MAT1072 MATHEMATICS II MIDTERM EXERCISES

① $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = ?$

exam

Soln: $S_n = \sum_{k=1}^{n} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k^2+k}} = \sum_{k=1}^{n} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}.\sqrt{k+1}} = \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)$

$= \left(1-\cancel{\frac{1}{\sqrt{2}}}\right)+\left(\cancel{\frac{1}{\sqrt{2}}}-\cancel{\frac{1}{\sqrt{3}}}\right)+\left(\cancel{\frac{1}{\sqrt{3}}}-\cancel{\frac{1}{\sqrt{4}}}\right)+\dots+\left(\cancel{\frac{1}{\sqrt{n-1}}}-\cancel{\frac{1}{\sqrt{n}}}\right)+\left(\cancel{\frac{1}{\sqrt{n}}}-\frac{1}{\sqrt{n+1}}\right)$

$= 1-\frac{1}{\sqrt{n+1}}$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n+1}}\right) = 1.$$

② Find the characteristic of the series $\sum_{n=2}^{\infty} n.\ln(1-n^{-1})$

exam

Soln: $a_n = n.\ln\left(1-\frac{1}{n}\right)$; $\forall n \geq 2$ $\left(1-\frac{1}{n}\right)^n > 0$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(1-\frac{1}{n}\right)^n = \ln\left(\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n\right) = \ln e^{-1} = -1 \neq 0$$

So by the nth term test, the series is divergent.

③ Let $\sum_{n=1}^{\infty} a_n$ be the series with terms $a_1 = \frac{1}{2}$, $a_{n+1} = \frac{1}{2}\left(1+\frac{1}{n}\right)a_n$

exam

for $n \geq 1$. Determine the characteristic (convergent or divergent) of the series.

Soln: Let us apply the Ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2} < 1$$

So the serie is convergent.

④ Find the values of x which makes the following series convergent.

exam

$$\sum_{n=0}^{\infty} n!.(x-1)^n$$

Soln: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!.(x-1)^{n+1}}{n!.(x-1)^n} \right| = \lim_{n\to\infty} (n+1).|x-1| = \infty >> 1$

So, for $x \neq 1$ the series is divergent.

Only for $x=1$, the series is convergent.

⑤ Find the interval of convergence and the sum of the series

exam

$$\sum_{n=0}^{\infty} \frac{(x+3)^n}{4^{n+1}}.$$

Soln: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x+3)^{n+1}}{4^{n+2}} \cdot \frac{4^{n+1}}{(x+3)^n} \right| = \lim_{n\to\infty} \frac{|x+3|}{4} = \frac{|x+3|}{4} < 1$

$\Rightarrow -4 < x+3 < 4 \Rightarrow -7 < x < 1$

For $x=1$, $\sum_{n=0}^{\infty} \frac{4^n}{4^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{4}$, divergent

For $x=-7$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{4}$, divergent

So, the interval of convergence $(-7, 1)$

$\sum_{n=0}^{\infty} \frac{1}{4} \cdot \left(\frac{x+3}{4}\right)^n$, $a = \frac{1}{4}$, $r = \frac{x+3}{4}$

$$= \frac{a}{1-r} = \frac{1/4}{1-\frac{x+3}{4}} = \frac{1}{4} \cdot \frac{4}{4-x-3} = \frac{1}{1-x}$$

⑥ If $n \geq 0$ is an integer, find the summation of $\sum_{k=n}^{\infty} 3^{n-k}$

exam

Soln: $\sum_{k=n}^{\infty} 3^{n-k} = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

$a=1$
$r=\frac{1}{3}$

⑦ Investigate the alternating series $\sum_{k=3}^{\infty} (-1)^k \frac{\ln k+1}{\sqrt{k}}$ for absolute and conditional convergence.

exam

Soln: $\sum_{k=3}^{\infty} \left| (-1)^k \frac{\ln k+1}{\sqrt{k}} \right| = \sum_{k=3}^{\infty} \frac{\ln k+1}{\sqrt{k}}$

Let us choose $\sum b_k = \sum_{k=3}^{\infty} \frac{1}{\sqrt{k}}$ (by p-series test, $p=\frac{1}{2}<1$, divergent)

to apply limit comparison test.

