partial_\mu \partial_\nu (F^{\mu\nu} + F^{\nu\mu})$

$$\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\mu \partial_\nu F^{\mu\nu}$$

$$\Rightarrow \partial_\mu \partial_\nu F^{\mu\nu} = 0 \Rightarrow \partial_\mu J^\mu = 0$$

, which is the law of conservation of charge

The other two Maxwell Equations:

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t}$$

$$(\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times (\vec{\nabla} \phi) - \frac{\partial \vec{\nabla} \times \vec{A}}{\partial t})$$

$$\vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t} \text{ is generalized to}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \text{ where}$$

$$A^\mu = \begin{bmatrix} A^t = \frac{\phi}{c} \\ A^x \\ A^y \\ A^z \end{bmatrix}$$

Also $\partial^\alpha F^{\mu\nu} + \partial^\nu F^{\alpha\mu} + \partial^\mu F^{\nu\alpha} = 0$

$$[(\mu \alpha \nu \nu \rightarrow \nu \alpha \mu \rightarrow \mu \nu \alpha)]$$

Chapter 8: Geodesics

A geodesic through spacetime is defined in one of two ways

1) as the worldline of longest proper time between two events, or

2) as the straightest possible worldline

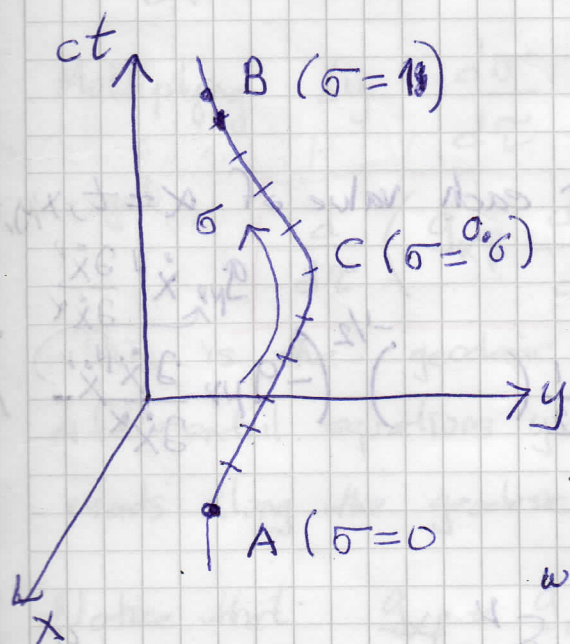
We will focus on the 1st definition.

It is proved in Box 8.1 that in flat spacetime an inertial clock passing through two events measures the longest possible proper time.

This will be assumed to be true in curved spacetimes as well.

Timelike Geodesics:

Consider two events A and B that have a timelike separation ($ds^2 < 0$), and consider possible timelike worldlines between them.



The spacetime (coordinates) of events along a worldline can be specified as a function of a parameter σ

At event A: $\sigma = 0$

At event B: $\sigma = 1$

The proper time measured along this worldline is then

$$\tau_{AB} = \int \sqrt{-ds^2} = \int_0^1 \sqrt{-g_{\mu\nu}(x^\mu(\sigma)) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma}} d\sigma$$

Recall: In mechanics the trajectory of $q_i(t)$ of a system is found by extremising the action S

$$S \equiv \int_{t_A}^{t_B} L(q_i, \dot{q}_i) dt ; \quad \frac{dS}{dt} = 0 \quad \text{gives}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad (\text{Euler-Lagrange equations})$$

In relativity the situation is similar. We have

$$L(x^\alpha, \dot{x}^\alpha) \equiv \sqrt{-g_{\mu\nu}(x^\alpha) \dot{x}^\mu \dot{x}^\nu},$$

$$\text{where } \dot{x}^\alpha \equiv \frac{dx^\alpha}{d\sigma}$$

We want to find the worldline $x^\alpha(\sigma)$ that makes the integral of this Lagrangian extreme.

The E-L equations for this Lagrangian are

$$\frac{d}{d\sigma} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) - \frac{\partial L}{\partial x^\alpha} = 0$$

In 4D, we have one equation for each value of $\alpha = t, x, y, z$, (altogether 4 eqns.)

$$\text{Now, } \frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial}{\partial \dot{x}^\alpha} \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} = \frac{1}{2} \left(\frac{\partial}{\partial \dot{x}^\alpha} \left(-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right)^{-1/2} \right) \left(-g_{\mu\nu} \frac{\partial \dot{x}^\mu \dot{x}^\nu}{\partial \dot{x}^\alpha} \right)$$

(note that $\frac{\partial g_{\mu\nu}}{\partial \dot{x}^\alpha} = 0$)

$$\Rightarrow \text{Since } \frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha} = \delta^\mu_\alpha, \quad \frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha} = \delta^\nu_\alpha$$

