MAT1072 MATHEMATICS II QUESTIONS & SOLUTIONS

① A particle moves in space with the position vector

(exam)

$$\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + (t\sqrt{3})\vec{k}$$

For which values of t, does the angle between its position and acceleration vectors become $\frac{2\pi}{3}$?

Solution: $\vec{v}(t) = \frac{d\vec{r}}{dt} = \langle -\sin t, \cos t, \sqrt{3} \rangle$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \langle -\cos t, -\sin t, 0 \rangle$$

Note that; $u.v = |u|.|v|.\cos\theta$

$$\Rightarrow \cos\theta = \frac{\vec{r}.\vec{a}}{|\vec{r}|.|\vec{a}|} = \frac{-\cos^2 t - \sin^2 t}{\sqrt{1+3t^2}\cdot\sqrt{1}} = -\frac{1}{\sqrt{1+3t^2}}$$

If $\theta = \frac{2\pi}{3}$, then $\cos\theta = -\frac{1}{2}$ follows.

$$\Rightarrow -\frac{1}{\sqrt{1+3t^2}} = -\frac{1}{2} \Rightarrow 1+3t^2 = 4 \Rightarrow t = \mp 1$$

② Let $\vec{r}(t) = at^2\vec{i} + bt\vec{j} + c\ln t\vec{k}$, $1 \le t \le T$, $a,b,c \in \mathbb{R}$. Find the arclength of the curve while $b^2 = 4ac$.

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Solution: $L = \int_1^T |\vec{v}(t)|\,dt = \int_1^T |\vec{r}\,'(t)|\,dt = \int_1^T \left|2at\vec{i} + b\vec{j} + \frac{c}{t}\vec{k}\right| dt$

$$= \int_1^T \sqrt{4a^2t^2 + b^2 + \frac{c^2}{t^2}}\,dt \underset{b^2=4ac}{=} \int_1^T \sqrt{4a^2t^2 + 4ac + \frac{c^2}{t^2}}\,dt$$

$$= \int_1^T \left(2at + \frac{c}{t}\right) dt = at^2 + c\ln t \Big|_1^T = aT^2 + c\ln T - a = a(T^2-1) + c\ln T$$

(3) Let $\vec{r}(t) = (1+t^2)^{3/2}\vec{i} + (3-t^2)^{3/2}\vec{j} + (4t^2)\vec{k}$. Find the arclength
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of the curve for $-1 \le t \le 1$.

<u>Solution:</u> $L = \int_a^b |\vec{v}(t)|\,dt$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \left< \frac{3}{2}(1+t^2)^{1/2} \cdot 2t,\ \frac{3}{2}(3-t^2)^{1/2} \cdot (-2t),\ 8t \right>$$

$$= \left< 3t\sqrt{1+t^2},\ -3t\sqrt{3-t^2},\ 8t \right>$$

$$\Rightarrow |\vec{v}(t)| = \sqrt{9t^2(1+t^2) + 9t^2(3-t^2) + 64t^2} = \sqrt{100t^2} = 10|t|$$

$$\Rightarrow L = \int_{-1}^{1} 10|t|\,dt = 10\int_{-1}^{0} -t\,dt + 10\int_{0}^{1} t\,dt = 10\left(-\frac{t^2}{2}\Big|_{-1}^{0} + \frac{t^2}{2}\Big|_{0}^{1}\right)$$

$$= -(-5) + 5 = 10.$$

(4) Let $\vec{v}(t) = (t+1)\vec{i} + (t^2-1)\vec{j} + (2t)\vec{k}$ be the velocity vector of a
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moving particle in space (at time t).

i) Calculate the angle between its velocity and acceleration vectors at $t=1$.

ii) If this particle on the point $(1,-1,2)$ at $t=0$, then find the position vector $\vec{r}(t)$ of it.