$\lim_{k\to\infty} \frac{a_n}{b_n} = \lim_{k\to\infty} \frac{\ln k+1}{\cancel{\sqrt{k}}} \cdot \cancel{\sqrt{k}} = \infty$, So $\sum_{k=3}^{\infty} \frac{\ln k+1}{\sqrt{k}}$ diverges.

Thus $\sum_{k=3}^{\infty} (-1)^k \frac{\ln k+1}{\sqrt{k}}$ does not converge absolutely

Now apply the alternating series test

i) $\frac{\ln k+1}{\sqrt{k}} > 0$ for $k \geq 3$ ✓

ii) $a_{k+1} \leq a_k$?

Let $a_k = f(x)$ for $x \geq 3$ such that $f(x) = \frac{\ln x+1}{\sqrt{x}}$

$f'(x) = \frac{1-\ln x}{2x\sqrt{x}} \Rightarrow f'(x) < 0$ (since $\ln x > 1$ for $x \geq 3$) $\Rightarrow f(x)$ is decreasing $\Rightarrow a_{k+1} \leq a_k$ ✓

iii) $\lim_{k\to\infty} \frac{\ln k+1}{\sqrt{k}} \overset{L.H}{=} \lim_{k\to\infty} \frac{1/k}{\frac{1}{2}\frac{1}{\sqrt{k}}} = \lim_{k\to\infty} \frac{2}{\sqrt{k}} = 0$ ✓

Hence the alternating series converges.

Therefore it converges conditionally.

⑧ Express the power series representation of the function

exam

$g(x) = x.\ln(1+x^2)$ by using the power series expansion of $\frac{1}{1-x}$

for $|x|<1$ and find the interval of convergence of the series.

Soln: For $|x|<1$ $\quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$x \leftrightarrow -x^2 \longrightarrow \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n \longrightarrow \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n . x^{2n} \quad |-x^2|<1$$

$$\longrightarrow \frac{x}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n . x^{2n+1}$$

$$\longrightarrow \int_0^x \frac{t}{1+t^2} dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n+1} dt$$

$$\Rightarrow \frac{1}{2} \ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n+2}}{2n+2} \Rightarrow \underbrace{x.\ln(1+x^2)}_{g(x)} = \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n+3}}{n+1}$$

$$\Rightarrow g(x) = x.\ln(1+x^2) = \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n+3}}{n+1}, \quad |x|<1$$

It is alternating series for $x = \mp 1$ and conditionally convergent.

The interval of convergence $[-1, 1]$.

⑨ Find the summation of the series $\sum_{n=0}^{\infty} \frac{(n+1).(n+2)}{2^n}$ by using the

exam

power series expansion of $\frac{1}{1-x}$ for $|x|<1$.

Soln: $f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \rightarrow f'(x) = \frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k . x^{k-1}$

$$\rightarrow f''(x) = \frac{2}{(1-x)^3} = \sum_{k=2}^{\infty} k.(k-1). x^{k-2}$$

For $k = n+2$, $\quad \sum_{n=0}^{\infty} (n+2).(n+1). x^n = \frac{2}{(1-x)^3}$

$$x = \frac{1}{2} \Rightarrow \sum_{n=0}^{\infty} \frac{(n+2).(n+1)}{2^n} = \frac{2}{(1-\frac{1}{2})^3} = 16.$$

(10) Determine whether the series $\sum_{n=1}^{\infty} \frac{1+\sin n}{n!}$ is convergent or divergent.

exam

Soln: $-1 \leq \sin n \leq 1$

$$0 \leq 1+\sin n \leq 2$$

$$0 \leq \frac{1+\sin n}{n!} \leq \frac{2}{n!}$$

★ $n! = n.(n-1).(n-2) \dots 3.2.1 \geq 2.2.2 \dots 1 = 2^{n-1} \Rightarrow \frac{1}{n!} \leq \frac{1}{2^{n-1}} \Rightarrow \frac{2}{n!} \leq \frac{2}{2^{n-1}}$

by ★ $\Rightarrow 0 \leq \frac{1+\sin n}{n!} \leq \frac{2}{2^{n-1}}$

$\sum_{n=1}^{\infty} 2.\left(\frac{1}{2}\right)^{n-1}$ is geometric series with $r = \frac{1}{2} < 1$, so is convergent.