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}^\alpha} &= \frac{1}{2L} \left(-g_{\alpha\nu} \dot{x}^\nu - g_{\mu\alpha} \dot{x}^\mu \right) = \frac{1}{2L} \left(-g_{\alpha\nu} \dot{x}^\nu - g_{\mu\alpha} \dot{x}^\mu \right) \\ &= \frac{1}{L} \left(-g_{\alpha N} \dot{x}^N \right) = -\frac{1}{L} g_{\alpha N} \frac{dx^N}{d\sigma} \end{aligned}$$

$$d\tau^2 = -g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu d\sigma^2$$

But notice that $\tau = \int \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\sigma$

$$\frac{d\tau}{d\sigma} = \frac{d}{d\sigma} \int L d\sigma = L$$

$$\Rightarrow \frac{\partial L}{\partial \dot{x}^\alpha} = -g_{\alpha N} \frac{dx^N}{d\sigma} \frac{d\sigma}{d\tau} = -g_{\alpha N} \frac{dx^N}{d\tau}$$

Similarly $\frac{\partial L}{\partial x^\alpha} = \frac{1}{2} \left(-\frac{1}{2} \left(-\frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} \right)$

$$= -\frac{1}{2L} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} = -\frac{1}{2} \frac{1}{\frac{d\tau}{d\sigma}} \left(\right)$$

$$= -\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{d\sigma} \frac{d\sigma}{d\tau} \frac{dx^\nu}{d\sigma} = -\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\sigma}$$

The E-L equations become

$$\frac{d}{d\sigma} \left(-g_{\alpha N} \frac{dx^N}{d\tau} \right) + \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\sigma} = 0$$

Multiplying by $\frac{d\sigma}{d\tau}$ and renaming $N \rightarrow \beta$ in the first term

we get, $\frac{d}{d\tau} \left(g_{\alpha\beta} \frac{dx^\beta}{d\tau} \right) - \frac{1}{2} \left(\partial_\alpha g_{\mu\nu} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$

This is the geodesic equation. Solving this set of differential equations yields the coordinates $x^\alpha(\tau)$ of events along the geodesic.

Notice that $g_{\alpha\beta} = g_{\alpha\beta}(x^\gamma(\tau))$

$$\frac{d}{d\tau} \left(g_{\alpha\beta} \frac{dx^\beta}{d\tau} \right) = \left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \frac{dx^\gamma}{d\tau} \right) \frac{dx^\beta}{d\tau} + g_{\alpha\beta} \frac{d^2 x^\beta}{d\tau^2}$$

rename $\gamma \rightarrow \mu$ and $\beta \rightarrow \nu$ and then put this in the geodesic equation above and multiply by the inverse metric $g^{\alpha\gamma}$:

$$g^{\mu\alpha} \left[\left(\frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right) \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} \right]$$

$$- \frac{1}{2} g^{\mu\alpha} \partial_\alpha g_{\mu\nu} \frac{dx^\nu}{d\tau} = 0$$

Using $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$ and $\delta^\mu_\nu \frac{d^2 x^\nu}{d\tau^2} = \frac{d^2 x^\mu}{d\tau^2}$

this equation can be written as

$$\frac{d^2 x^\mu}{d\tau^2} + g^{\mu\alpha} \left(\partial_\mu g_{\alpha\nu} - \frac{1}{2} \partial_\alpha g_{\mu\nu} \right) \frac{dx^\nu}{d\tau} = 0$$

or $\frac{d^2 x^\mu}{d\tau^2} + g^{\mu\alpha} \left(\partial_\mu g_{\alpha\nu} - \frac{1}{2} \partial_\alpha g_{\mu\nu} \right) \frac{dx^\nu}{d\tau} = 0$

Special case: In special relativity $g_{\mu\nu} = \eta_{\mu\nu} = \text{constant}$
then $\frac{d^2 x^\mu}{d\tau^2} = 0$ (acceleration)

A hint for solving the geodesic equation:

The definition of proper time $d\tau = \frac{1}{c} \sqrt{-ds^2} = \frac{1}{c} \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}$

and $\underline{U} \cdot \underline{U} = U^2 = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$

$$= - \left(\frac{cd\tau}{d\tau} \right)^2$$

can be used to obtain one of the geodesic equations,

$$g^{\alpha\beta} \left[\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} \right) \frac{dx^\gamma}{d\tau} + g_{\alpha\gamma} \frac{d^2 x^\gamma}{d\tau^2} \right]$$

$$- \frac{1}{2} g^{\alpha\gamma} \partial_\alpha g_{\gamma\beta} \frac{dx^\beta}{d\tau} = 0$$

Using $g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta$ and $\delta^\alpha_\beta \frac{d^2 x^\beta}{d\tau^2} = \frac{d^2 x^\alpha}{d\tau^2}$

this equation can be written as

$$g^{\alpha\beta} \frac{d^2 x^\beta}{d\tau^2} + g^{\alpha\beta} \left(\partial_\beta g_{\alpha\gamma} - \frac{1}{2} \partial_\alpha g_{\gamma\beta} \right) \frac{dx^\gamma}{d\tau} = 0$$

or $\frac{d^2 x^\beta}{d\tau^2} + g^{\beta\alpha} \left(\partial_\alpha g_{\beta\gamma} - \frac{1}{2} \partial_\beta g_{\gamma\alpha} \right) \frac{dx^\gamma}{d\tau} = 0$

