D.1. $2yy'' = 3yy' + (y')^2 \quad (y \geq 0)$ x missing (x yok)

$$\left.\begin{aligned} y' &= p \\ y'' &= p\frac{dp}{dy}\end{aligned}\right\} \quad 2yp\frac{dp}{dy} = 3yp + p^2 \Rightarrow p\left[2y\frac{dp}{dy} - 3y - p\right] = 0$$

1°) $p = 0 \Rightarrow y = c$ (not solution)

2°) $2y\dfrac{dp}{dy} - 3y - p = 0$ $2y\dfrac{dp}{dy} - p = 3y$

$$\Rightarrow \frac{dp}{dy} - \frac{p}{2y} = \frac{3}{2} \quad \text{Lineer D.}$$

$$\lambda(y)=e^{\int -\frac{dy}{2y}}=e^{-\frac{1}{2}\ln y}=\frac{1}{\sqrt{y}}$$

$$p=\sqrt{y}\left[\int \frac{1}{\sqrt{y}}\cdot\frac{3}{2}\,dy+c_1\right]=\sqrt{y}\left[3\sqrt{y}+c_1\right]=3y+c_1\sqrt{y}$$

$$p=\frac{dy}{dx}=3y+c_1\sqrt{y}\Rightarrow\int\frac{dy}{3y+c_1\sqrt{y}}=\int dx$$

$$\left.\begin{array}{l} y=t^2 \\ dy=2t\,dt \end{array}\right\}\int\frac{2t\,dt}{3t^2+c_1t}=\int\frac{2\,dt}{3t+c_1}=\frac{2}{3}\ln(3t+c_1)=\frac{2}{3}\ln(3\sqrt{y}+c_1)$$

$$\boxed{\frac{2}{3}\ln(3\sqrt{y}+c_1)=x+c_2}$$

2. $x^2y''+xy'+4y=25x\ln x$ (Euler)

$$\left.\begin{array}{l} x=e^t,\ (t=\ln x) \\ y'=e^{-t}Dy \\ y''=e^{-2t}D(D-1)y \end{array}\right\}\begin{array}{l} \underbrace{e^{2t}\cdot e^{-2t}}_{1}D(D-1)y+\underbrace{e^{t}\cdot e^{-t}}_{1}Dy+4y=25e^t\cdot t \\ \left[D(D-1)+D+4\right]y=25te^t \end{array}$$

$$(D^2+4)y=25te^t$$

$$r^2+4=0\Rightarrow r_{1,2}=\mp 2i\rightarrow y_h=c_1\cos 2t+c_2\sin 2t$$

$$\left.\begin{array}{l} y_p=(at+b)e^t \\ y_p'=ae^t+(at+b)e^t \\ y_p''=ae^t+ae^t+(at+b)e^t \end{array}\right\}\begin{array}{l} (2a+(at+b)+4(at+b))\cancel{e^t}=25t\,\cancel{e^t} \\ 5at+5b+2a=25t \\ 5a=25 \qquad 5b+2a=0 \\ a=5 \qquad b=-2 \end{array}$$

$$y_p=(5t-2)e^t$$

$$y=c_1\cos 2(\ln x)+c_2\sin 2(\ln x)+(5\ln x-2)x$$

3. $2yy'' = (y')^2 + 1$ (x missing)

$$\left.\begin{array}{l} y' = p \\ y'' = p\dfrac{dp}{dy} \end{array}\right\} \quad 2yp\frac{dp}{dy} = p^2 + 1 \Rightarrow 2yp\,dp = (p^2+1)dy$$

$$\frac{2p\,dp}{p^2+1} = \frac{dy}{y} \Rightarrow \ln(p^2+1) = \ln y + \ln c_1$$

$$p^2 + 1 = c_1 y$$

$$p = \sqrt{c_1 y - 1} \Rightarrow \frac{dy}{dx} = \sqrt{c_1 y - 1} \Rightarrow \int \frac{dy}{\sqrt{c_1 y - 1}} = \int dx$$

$$\left.\begin{array}{l} c_1 y - 1 = t^2 \\ c_1\,dy = 2t\,dt \end{array}\right\} \quad \int \frac{2t\,dt}{c_1 t} = \frac{2}{c_1} t = \frac{2}{c_1}\sqrt{c_1 y - 1}$$

$$\frac{2}{c_1}\sqrt{c_1 y - 1} = x + c_2$$

4. $y'' + 4y = \tan 2x$

$$r^2 + 4 = 0 \Rightarrow r_{1,2} = \mp 2i \Rightarrow y_h = c_1 \cos 2x + c_2 \sin 2x$$

$$\begin{array}{l} c_1' \cos 2x + c_2' \sin 2x = 0 \quad / 2\sin 2x \\ -2c_1' \sin 2x + 2c_2' \cos 2x = \tan 2x \quad / \cos 2x \end{array}$$