<u>Solution:</u> i) $\vec{a}(t) = \frac{d\vec{v}}{dt} = \vec{i} + 2t\vec{j} + 2\vec{k}$

at $t=1 \Rightarrow$ $\vec{a}(1) = \vec{i} + 2\vec{j} + 2\vec{k} \Rightarrow |\vec{a}(1)| = \sqrt{1+4+4} = 3$

$\vec{v}(1) = 2\vec{i} + 0\vec{j} + 2\vec{k} \Rightarrow |\vec{v}(1)| = \sqrt{4+4} = 2\sqrt{2}$

$$\Rightarrow \cos\theta = \frac{\vec{a}(1) \cdot \vec{v}(1)}{|\vec{a}(1)|\,|\vec{v}(1)|} = \frac{2+4}{6\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

ii) $\vec{r}(t) = \int \vec{v}(t)\,dt = \left(\frac{t^2}{2}+t\right)\vec{i} + \left(\frac{t^3}{3}-t\right)\vec{j} + t^2\vec{k} + C$

at $t=0$ particle on $(1,-1,2) \Rightarrow \vec{r}(0) = \vec{i}-\vec{j}+2\vec{k}$

thus $\vec{r}(0) = C = \vec{i}-\vec{j}+2\vec{k}$

Hence, $\vec{r}(t) = \left(\frac{t^2}{2}+t+1\right)\vec{i} + \left(\frac{t^3}{3}-t-1\right)\vec{j} + (t^2+2)\vec{k}$

⑤ Find the length of the curves

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i) $r = e^{a\theta}$, for $-\pi < \theta < \pi$

ii) $r = a.\sin^2\frac{\theta}{2}$, for $0 \le \theta \le \pi$, $a > 0$.

Solution: i) $L = \int_{-\pi}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{-\pi}^{\pi} \sqrt{e^{2a\theta} + a^2 e^{2a\theta}}\, d\theta$

$$= (1+a^2)^{1/2} \int_{-\pi}^{\pi} e^{a\theta}\, d\theta = (1+a^2)^{1/2} \frac{e^{a\theta}}{a}\Big|_{-\pi}^{\pi} = \frac{\sqrt{1+a^2}}{a}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$= 2\,\frac{\sqrt{1+a^2}}{a}\sinh(a\pi)$$

ii) $L = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$r^2 = a^2\sin^4\frac{\theta}{2}$, $\frac{dr}{d\theta} = a.\left(2.\sin\frac{\theta}{2}.\cos\frac{\theta}{2}\right).\frac{1}{2} = a.\sin\frac{\theta}{2}.\cos\frac{\theta}{2}$,

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\sin^4\frac{\theta}{2} + a^2.\sin^2\frac{\theta}{2}.\cos^2\frac{\theta}{2} = a^2\left(\sin^4\frac{\theta}{2} + \sin^2\frac{\theta}{2}\left(1-\sin^2\frac{\theta}{2}\right)\right)$$

$$= a^2\left(\sin^2\frac{\theta}{2}\right)$$

$$\Rightarrow L = \int_0^{\pi} \sqrt{a^2\sin^2\frac{\theta}{2}}\, d\theta = \int_0^{\pi} a\left|\sin\frac{\theta}{2}\right| d\theta = a.\int_0^{\pi} \sin\frac{\theta}{2}\, d\theta = -2a\cos\frac{\theta}{2}\Big|_0^{\pi}$$

$$= -2a(0-1) = 2a \Rightarrow L = 2a$$

⑥ Let $P_1 : x+2y-z$, $P_2 : 2x+y+z=4$

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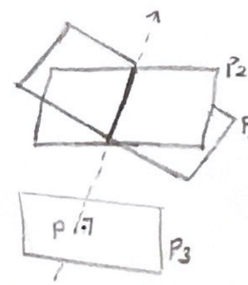
$\ell : x = 1+t$, $y = 2-t$, $z = 1-t$ be given.

(i) Find the equation of the plane which passes through the point $P(1,-2,1)$ and perpendicular to the line of intersection of the planes P_1 and P_2.

Solution i) normal of the plane P_1 : $\vec{n_1} = \langle 1, 2, -1 \rangle$

normal of the plane P_2 : $\vec{n_2} = \langle 2, 1, 1 \rangle$

The line of intersection of the planes P_1 and P_2 is perpendicular to both planes' normal vectors $\vec{n_1}$ and $\vec{n_2}$ and thus parallel to $\vec{n_1} \times \vec{n_2}$.