Therefore $\sum_{n=1}^{\infty} \frac{1+\sin n}{n!}$ is convergent.

(11) Find the summation of the series $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

exam

Soln: $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \sum_{n=1}^{\infty} \frac{n+1-1}{(n+1)!} = \sum_{n=1}^{\infty} \left(\frac{n+1}{(n+1)!} - \frac{1}{(n+1)!}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)$

$$S_k = \sum_{n=1}^{k} \frac{n}{(n+1)!} = \left(1 - \cancel{\frac{1}{2!}}\right) + \left(\cancel{\frac{1}{2!}} - \cancel{\frac{1}{3!}}\right) + \dots + \left(\cancel{\frac{1}{(k-1)!}} - \cancel{\frac{1}{k!}}\right) + \left(\cancel{\frac{1}{k!}} - \frac{1}{(k+1)!}\right)$$

$$S_k = 1 - \frac{1}{(k+1)!} \Rightarrow S = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \left(1 - \frac{1}{(k+1)!}\right) = 1.$$

(12) Find the summation of the series $\sum_{n=0}^{\infty} (-1)^n . \frac{\pi^{2n}}{(2n)!.9^n}$

exam

Soln: $\sum_{n=0}^{\infty} (-1)^n . \frac{\pi^{2n}}{(2n)!.9^n} = \sum_{n=0}^{\infty} (-1)^n . \frac{\pi^{2n}}{3^{2n}.(2n)!} = \sum_{n=0}^{\infty} (-1)^n . \frac{(\pi/3)^{2n}}{(2n)!}$

we know that $\cos x = \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n}}{(2n)!}$

So, for $x = \frac{\pi}{3}$, $\sum_{n=0}^{\infty} (-1)^n . \frac{(\pi/3)^{2n}}{(2n)!} = \cos\frac{\pi}{3} = \frac{1}{2}$

(13) Find the Maclaurin series of the function $L(x) = \int_0^x \cos(t^2)\,dt$.

exam

Soln: $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$x \to t^2$, $\cos(t^2) = \sum_{n=0}^{\infty} (-1)^n . \frac{t^{4n}}{(2n)!}$

$$L(x) = \int_0^x \cos t^2 dt = \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n . \frac{t^{4n}}{(2n)!} \right) dt = \sum_{n=0}^{\infty} (-1)^n . \frac{x^{4n+1}}{(2n)!} . \frac{1}{(4n+1)}$$

(14) Find the Maclaurin series of $g(t) = \int_0^t \frac{e^x - 1}{x} dx$.

Soln: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$g(t) = \int_0^t \left(\frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - 1}{x} \right) dx = \int_0^t \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots \right) dx$$

$$= x + \frac{x^2}{2.2!} + \frac{x^3}{3.3!} + \cdots \Big|_0^t = t + \frac{t^2}{2.2!} + \frac{t^3}{3.3!} + \cdots = \sum_{n=1}^{\infty} \frac{t^n}{n.n!}$$

(15) Find the power series expansion of the function $f(x) = \ln(3+x)$ and find the interval of convergence.

Soln: $f(x) = \ln(3+x) = \int \frac{dx}{3+x} = \frac{1}{3} . \int \frac{dx}{1 + \frac{x}{3}} = \frac{1}{3} \int \sum_{n=0}^{\infty} (-1)^n . \left(\frac{x}{3}\right)^n dx$, $\left|\frac{x}{3}\right| < 1$

$$\Rightarrow f(x) = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n . x^{n+1}}{3^n (n+1)} + C$$

$f(0) = \ln 3 \to C = \ln 3$

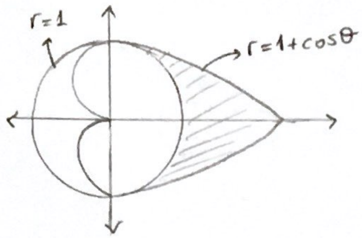
$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n . x^{n+1}}{3^{n+1}(n+1)} + \ln 3 \quad , \ |x| < 3$$