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$$c_2'\underbrace{(2\sin^2 2x + 2\cos^2 2x)}_{2} = \sin 2x \Rightarrow c_2' = \frac{1}{2}\sin 2x \Rightarrow c_2 = \frac{-1}{4}\cos 2x + k_2$$

$$c_1' = -c_2'\frac{\sin 2x}{\cos 2x} = -\frac{1}{2}\sin 2x \cdot \frac{\sin 2x}{\cos 2x} \Rightarrow c_1 = -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$c_1 = -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx = -\frac{1}{2}\left[\int \sec 2x\,dx - \int \cos 2x\,dx\right]$$

$$c_1 = -\frac{1}{2}\left[\boxed{\tfrac{1}{2}}\ln|\sec 2x + \tan 2x| - \frac{1}{2}\sin 2x\right] + k_1$$

$$y = -\frac{1}{4}\ln|\sec 2x + \tan 2x| \cdot \cos 2x + \cancel{\frac{1}{4}\sin 2x \cdot \cos 2x} + k_1 \cos 2x - \cancel{\frac{1}{4}\sin 2x \cos 2x}$$
$$+ k_2 \sin 2x$$

5. $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^2}$

$r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow r_1 = r_2 = -2 \Rightarrow y_h = c_1 e^{-2x} + c_2 x e^{-2x}$

$$2 / \quad \cancel{c_1' e^{-2x}} + c_2' x e^{-2x} = 0$$

$$-2\cancel{c_1' e^{-2x}} + c_2' e^{-2x} - 2c_2' x e^{-2x} = \frac{e^{-2x}}{x^2}$$

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$$c_2' e^{-2x} = \frac{e^{-2x}}{x^2} \Rightarrow c_2' = \frac{1}{x^2} \Rightarrow c_2 = -\frac{1}{x} + k_2$$

$$c_1' = -c_2' x = -\frac{1}{x^2} \cdot x = \frac{-1}{x} \Rightarrow c_1 = -\ln x + k_1$$

$$y = -\ln x \cdot e^{-2x} + k_1 e^{-2x} - e^{-2x} + k_2 x e^{-2x}$$

6. $x y'' - 2y' = \dfrac{1}{x^2}$ (Euler or y missing)

$$\left.\begin{aligned} y' &= p \\ y'' &= \frac{dp}{dx} \end{aligned}\right\} \quad x \frac{dp}{dx} - 2p = \frac{1}{x^2} \Rightarrow \frac{dp}{dx} - \frac{2p}{x} = \frac{1}{x^3} \quad \text{Linear}$$

$$\lambda(x) = e^{\int -\frac{2dx}{x}} = e^{-2\ln x} = \frac{1}{x^2}$$

$$p = x^2 \left[\int \frac{1}{x^2} \cdot \frac{1}{x^3} dx + c_1 \right] = x^2 \left[\frac{1}{4x^4} + c_1 \right] = \frac{1}{4x^2} + \frac{c_1}{x^2}$$

$$p = \frac{dy}{dx} = \frac{1}{4x^2} + \frac{c_1}{x^2} \Rightarrow \int dy = \int \left(\frac{1}{4x^2} + \frac{c_1}{x^2} \right) dx + c_2$$

$$y = \frac{-1}{4x} - \frac{c_1}{x} + c_2$$

7. $x^2y'' + xy' + 4y = 4\cos(2\ln x)$

$$\left.\begin{aligned} x &= e^t \\ y' &= e^{-t}Dy \\ y'' &= e^{-2t}D(D-1)y \end{aligned}\right\} \quad \begin{aligned} &\underbrace{e^{2t}\cdot e^{-2t}}_{1}D(D-1)y + \underbrace{e^{t}\cdot e^{-t}}_{1}Dy + 4y = 4\cos 2t \\ &[D(D-1)+D+4]y = 4\cos 2t \end{aligned}$$

$$(D^2+4)y = 4\cos 2t$$

$$r^2+4=0 \Rightarrow r_{1,2} = \mp 2i \Rightarrow y_h = c_1\cos 2t + c_2\sin 2t$$

$2\sin 2t$ / $c_1'\cos 2t + c_2'\sin 2t = 0$

$\cos 2t$ / $-2c_1'\sin 2t + 2c_2'\cos 2t = 4\cos 2t$

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$$c_2'\underbrace{(2\sin^2 2t + 2\cos^2 2t)}_{2} = 4\cos^2 2t \Rightarrow c_2' = 2\cos^2 2t$$

$$c_2' = (1+\cos 4t) \Rightarrow c_2 = t + \frac{1}{4}\sin 4t + k_2$$

$$c_1' = -c_2'\frac{\sin 2t}{\cos 2t} = -2\cdot\cos^{\cancel{2}} 2t\cdot\frac{\sin 2t}{\cancel{\cos 2t}} = -\sin 4t$$

$$c_1 = \frac{1}{4}\cos 4t + k_1$$

$$y = \underbrace{\frac{1}{4}\cos 4t\cdot\cos 2t} + k_1\cos 2t + t\sin 2t + \underbrace{\frac{1}{4}\sin 4t\sin 2t} + k_2\sin 2t$$

$$y = \frac{1}{4}\cos 2t + t\sin 2t + k_1\cos 2t + k_2\sin 2t$$

$$y = \frac{1}{4}\cos 2(\ln x) + \ln x\sin 2(\ln x) + k_1\cos 2(\ln x) + k_2\sin 2(\ln x)$$