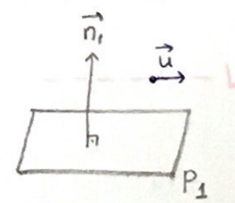
$$\Rightarrow \vec{v} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{vmatrix} = \langle 3, -3, -3 \rangle$$

Thus, the equation of the plane which passes through $P(1,-2,1)$ and is normal to $\vec{v}$ is;

$$3(x-1) - 3(y+2) - 3(z-1) = 0 \Rightarrow x - y - z = 2.$$

(ii) Investigate whether the plane P_1 is parallel to ℓ or not.

Solution ii) $(1,2,1)$ is on L and $1+2.2-1 = 4 \neq 1$, so L is <u>not</u> on P_1.



normal of the plane P_1 : $\vec{n_1} = \langle 1, 2, -1 \rangle$

direction of the line ℓ : $\vec{u} = \langle 1, -1, -1 \rangle$

If P_1 is parallel to L then $\vec{n_1}$ and $\vec{u}$ are orthogonal.

$\vec{n_1} \cdot \vec{u} = 1 - 2 + 1 = 0 \Rightarrow \vec{n_1} \perp \vec{u}$ (and L is not on P_1)

$\Rightarrow P_1$ and L are parallel.

⑦ Find the equation of the line which is perpendicular to the plane $2x-y+7z=12$ and passing through $P(-1,0,1)$.
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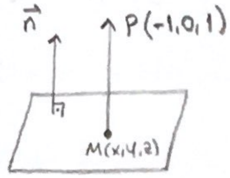
Solution:



$\vec{n} = \langle 2,-1,7 \rangle$

$\vec{PM} \parallel \vec{n} \Rightarrow \vec{PM} = t.\vec{n}$

$\Rightarrow \langle x+1, y, z-1 \rangle = t.\langle 2,-1,7 \rangle$

$$\Rightarrow \left.\begin{array}{l} x+1 = 2t \\ y = -t \\ z-1 = 7t \end{array}\right\} \quad \begin{array}{l} x = 2t-1 \\ y = -t \\ z = 7t+1 \end{array}$$

⑧ Find the equation of the plane P_2 which passes through the points $A_1(1,1,1)$ and $A_2(2,0,3)$ and perpendicular to the plane
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$P_1 : x+2y-3z=0$.

Soln: normal of the plane P_1: $\vec{n_1} = \langle 1,2,-3 \rangle$

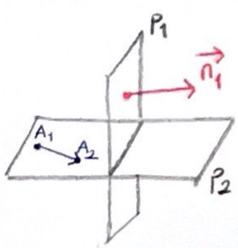
Since the plane P_2 is perpendicular to P_1, so $\vec{n_1}$ is parallel to P_2.

$\vec{A_1A_2} = \langle 1,-1,2 \rangle$ lies on P_2.

$$\Rightarrow \vec{n_2} = \vec{n_1} \times \vec{A_1A_2} = \begin{vmatrix} i & j & k \\ 1 & 2 & -3 \\ 1 & -1 & 2 \end{vmatrix} = \vec{i} - 5\vec{j} - 3\vec{k}$$ is the normal to P_2.

Thus the equation of the P_2 is

$$(x-1) - 5(y-1) - 3(z-1) = 0 \Rightarrow x-5y-3z = -7.$$



⑨ For which value of λ, the planes $\lambda x - y + 3z = 1$ and $2x + 3y - \lambda z = 4$ are perpendicular (orthogonal to each other)?

Solution: $\vec{n_1}$ is the normal of $\lambda x - y + 3z = 1 \Rightarrow \vec{n_1} = \langle \lambda, -1, 3 \rangle$

$\vec{n_2}$ " " $2x + 3y - \lambda z = 4 \Rightarrow \vec{n_2} = \langle 2, 3, -\lambda \rangle$

these planes are perpendicular $\Leftrightarrow$ their normal vectors are perpendicular.