Special case: In special relativity $g_{\mu\nu} = \eta_{\mu\nu} = \text{constant}$
then $\frac{d^2 x^\beta}{d\tau^2} = 0$ (acceleration)

A hint for solving the geodesic equation:

The definition of proper time $\frac{d\tau}{ds} = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}}$

and $\underline{u} \cdot \underline{u} = u^2 = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$

$$= - \left(\frac{cd\tau}{ds} \right)^2$$

can be used to obtain one of the geodesic equations,

Lightlike geodesics: Since the proper time measured along a photon worldline is always $\tau_{AB}=0$, τ cannot be a good parameter for a lightlike geodesic.

For a massive particle ($m \neq 0$) $E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$ (energy)

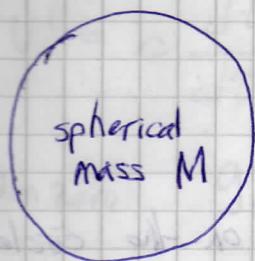
and the i -component of momentum

$$p^i = \frac{mv_i}{\sqrt{1 - \frac{v^2}{c^2}}}$$

As $m \rightarrow 0$, the particle becomes photon like

So the $m \rightarrow 0$ limit is taken carefully to find the photon geodesics.

Chapter 9: The Schwarzschild Metric



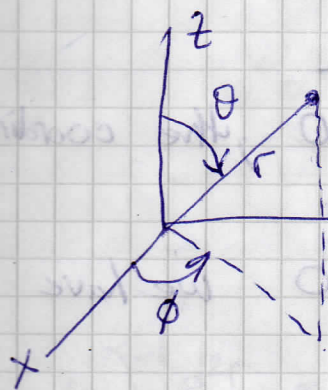
spherical
mass M

Empty space (= vacuum)

The spherically symmetric, time independent solution to the Einstein field equations ^{in vacuum} describing the spacetime ~~in vacuum~~ surrounding a spherical static object of mass M is the Schwarzschild solution.

The infinitesimal interval squared in flat spacetime using the spherical coordinates is

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$



(Minkowski metric in spherical coordinates)

The interval squared ds^2 for a spherical mass distribution is found by solving the Einstein field equations (first done by Karl Schwarzschild in 1916)

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

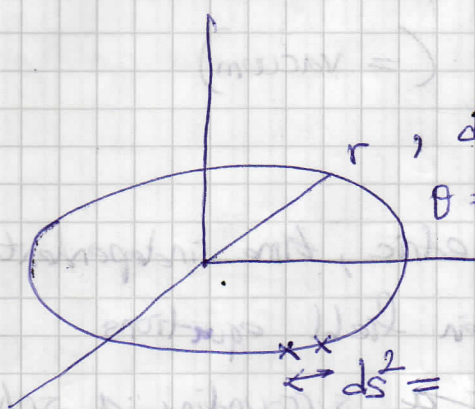
where r_s is a constant with units of length called the Schwarzschild radius.

ds^2 is time-independent

$ds^2 \rightarrow ds^2(\text{Minkowski})$ as $r \rightarrow \infty$

The meaning of the radial coordinate r :

Consider a circle of ^{constant} radius r in the equatorial plane $\theta = \frac{\pi}{2}$ at an instant of time.



$$\theta = \frac{\pi}{2}, d\theta = 0$$

$dt = 0$ for all events on the circle.

$$ds^2 = 0 + 0 + 0 + r^2 d\phi^2$$

$ds = r d\phi$ for such a circle

(infinitesimal distance between neighboring points)

$$\text{Circumference of the circle} = C = \oint ds = \int_0^{2\pi} r d\phi = 2\pi r$$

$$\Rightarrow r = \frac{C}{2\pi}$$

In the flat space-time given by $r_s = 0$, the coordinate r is the distance from the origin.

When $r_s \neq 0$, for $cdt = d\theta = d\phi = 0$ we have

$$ds^2 = \left(1 - \frac{r_s}{r}\right)^{-1} dr^2$$

So, the infinitesimal radial distance is

$$ds = dr = \left(1 - \frac{r_s}{r}\right)^{-1/2} dr$$

and $dr > dr$ and r no longer measures radial distance

The meaning of r_s : It follows from Eq. 8.13 that

The geodesic equation $g_{\alpha\beta} \frac{d^2 x^\beta}{d\tau^2} = - \left(\partial_\nu g_{\alpha\mu} - \frac{1}{2} \partial_\alpha g_{\mu\nu} \right) U^\mu U^\nu$

for $\beta = r$ $g_{\alpha r} \frac{d^2 r}{d\tau^2} = - \left(\partial_\nu g_{\alpha\mu} - \frac{1}{2} \partial_\alpha g_{\mu\nu} \right) U^\mu U^\nu$

in the rest frame of the object

$$U^\mu = \begin{bmatrix} -U^t \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$cd\tau = \sqrt{\left(1 - \frac{r_s}{r}\right)} cdt ; \quad \frac{dt}{d\tau} = \left(1 - \frac{r_s}{r}\right)^{-1/2}$$

$$\Rightarrow v^t = c \frac{dt}{d\tau} = c \left(1 - \frac{r_s}{r}\right)^{-1/2}$$