$\Rightarrow$ The interval of convergence is $(-3, 3)$

(16) Find the area of the region outside of the circle $r=1$ and inside of the curve $r=1+\cos\theta$.

exam

Soln:



$1 = 1+\cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$$\frac{A}{2} = \int_0^{\pi/2} \frac{1}{2}\left[(1+\cos\theta)^2 - 1^2\right] d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta$$

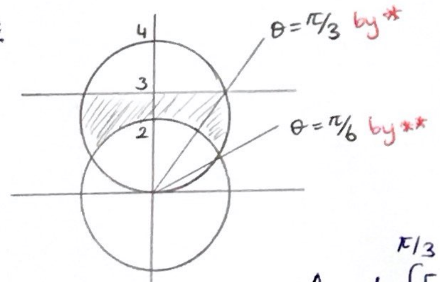
$$= \frac{1}{2}\left(2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2} = 1 + \frac{\pi}{8}$$

$\Rightarrow A = 2 + \frac{\pi}{4}$ unit square.

(17) Write the integral of a region which enclosed by $r=2$, $r=4\sin\theta$ and $r\sin\theta = 3$ by using polar coordinates.

exam

Soln:



$4\sin\theta = \frac{3}{\sin\theta} \Rightarrow \sin^2\theta = \frac{3}{4} \Rightarrow \sin\theta = \mp\frac{\sqrt{3}}{2}$

$\Rightarrow \sin\theta = \frac{\pi}{3}$ *

$4\sin\theta = 2 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ **

$$\frac{A}{2} = \frac{1}{2}\int_{\pi/6}^{\pi/3}\left[(4\sin\theta)^2 - 2^2\right]d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}\left[\left(\frac{3}{\sin\theta}\right)^2 - 2^2\right]d\theta$$

(18) Find the length of the curve $r=e^{a\theta}$ in the interval $-\pi \le \theta \le \pi$

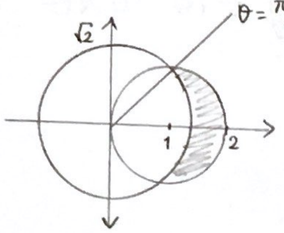
exam

Soln: $$L = \int_{-\pi}^{\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_{-\pi}^{\pi}\sqrt{e^{2a\theta} + a^2 e^{2a\theta}}\,d\theta = \sqrt{1+a^2}\int_{-\pi}^{\pi} e^{a\theta}\,d\theta$$

$$= \frac{\sqrt{1+a^2}}{a}\,e^{a\theta}\Big|_{-\pi}^{\pi} = \frac{\sqrt{1+a^2}}{a}\left(e^{a\pi} - e^{-a\pi}\right).$$

(19) Find the area of the region inside of the curve $r=2\cos\theta$ and
exam
outside of the curve $r=\sqrt{2}$.

Soln:



$2\cos\theta=\sqrt{2} \Rightarrow \theta=\frac{\pi}{4}$ *

$$\frac{A}{2}=\frac{1}{2}\int_0^{\pi/4}(4\cos^2\theta-2)\,d\theta=\frac{1}{2}\int_0^{\pi/4}(2+2\cos2\theta-2)\,d\theta$$

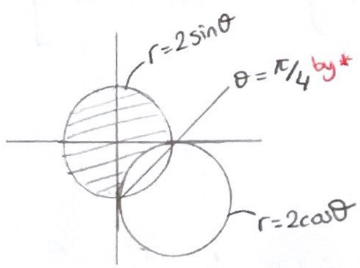
$$=\frac{1}{2}\sin2\theta\Big|_0^{\pi/4}=\frac{1}{2}$$

$\Rightarrow A=1$ unit square.

(20) Write the equations $x^2+y^2=2x$ and $x^2+y^2=2y$ in polar coordinates
exam
and sketch the graph.