So $\vec{n_1} . \vec{n_2} = 0 \Rightarrow 2\lambda - 3 - 3\lambda = 0 \Rightarrow \lambda = -3.$

⑩ For which value of λ, the planes $x + \lambda y + 2z = 3$ and $\lambda x + y - 2z = 1$ are parallel?

Soln: $\vec{n_1}$ for $x + \lambda y + 2z = 3 \Rightarrow \vec{n_1} = \langle 1, \lambda, 2 \rangle$

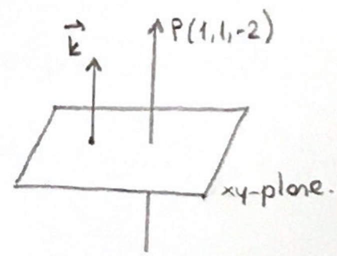
$\vec{n_2}$ for $\lambda x + y - 2z = 1 \Rightarrow \vec{n_2} = \langle \lambda, 1, -2 \rangle$

As these parallel so $\vec{n_1} \parallel \vec{n_2}$ also holds.

$$\Rightarrow \frac{1}{\lambda} = \frac{\lambda}{1} = \frac{2}{-2} \Rightarrow \lambda = -1.$$

⑪ Find the equation for the line which passes through the point $P(1, 1, -2)$ and perpendicular to the xy-plane.

Soln: Since the line is perpendicular to xy-plane so $\vec{v} = \langle 0, 0, 1 \rangle = \vec{k}$ is the direction for this line.



$\vec{v} = \langle 0, 0, 1 \rangle$, $P(1, 1, -2)$

Parametric equation at this line,

$x = 1, \quad y = 1, \quad z = -2 + t$

vector equation of this line,

$\vec{r}(t) = \vec{i} + \vec{j} + (-2 + t)\vec{k}$

(12) Let u be a differentiable function with
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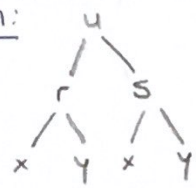
$u = f(s) + g(r)$ and $s = 5x + y$, $r = y - 5x$.

By using chain rule, show that $\frac{\partial^2 u}{\partial x^2} - 25\frac{\partial^2 u}{\partial y^2}$ equals to zero.

Soln:



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$u_x = u_s . 5 - u_r . 5$$

$$u_{xx} = 5(u_{ss}.5 - u_{sr}.5) - 5(u_{rs}5 - u_{rr}5)$$

$$= 25u_{ss} - 25u_{sr} - 25u_{rs} + 25u_{rr}$$

$$= 25u_{ss} - \underbrace{50u_{sr}}_{=0} + 25u_{rr} = 25f''(s) + 25g''(r)$$

$$u_y = u_s.1 + u_r.1$$

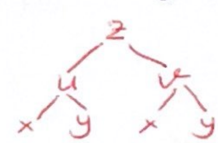
$$u_{yy} = u_{ss}.1 + \cancel{u_{sr}}^{0}.1 + \cancel{u_{rs}}^{0}.1 + u_{rr} = u_{ss} + u_{rr} = f''(s) + g''(r)$$

$$u_{xx} - 25u_{yy} = 25f''(s) + 25g''(r) - 25f''(s) - 25g''(r) = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} - 25\frac{\partial^2 u}{\partial y^2} = 0.$$

(13) Let g be a differentiable function and for $x, y > 0$, $z = g\left(\ln\frac{y}{x}, \frac{y}{x}\right)$.
(exam)
Show that $x z_x + y z_y = 0$ where $z_x = \frac{\partial z}{\partial x}$, $z_y = \frac{\partial z}{\partial y}$

Soln: $z = g(u, v)$, $u = \ln\frac{y}{x} = \ln y - \ln x$
$v = \frac{y}{x}$



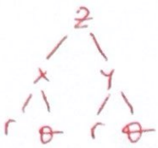
$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x} = g_u\cdot\left(-\frac{1}{x}\right) + g_v\cdot\left(-\frac{y}{x^2}\right)$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial y} = g_u\cdot\left(\frac{1}{y}\right) + g_v\cdot\frac{1}{x}$$

$$\Rightarrow x.z_x + y.z_y = -g_u - g_v\cdot\frac{y}{x} + g_u + g_v\cdot\frac{y}{x} = 0.$$

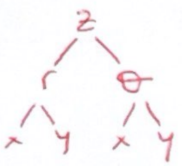
(14) Let f be a diff'ble function and $z=f(x,y)$. Find the representation of the equation $y.\frac{\partial z}{\partial x} - x.\frac{\partial z}{\partial y} = 0$ by substituding $x=r.\cos\theta$ and $y=r.\sin\theta$.