Non zero $g_{\alpha\beta}$: $g_{tt} = -\left(1 - \frac{r_s}{r}\right)$, $g_{rr} = \left(1 - \frac{r_s}{r}\right)^{-1}$

$$g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 \sin^2\theta$$

So, the geodesic equation gives

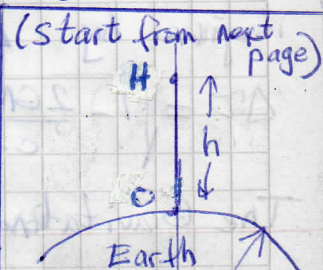
$$x=r \quad \leftarrow \quad g_{rr} \frac{d^2 r}{d\tau^2} = - \left(\partial_\nu g_{r\nu} - \frac{1}{2} \partial_r g_{\mu\nu} \right) U^\mu U^\nu$$

$$= - \left(\partial_t g_{rt} - \frac{1}{2} \partial_r g_{tt} \right) U^t U^t$$

$$= - \frac{1}{2} \frac{\partial}{\partial r} \left(1 - \frac{r_s}{r} \right) c^2 \left(\right)^{-1}$$

$$\left(1 - \frac{r_s}{r} \right)^{-1} \frac{d^2 r}{d\tau^2} = - \frac{1}{2} (+1) \frac{r_s}{r^2} c^2 \left(\right)^{-1}$$

$$\frac{d^2 r}{d\tau^2} = - \frac{\frac{c^2}{2} r_s}{r^2}$$



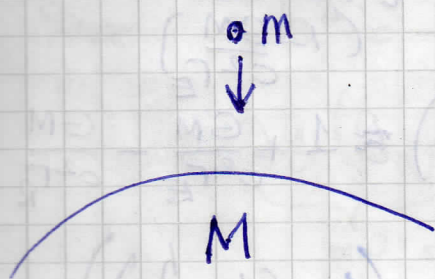
$$V_0 = -\frac{GM}{R_E} \quad \leftarrow \text{potential (not pot. energy)}$$

$$V_H = -\frac{GM}{R_E + h}, \quad V_H > V_0$$

$\Delta\tau_H > \Delta\tau_0$
Time (clock) runs faster at points of higher potential than points of lower potential.
"Gravitational Time Dilation"

Newtonian gravitation tells us that

$$F_G = -\frac{GMm}{r^2} = ma \Rightarrow a = -\frac{GM}{r^2}$$



$$\text{so, } \frac{d^2 r}{d\tau^2} = -\frac{GM}{r^2} \quad \leftarrow$$

$$= -\frac{\frac{c^2}{2} r_s}{r^2} \quad \leftarrow \Rightarrow r_s = \frac{2GM}{c^2}$$

M: the source of the gravitational field described by the Schwarzschild metric.

$$r_s^{\text{Sun}} = \frac{2GM_{\text{Sun}}}{c^2} = 2954 \text{ m} \approx 3 \text{ km}; \quad r_s^{\text{Earth}} = \frac{2GM_{\text{Earth}}}{c^2} = 8.90 \text{ mm} \approx 9 \text{ mm}$$

Since R_{Sun} and R_{Earth} are both $\gg r_s^{\text{Sun}}$ and r_s^{Earth} the space around such objects are very close to being flat.

• Typical ~~neutron~~ stars: $r_s^{\text{NS}} \approx 4.4 \text{ km}$, r -coordinate on the surface $R = 10$ to 12 km , so $R \approx 5 GM/c^2$. This means that we would expect to see significant deviations from the predictions of Newtonian gravity in the empty space around a neutron star.

* Black holes are objects with surface radial coordinates $R < r_s = \frac{2GM}{c^2}$ with $r_s = 2GM/c^2$, the Schwarzschild metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

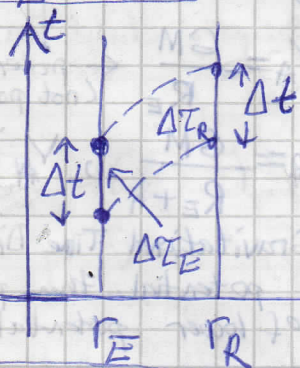
The meaning of the time coordinate: Consider a clock at rest ($dr = d\theta = d\phi = 0$)

The proper time $\Delta\tau$ this clock measures between two events at its location

$$\text{is given by } \Delta\tau = \int d\tau = \int \sqrt{-ds^2/c^2} = \int \sqrt{\left(1 - \frac{2GM}{c^2 r}\right)} dt^2 + 0 + 0 + 0$$

$$\Delta\tau = \sqrt{1 - \frac{2GM}{c^2 r}} \Delta t, \quad \Delta\tau = \Delta t \text{ at } r = \infty, \quad (\Delta t: \text{time measured at } r = \infty)$$