Soln: $\left.\begin{array}{l} x=r.\cos\theta \\ y=r.\sin\theta \end{array}\right\}$ $x^2+y^2=r^2\cos^2\theta+r^2\sin^2\theta=2x=2r\cos\theta$

$\Rightarrow r^2=2r\cos\theta \Rightarrow r=2\cos\theta$



$x^2+y^2=r^2\cos^2\theta+r^2\sin^2\theta=2y=2r\sin\theta$

$\Rightarrow r^2=2r\sin\theta \Rightarrow r=2\sin\theta$

(21) Find the area of the region inside of the curve $r=2\sin\theta$ and
exam
outside of the curve $r=2\cos\theta$. (In the previous example, check the graph.)

Soln: $2\cos\theta=2\sin\theta \Rightarrow \theta=\frac{\pi}{4}$ *

$$A=\frac{1}{2}\int_{\pi/4}^{\pi/2}(4\sin^2\theta-4\cos^2\theta)\,d\theta+\frac{1}{2}\int_{\pi/2}^{\pi}4\sin^2\theta\,d\theta$$

$$=2\int_{\pi/4}^{\pi/2}(-\cos2\theta)\,d\theta+\int_{\pi/2}^{\pi}(1-\cos2\theta)\,d\theta$$

$$=-\sin2\theta\Big|_{\pi/4}^{\pi/2}+(\theta-\sin2\theta)\Big|_{\pi/2}^{\pi}=1+\pi-\frac{\pi}{2}=1+\frac{\pi}{2}$$ unit square

(22) Find the length of the curve $r=\theta^2$ in the interval $0 \le \theta \le 2\pi$

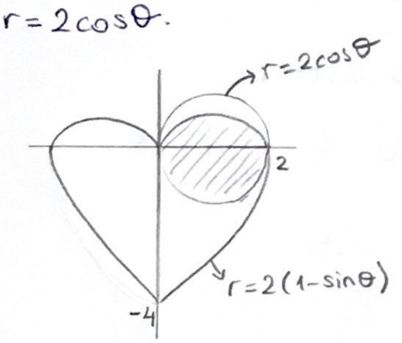
Soln: $r' = 2\theta$

$$L = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta = \int_0^{2\pi} \underbrace{|\theta|}_{=\theta \text{ since } \theta\in[0,2\pi]} \cdot \sqrt{\theta^2+4}\, d\theta = \int_0^{2\pi} \theta \cdot \sqrt{\theta^2+4}\, d\theta$$

$$\left.\begin{aligned} \theta^2+4 &= u^2 \\ 2\theta d\theta &= 2u du \end{aligned}\right\} \quad \begin{aligned} &\theta = 0 \Rightarrow u = 2 \\ &\theta = 2\pi \Rightarrow u = \sqrt{4\pi^2+4} = 2\sqrt{\pi^2+1} \end{aligned}$$

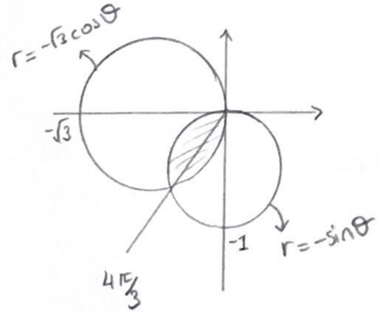
$$L = \int_2^{2\sqrt{\pi^2+1}} u^2 du = \left.\frac{u^3}{3}\right|_2^{2\sqrt{\pi^2+1}} = \frac{1}{3}\left(8(\pi^2+1)^{3/2} - 8\right) = \frac{8}{3}\left((\pi^2+1)^{3/2} - 1\right)$$

(23) Write the integral of the region enclosed by $r=2(1-\sin\theta)$ and $r=2\cos\theta$.



$$A = \frac{1}{2}\left[\int_{-\pi/2}^{0} 4\cos^2\theta\, d\theta + \int_0^{\pi/2} 4(1-\sin\theta)^2 d\theta\right]$$

(24) Write the integral of the region between the curve $r=-\sqrt{3}\cos\theta$ and $r=-\sin\theta$.



$$-\sqrt{3}\cos\theta = -\sin\theta \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{4\pi}{3}$$

$$A = \frac{1}{2}\left[\int_{\pi}^{4\pi/3} \sin^2\theta\, d\theta + \int_{4\pi/3}^{3\pi/2} 3\cos^2\theta\, d\theta\right]$$