Soln: $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}.\underbrace{(-r.\sin\theta)}_{=-y} + \frac{\partial z}{\partial y}.\underbrace{(r.\cos\theta)}_{=x}$$



$$= -y.\frac{\partial z}{\partial x} + x.\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial \theta} = 0$$

Second way: $r=\sqrt{x^2+y^2}$, $\theta = \arctan\frac{y}{x}$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}.\frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r}.\frac{x}{r} - \frac{\partial z}{\partial \theta}.\frac{y}{r^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}.\frac{y}{r} + \frac{\partial z}{\partial \theta}.\frac{x}{r^2}$$

$$\Rightarrow y.\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = -\frac{\partial z}{\partial \theta} = 0.$$

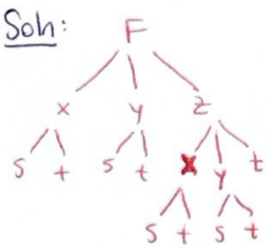
(15) Let $x(s,t) = 2s+3ts$

(exam) $y(s,t) = 3t+2st$ and $F(x,y,z) = zx + \sin(2yz)$

$z(x,y,t) = x+yt$

Compute $\frac{\partial F}{\partial t}$ at the point $(s,t)=(1,2)$ by using the chain rule.

Soln:



$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}.\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial t} + \frac{\partial z}{\partial t}\right)$$

$$= z.3s + 2z.\cos(2yz)(3+2s)$$

$$+ (x+2y\cos(2yz)).(3s+t(3+2s)+y)$$

$$\left.\frac{\partial F}{\partial t}\right|_{\substack{(1,2)\\(s,t)}} = 28.3.1 + 2.28.\cos(2.10.28).(3+2.1) + (8+20\cos(560)).(3+2(3+2)+10)$$

$x=8$, $z=28$
$y=10$,

$$= 84 + 280\cos(560) + (8+20\cos 560).23$$

(16) Find the directional derivative of $f(x,y) = 2xy - 3y^2$ at $P_0(5,5)$ in the direction of $\vec{u} = \langle 4,3 \rangle$

Soln: $(D_u f)_{P_0} = \nabla f \big|_{P_0} \cdot \frac{\vec{u}}{|\vec{u}|}$ where $\vec{u}$ is the unit vector in the direction of u.

$$\frac{\vec{u}}{|\vec{u}|} = \frac{4\vec{i} + 3\vec{j}}{5} = \langle \frac{4}{5}, \frac{3}{5} \rangle$$

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle 2y, 2x - 6y \rangle$$

$$\Rightarrow \nabla f \big|_{P_0} = \langle 10, -20 \rangle$$

$$(D_u f)_{P_0} = \langle 10, -20 \rangle \cdot \langle \frac{4}{5}, \frac{3}{5} \rangle = 8 - 12 = -4$$

(17) Find the directional derivative of $f(x,y,z) = xe^{2yz} + z^2$ at $P(3,0,-2)$ in the direction of $u = 2\vec{i} - 2\vec{j} - \vec{k}$.

Soln: $(D_u f)_{P_0} = \nabla f \big|_{P_0} \cdot \frac{\vec{u}}{|\vec{u}|}$

$$\frac{\vec{u}}{|\vec{u}|} = \frac{2\vec{i} - 2\vec{j} - \vec{k}}{3}$$

$$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz} + 2z \rangle$$

$$\nabla f \big|_P = \langle e^0, 2.3.(-2).e^0, -4 \rangle = \langle 1, -12, -4 \rangle$$

$$(D_u f)\big|_P = \langle 1, -12, -4 \rangle \cdot \langle \frac{2}{3}, -\frac{2}{3}, -\frac{1}{3} \rangle = \frac{2}{3} + \frac{24}{3} + \frac{4}{3} = 10.$$

(18) In which directions is the derivative of $f(x,y) = xy + y^2$ at $P(3,2)$ equal to zero?