The Gravitational Red Shift: Let two light signals be emitted by a source at r_E and received by a receiver at r_R



Since the Schwarzschild metric is independent of t $\Delta t_E = \Delta t_R = \Delta t$

$$\text{But } \Delta\tau_E = \sqrt{1 - \frac{2GM}{c^2 r_E}} \Delta t, \quad \Delta\tau_R = \sqrt{1 - \frac{2GM}{c^2 r_R}} \Delta t$$

$$\text{Since } f(\text{frequency}) = \frac{1}{\Delta\tau} = \frac{c}{\lambda}$$

$$\text{Similarly } \frac{\Delta\tau_R - \Delta\tau_E}{\Delta\tau_E} \approx \frac{gh}{c^2}; \quad \Delta\tau_R > \Delta\tau_E \Rightarrow \frac{\lambda_R}{\lambda_E} = \frac{\sqrt{1 - \frac{2GM}{c^2 r_R}} \Delta t}{\sqrt{1 - \frac{2GM}{c^2 r_E}} \Delta t} \approx \frac{\left(1 - \frac{GM}{c^2 r_R}\right)}{\left(1 - \frac{GM}{c^2 r_E}\right)}$$

$$\approx \left(1 - \frac{GM}{c^2 r_R}\right) \left(1 + \frac{GM}{c^2 r_E}\right) \approx 1 + \frac{GM}{c^2 r_E} - \frac{GM}{c^2 r_R}$$

$$r_R = r_E + h, \text{ with } h \ll r_E$$

$$\Rightarrow \frac{\lambda_R}{\lambda_E} \approx 1 + \frac{GM}{c^2} \left(\frac{1}{r_E} - \frac{1}{r_E(1 + \frac{h}{r_E})} \right) = 1 + \frac{GM}{c^2 r_E} \left(1 - \left(1 - \frac{h}{r_E}\right) \right)$$

$$\approx 1 + \frac{GM}{c^2 r_E^2} \cdot h. \quad \text{Since } r_E \approx R_E \text{ and } \frac{GM}{R_E^2} = g \quad \left[\begin{array}{l} \text{observed by} \\ \text{using Mössbauer} \\ \text{Effect} \end{array} \right]$$

$$\Rightarrow \frac{\lambda_R}{\lambda_E} \approx 1 + \frac{gh}{c^2}; \quad \lambda_R \approx \left(1 + \frac{gh}{c^2}\right) \lambda_E \Rightarrow \frac{\lambda_R - \lambda_E}{\lambda_E} \approx \frac{gh}{c^2} > 0 \Rightarrow \text{Red Shift}$$

Chapter 10 : Particle Orbits

The geodesic equation: $\frac{d}{d\tau} \left(g_{\mu\nu} \frac{dx^\nu}{d\tau} \right) - \frac{1}{2} \partial_\mu g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$

The Schwarzschild Metric:

$$ds^2 = - \left(1 - \frac{2GM}{c^2 r} \right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

$$= g_{\mu\nu} dx^\mu dx^\nu$$

$$\Rightarrow g_{tt} = - \left(1 - \frac{2GM}{c^2 r} \right), \quad g_{rr} = \left(1 - \frac{2GM}{c^2 r} \right)^{-1} \leftarrow g_{11}$$

$$g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 \sin^2 \theta \leftarrow g_{33}$$

$$g_{22}$$

Conserved Quantities: The $\mu=t$ component of the geodesic equation becomes

$$\frac{d}{d\tau} \left(g_{t\nu} \frac{dx^\nu}{d\tau} \right) - \frac{1}{2} \partial_t g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

$$\frac{d}{d\tau} \left(g_{tt} \frac{dx^t}{d\tau} + g_{ti} \frac{dx^i}{d\tau} \right) - \frac{1}{2} \partial_t g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

$\uparrow$
 $g_{\mu\nu}$ is diagonal

Since $ds^2 = -c^2 d\tau^2 = -g_{\alpha\beta} dx^\alpha dx^\beta$

$$\partial_t \left(g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \right) = 0$$

= 0 since $g_{\alpha\beta}$ is time independent

$$\Rightarrow \frac{d}{d\tau} \left(g_{tt} \frac{cdt}{d\tau} \right) = 0 \quad \Rightarrow -g_{tt} \frac{dt}{d\tau} = \text{constant}$$

$$\Rightarrow \left(1 - \frac{2GM}{c^2 r} \right) \frac{dt}{d\tau} \equiv e$$

e is conserved along all geodesic trajectories in Schwarzschild space-time

Consider $\lim_{r \rightarrow \infty} e = \lim_{r \rightarrow \infty} \frac{dt}{d\tau}$