Soln: $(D_u f)|_P = \nabla f \cdot \vec{u} = |\nabla f| \cos\theta$

$D_u f = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \theta = \pi/2$ which means $\nabla f \perp u$.

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = y\vec{i} + (x+2y)\vec{j}$$

$$\nabla f|_P = 2\vec{i} + 7\vec{j}$$

$$(2i + 7j) \cdot (ai + bj) = 0 \Rightarrow 2a + 7b = 0 \Rightarrow a = -\frac{7b}{2}$$

Since u is a unit $\Rightarrow a^2 + b^2 = 1 \Rightarrow \frac{49}{4}b^2 + b^2 = 1 \Rightarrow b = \mp \frac{2}{\sqrt{53}}$

Thus $\vec{u} = \frac{7\vec{i}}{\sqrt{53}} - \frac{2\vec{j}}{\sqrt{53}}$ and $\vec{u} = -\frac{7\vec{i}}{\sqrt{53}} + \frac{2\vec{j}}{\sqrt{53}}$

(19) Find the tangent plane and normal line of $\cos \pi x - x^2 y + e^{xz} + yz = 4$ at $P(0,1,2)$.

Soln: $F(x,y,z) = \cos(\pi x) - x^2 y + e^{xz} + yz - 4 = 0$

$$\nabla F = (-\pi \sin(\pi x) - 2xy + z e^{xz})\vec{i} + (-x^2 + z)\vec{j} + (x \cdot e^{xz} + y)\vec{k}$$

$$\nabla F|_{(0,1,2)} = \underbrace{2}_{A}\vec{i} + \underbrace{2}_{B}\vec{j} + \underbrace{\vec{k}}_{C=1}$$ (normal to the tangent plane parallel to the normal line)

Tangent plane: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

$$2(x - x_0) + 2(y-1) + (z-2) = 0 \Rightarrow 2x + 2y + z = 4$$

Normal line: $\begin{cases} x = 0 + 2t \\ y = 1 + 2t \\ z = 2 + t \end{cases}$

(20) Find and classify the critical points of $f(x,y) = 4xy - x^4 - y^4$

Soln: i) Domain of f : $\mathbb{R}^2$

ii) $f_x = 4y - 4x^3 = 0 \Rightarrow y = x^3$
$f_y = 4x - 4y^3 = 0 \Rightarrow x = y^3$

$$\left.\begin{array}{l} f_x = 4y - 4x^3 = 0 \Rightarrow y = x^3 \\ f_y = 4x - 4y^3 = 0 \Rightarrow x = y^3 \end{array}\right\} \quad \begin{array}{l} x = x^9 \Rightarrow x(x^8 - 1) = 0 \\ \Rightarrow x(x^4+1).(x^4-1) = 0 \\ \Rightarrow (x^4+1)(x^2+1).(x+1).(x-1) = 0 \end{array}$$

$\Rightarrow x = 0$ or $x = 1$ or $x = -1$

$A(0,0)$, $B(1,1)$, $C(-1,-1)$ are critical points.

iii) Now check,

$$\left.\begin{array}{l} f_{xx} = -12x^2 \\ f_{yy} = -12y^2 \\ f_{zz} = 4 \end{array}\right\} \quad \begin{array}{l} \Delta = f_{xx}.f_{yy} - (f_{xy})^2 \\ = 144x^2y^2 - 16 \end{array}$$

at $A(0,0)$:

$\Delta = -16 < 0 \rightarrow A(0,0)$ is a saddle point.

at $B(1,1)$

$\Delta = 144 - 16 > 0$ and $f_{xx}(1,1) = -12 < 0 \rightarrow B(1,1)$ is a local max.

at $C(-1,-1)$

$\Delta = 144 - 16 > 0$ and $f_{xx}(-1,-1) = -12 < 0 \rightarrow C(-1,-1)$ is a local max.

(21) Find and classify the critical points of $f(x,y) = 3xy \cdot e^{-(x^2+y^2)}$.

Soln: i) Domain of f : $\mathbb{R}^2$

ii) $f_x = 3y\, e^{-(x^2+y^2)} + 3xy(-2x)\, e^{-(x^2+y^2)} = 0$

$\Rightarrow e^{-(x^2+y^2)}(3y - 6x^2y) = 0 \Rightarrow y(1-2x^2) = 0$ (*)

$f_y = 3x\, e^{-(x^2+y^2)} + 3xy(-2y)\, e^{-(x^2+y^2)} = 0$

$\Rightarrow x(1-2y^2) = 0$ (**)

By (*) and (**) $\quad y = 0 \Rightarrow x = 0$

$1 - 2x^2 = 0 \Rightarrow x = \mp \frac{1}{\sqrt{2}} \Rightarrow y = \mp \frac{1}{\sqrt{2}}$.

So points;

$A_1(0,0)$, $A_2\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $A_3\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $A_4\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $A_5\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$f_{xx} = 3y(-2x)\, e^{-(x^2+y^2)} - 12xy\, e^{-(x^2+y^2)} - 6x^2y(-2x)\, e^{-(x^2+y^2)}$

$= (-18xy + 12x^3y)\, e^{-(x^2+y^2)}$

$f_{yy} = (-18xy + 12xy^3)\, e^{-(x^2+y^2)}$

$f_{xy} = (3 - 6y^2 - 6x^2 + 12x^2y^2)\, e^{-(x^2+y^2)}$

* $A_1(0,0)$, $\Delta = f_{xx} f_{yy} - f_{xy}^2 = 0 - (-3)^2 < 0 \Rightarrow$ saddle point

* A_2, $\Delta = \frac{36}{e^2} > 0$ and $f_{xx} < 0 \Rightarrow$ local max

* A_3, $\Delta = \frac{36}{e^2} > 0$ and $f_{xx} > 0 \Rightarrow$ local min

* A_4, $\Delta = \frac{36}{e^2} > 0$ and $f_{xx} > 0 \Rightarrow$ local min

* A_5, $\Delta = \frac{36}{e^2} > 0$, and $f_{xx} < 0 \Rightarrow$ local max.

(22) Find the critical points of $f(x,y)=\sqrt{1+x^2y^2}$ and extreme values at that points.
(exam)

Soln: i) f is defined for all (x,y) $\rightarrow$ Domain of $f: \mathbb{R}^2$

ii) $$\left.\begin{aligned} f_x(x,y) &= \frac{xy^2}{\sqrt{1+x^2y^2}} = 0 \\ f_y(x,y) &= \frac{x^2y}{\sqrt{1+x^2y^2}} = 0 \end{aligned}\right\} \quad x=0 \text{ or } y=0.$$

Thus, $\forall x,y \in \mathbb{R}$ the points $(x,0)$ and $(0,y)$ are critical points.
(because of $f_x(x,0)=f_y(x,0)=0$ and $f_x(0,y)=f_y(0,y)=0$)

iii) $$f_{xx} = \frac{y^2}{(1+x^2y^2)^{3/2}}, \quad f_{yy} = \frac{x^2}{(1+x^2y^2)^{3/2}}, \quad f_{xy} = \frac{xy(2+x^2y^2)}{(1+x^2y^2)^{3/2}}$$

For $(x,0) \Rightarrow \Delta = f_{xx}(x,0)\cdot f_{yy}(x,0) - f_{xy}^2(x,0) = 0$.
So, second derivative test is inconclusive.

For $(0,y) \Rightarrow \Delta = 0$. So, the second derivative test is inconclusive.

But for all (x,y), $\sqrt{1+x^2y^2} \geq 1$ and $f(x,0)=f(0,y)=1$.

So $f(x,y) \geq f(x,0)$
$f(x,y) \geq f(0,y)$

$\Rightarrow$ The points $(x,0)$ and $(0,y)$ are local minimum points.

So f has local minimum value 1 at these critical points.