Proper time (r, θ, ϕ constant) $d\tau = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} dt$
 $\xrightarrow{r \rightarrow \infty} dt$

Recall: For flat spacetime (namely for $r \rightarrow \infty$)

$$p^t = m U^t = mc \frac{dt}{d\tau} = \frac{E}{c}$$

$$\frac{dt}{d\tau} = \frac{E}{mc^2} \quad (\text{at } r = \infty)$$

The $\mu = \phi$ component:

$$\frac{d}{d\tau} \left(g_{\phi\nu} \frac{dx^\nu}{d\tau} \right) - \frac{1}{2} \left(\partial_\phi g_{\alpha\beta} \right) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

since the $g_{\mu\nu}$ are diagonal and independent of ϕ

$$-\partial_\phi (g_{\alpha\beta}) = 0 \Rightarrow \frac{d}{d\tau} \left(g_{\phi\phi} \frac{dx^\phi}{d\tau} \right) = 0$$

$$\Rightarrow g_{\phi\phi} \frac{d\phi}{d\tau} = \text{constant}$$

$$r^2 \sin^2 \theta \frac{d\phi}{d\tau} \equiv l. \quad l \text{ is conserved along all geodesic trajectories in Schwarzschild spacetime.}$$

For $\mu = \theta$ the geodesic equation becomes (Box 10.1)

$$r^2 \frac{d^2 \theta}{d\tau^2} + 2r \frac{dr}{d\tau} \frac{d\theta}{d\tau} - r^2 \sin \theta \cos \theta \left(\frac{d\phi}{d\tau} \right)^2 = 0$$

$$\Rightarrow \theta = \frac{\pi}{2} \text{ is a solution.}$$

So for $\theta = \frac{\pi}{2}$ l reduces to $l = r^2 \frac{d\phi}{d\tau}$

What is l ?

In Newtonian mechanics angular momentum $\vec{L} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v}$

$$\vec{v} = \frac{dr}{dt} \hat{e}_r + r \frac{d\phi}{dt} \hat{e}_\phi ; \quad \vec{r} \times \vec{v} = r^2 \omega \underbrace{\hat{r} \times \hat{e}_\phi}_{\hat{k}}$$

ω

$$\Rightarrow L = m r^2 \omega ; \quad \frac{L}{m} = r^2 \omega = r^2 \frac{d\phi}{dt}$$

so l is the relativistic angular momentum per mass.

The radial equation of motion:

For $\mu = r$ the geodesic equation is a bit complicated.

Instead, we use $\underline{U} \cdot \underline{U} = g_{\mu\nu} U^\mu U^\nu = -c^2$

$$g_{tt} \left(\frac{cdt}{d\tau} \right)^2 + g_{rr} \left(\frac{dr}{d\tau} \right)^2 + g_{\theta\theta} \left(\frac{d\theta}{d\tau} \right)^2 + g_{\phi\phi} \left(\frac{d\phi}{d\tau} \right)^2 = -c^2$$

$$-\left(1 - \frac{2GM}{c^2 r}\right) \left(\frac{cdt}{d\tau} \right)^2 + \left(\quad \right)^{-1} \left(\frac{dr}{d\tau} \right)^2 + 0 + r^2 \left(\frac{d\phi}{d\tau} \right)^2 = -c^2$$

$$-c^2 \left(\quad \right) e^2 \left(\quad \right)^{-2} + \left(\quad \right)^{-1} \left(\frac{dr}{d\tau} \right)^2 + r^2 \frac{l^2}{r^4} = -c^2$$

$$e^2 \left(1 - \frac{2GM}{c^2 r}\right)^{-1} - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \left(\frac{dr}{cd\tau} \right)^2 - \frac{l^2}{c^2 r^2} = 1$$

$\uparrow$
unitless

$\uparrow$ unitless

$\uparrow$ unitless

Multiplying through by $\left(1 - \frac{2GM}{c^2 r}\right)$

$$\Rightarrow e^2 - \left(\frac{dr}{cd\tau} \right)^2 - \left(1 - \frac{2GM}{c^2 r}\right) \frac{l^2}{c^2 r^2} = \left(1 - \frac{2GM}{c^2 r}\right)$$

$$(e^2 - 1) = \left(\frac{dr}{cd\tau} \right)^2 + \frac{l^2}{c^2 r^2} - \frac{2GM}{c^4 r^3} - \frac{2GM}{c^2 r}$$

$$\frac{1}{2}(e^2 - 1) = \frac{1}{2} \left(\frac{dr}{cd\tau} \right)^2 - \frac{GM}{c^2 r} + \frac{l^2}{2c^2 r^2} - \frac{GM l^2}{c^4 r^3} \equiv \tilde{E}$$

Interpretation: $\tilde{K} \equiv \frac{1}{2} \left(\frac{dr}{cd\tau} \right)^2$ = effective radial kinetic energy per unit mass energy

$$\tilde{V}(r) \equiv -\frac{GM}{c^2 r} + \frac{l^2}{2c^2 r^2} - \frac{GM l^2}{c^4 r^3} \quad \left| \begin{array}{l} \text{effective potential energy per unit mass energy} \\ = mc^2 = c^2, m=1 \end{array} \right.$$

$$\tilde{E} = \tilde{K} + \tilde{V}(r)$$

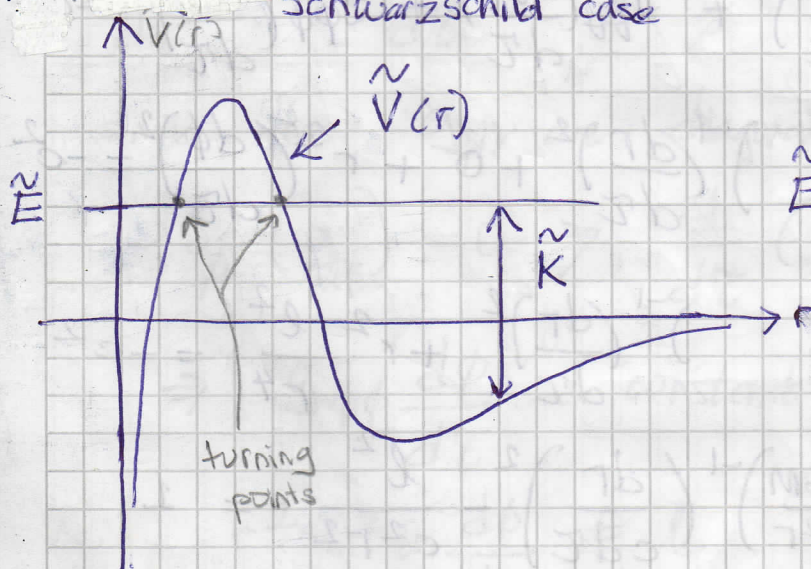
↳ energy / c^2 unit mass

For Newtonian orbits (Box. 10.3)

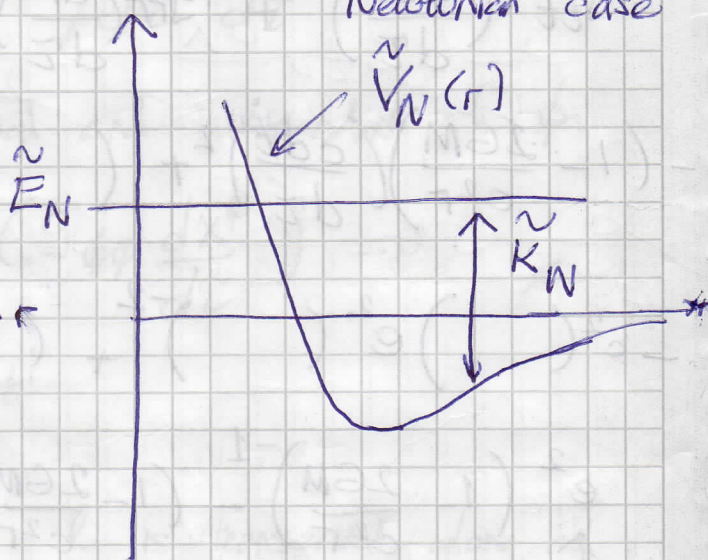
$$\tilde{E}_N \equiv \frac{1}{2} \left(\frac{dr}{cdt} \right)^2 + \tilde{V}_N, \text{ where } \tilde{V}_N(r) \equiv -\frac{GM}{c^2 r} + \frac{l^2}{2r^2 c^2}$$

Graphical Interpretation of the possible orbits:

Schwarzschild case



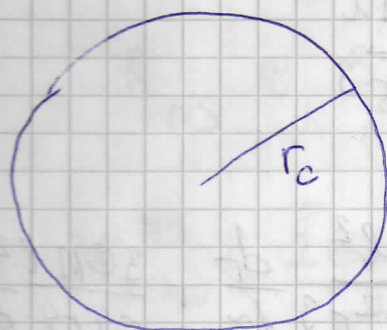
Newtonian case



Properties:

- $\tilde{K} = 0$ when $\tilde{E} = \tilde{V}(r)$: such points are turning points
- $\tilde{E} < 0$ orbits correspond to bound orbits
- $\tilde{E} > 0$ orbits correspond to unbound orbits
- Radii where $\frac{d\tilde{V}}{dr} = 0$ are possible circular orbit radii.

Circular orbits:



$\tilde{V} = \text{constant}$, $\frac{d\tilde{V}}{dr} = 0$ gives

$$\frac{GM}{c^2 r_c^2} - \frac{l^2}{c^2 r_c^3} + \frac{3GMl^2}{c^4 r_c^4} = 0$$

Multiplying by r_c^2 : $\frac{GM}{c^2} - \frac{l^2}{c^2 r_c} + \frac{3GMl^2}{c^4 r_c^2} = 0$

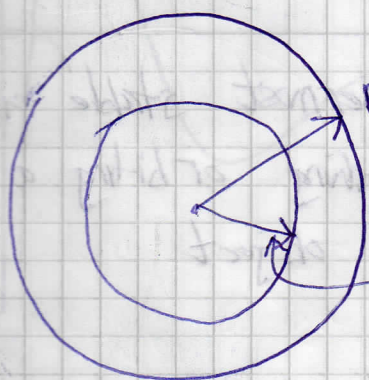
Letting $u_c = \frac{1}{r_c}$: $\frac{GM}{c^2} - \frac{l^2}{c^2} u_c + \frac{3GMl^2}{c^4} u_c^2 = 0$

$$ax^2 + bx + c = 0$$

$$x = u_c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1}{\frac{6GMl^2}{c^4}} \left[\frac{l^2}{c^2} \pm \left(\frac{l^4}{c^4} - 12 \frac{GMl^2}{c^4} \cdot \frac{GM}{c^2} \right)^{1/2} \right]$$

$$\frac{1}{r_c} = \frac{1}{\frac{6GM}{c^2}} \left[1 \pm \left(1 - 12 \left(\frac{GM}{cl} \right)^2 \right)^{1/2} \right]$$

$$\Rightarrow r_c = \frac{6GM/c^2}{1 \pm \sqrt{1 - 12 \left(\frac{GM}{cl} \right)^2}}$$



$$r_c^{\text{outer}} = \frac{6GM/c^2}{1 - \sqrt{\quad}} ; (\sqrt{\quad} > 0)$$

$$r_c^{\text{inner}} = \frac{6GM/c^2}{1 + \sqrt{\quad}} ; (\sqrt{\quad} > 0)$$

(Note $[GM/c^2] = \text{Length}$)

$$[l] = \frac{(\text{Length})^2}{\text{Time}}$$

Equations for radial acceleration:

$$\tilde{E} = \frac{1}{2} \left(\frac{dr}{cd\tau} \right)^2 - \frac{GM}{c^2 r} + \frac{l^2}{2c^2 r^2} - \frac{GM l^2}{c^4 r^3}$$

Take the τ -derivative:

$$0 = 2 \cdot \frac{1}{2} \frac{dr}{d\tau} \left(\frac{d^2 r}{c^2 d\tau^2} \right) + \frac{GM}{c^2 r^2} \frac{dr}{d\tau} - \frac{l^2}{c^2 r^3} \frac{dr}{d\tau} + \frac{3GM l^2}{c^4 r^4} \frac{dr}{d\tau}$$

$$\Rightarrow \frac{d^2 r}{d\tau^2} = - \frac{GM}{r^2} + \frac{l^2}{r^3} - \frac{3GM l^2}{c^2 r^4} \quad (\text{Schwarzschild})$$

$$\text{similarly } \frac{d^2 r}{dt^2} = - \frac{GM}{r^2} + \frac{l^2}{r^3} \quad (\text{Newtonian})$$

For an object moving radially $\frac{d\phi}{d\tau} = 0 \Rightarrow l = 0$

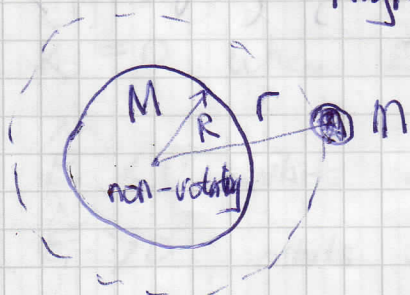
$$\Rightarrow \frac{d^2 r}{d\tau^2} = - \frac{GM}{r^2} : \text{relativistic acceleration.}$$

Astrophysical Implications:

In Newtonian gravity stable circular orbits exist for all radii,

But in Schwarzschild spacetime there are no stable circular orbits with $r < 6GM/c^2$

$r = \frac{6GM}{c^2}$ is the radius of the innermost stable circular orbit (ISCO) for anything orbiting a highly compact non-rotating object



• Compact objects, such as neutron stars, exist whose radii are smaller than $\frac{6GM}{c^2}$. $R \approx$

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For typical neutron stars $M_{NS} \sim 1.4 M_{\odot}$ ← solar mass

$$R_{NS} \approx 5 \frac{1.4 \times G M_{\odot}}{c^2} = 5 \times 1477 \text{ m} = 10339 \text{ m} \sim 10 \text{ km}$$

1477 m

$$r_{schw}^{NS} = \frac{2 G M_{NS}}{c^2} = \frac{2 \times 1.4 G M_{\odot}}{c^2} = 4135 \text{ m} \sim 4 \text{ km}$$

↑
for a NS star

since $r_{schw}^{NS} \sim 4 \text{ km} < R_{NS} \sim 10 \text{ km}$, neutron stars are not black holes. (For a black hole $R^{BH} < r_{schw}^{BH}$ must be satisfied.)

Accretion Disk:

(accrete: katılmalarla büyümek, artmak)
/akrikt/

objects falling toward a compact object will bounce off of the potential barrier $\tilde{V}(r)$ before reaching the object.

Such objects (particles) organize themselves into a flat disc around the object, called the accretion disk.