8. $2yy'' - (y')^2 + 9 = 0$ (x missing)

$$\left.\begin{aligned} y' &= p \\ y'' &= p\frac{dp}{dy} \end{aligned}\right\} \quad \begin{aligned} &2yp\frac{dp}{dy} - p^2 + 9 = 0 \Rightarrow 2yp\frac{dp}{dy} = p^2 - 9 \\ &\frac{2p\,dp}{p^2-9} = \frac{dy}{y} \Rightarrow \ln(p^2-9) = \ln y + \ln c_1 \end{aligned}$$

$$p^2 - 9 = c_1 y \Rightarrow p = \sqrt{c_1 y + 9}$$

$$\frac{dy}{dx} = \sqrt{c_1 y + 9} \Rightarrow \int \frac{dy}{\sqrt{c_1 y + 9}} = \int dx$$

$$\left.\begin{aligned} c_1 y + 9 &= t^2 \\ c_1 dy &= 2t\,dt \end{aligned}\right\} \int \frac{2t\,dt}{c_1 \cdot t} = \frac{2}{c_1} t = \frac{2}{c_1}\sqrt{c_1 y + 9}$$

$$\frac{2}{c_1}\sqrt{c_1 y + 9} = x + c_2$$

9. $y^{IV} + y''' = \sin 2t$

$$r^4 + r^3 = 0 \Rightarrow r^3(r+1) = 0 \rightarrow r_1 = r_2 = r_3 = 0,\ r_4 = -1$$

$$y_h = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t}$$

$$\left.\begin{aligned} y_p &= A\cos 2t + B\sin 2t \\ y_p' &= -2A\sin 2t + 2B\cos 2t \\ y_p'' &= -4A\cos 2t - 4B\sin 2t \\ y_p''' &= 8A\sin 2t - 8B\cos 2t \\ y_p^{IV} &= 16A\cos 2t + 16B\sin 2t \end{aligned}\right\}$$

$$8A\sin 2t - 8B\cos 2t + 16A\cos 2t + 16B\sin 2t = \sin 2t$$

$$\left.\begin{aligned} 8A + 16B &= 1 \\ 2/\ \ 16A - 8B &= 0 \\ + \quad\quad & \\ 40A &= 1 \\ A = \frac{1}{40}&,\ B = \frac{1}{20} \end{aligned}\right\} y_p = \frac{1}{40}\cos 2t + \frac{1}{20}\sin 2t$$

$$y = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t} + \frac{1}{40}\cos 2t + \frac{1}{20}\sin 2t$$

10. $y''y^3 = 1$ (x missing)

$$\left.\begin{aligned} y' &= p \\ y'' &= p\frac{dp}{dy} \end{aligned}\right\} \quad p\frac{dp}{dy} \cdot y^3 = 1 \Rightarrow \int p\,dp = \int \frac{dy}{y^3}$$

$$\frac{p^2}{2} = -\frac{1}{2y^2} + c_1$$

$$p = \sqrt{2c_1 - \frac{1}{y^2}} = \frac{\sqrt{2c_1y^2 - 1}}{y}$$

$$p = \frac{dy}{dx} = \frac{\sqrt{2c_1y^2-1}}{y} \Rightarrow \int \frac{y\,dy}{\sqrt{2c_1y^2-1}} = \int dx \qquad \begin{aligned} 2c_1y^2 - 1 &= t^2 \\ 4c_1y\,dy &= 2t\,dt \end{aligned}$$

$$\int \frac{2t\,dt}{4c_1 \cdot t} = \frac{1}{2c_1}t = \frac{1}{2c_1}\sqrt{2c_1y^2-1}$$

$$\frac{1}{2c_1}\sqrt{2c_1y^2-1} = x + c_2$$

11. $y'' = -2x(y')^2$, $y(0) = 2$, $y'(0) = -1$ (y missing)

$$\left.\begin{aligned} y' &= p \\ y'' &= \frac{dp}{dx} \end{aligned}\right\} \quad \frac{dp}{dx} = -2xp^2 \Rightarrow \int \frac{dp}{p^2} = -\int 2x\,dx$$

$$-\frac{1}{p} = -x^2 + c_1 \qquad (y'(0) = -1) \Rightarrow \frac{-1}{(-1)} = -0 + c_1 \Rightarrow \boxed{c_1 = 1}$$

$$-\frac{1}{p} = -x^2 + 1 \Rightarrow p = \frac{1}{x^2-1}$$

$$\frac{dy}{dx} = \frac{1}{x^2-1} \Rightarrow \int dy = \underbrace{\int \frac{dx}{x^2-1}}_{I} + c_2$$

$$I \Rightarrow \frac{A}{x-1} + \frac{B}{x+1} = \frac{1}{x^2-1}$$

$$\left.\begin{aligned} A + B &= 0 \\ A - B &= 1 \end{aligned}\right\} \quad A = \frac{1}{2},\ B = -\frac{1}{2}$$

$$y = \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x+1) + c_2$$

12. $x^2 y'' + 2xy' - 1 = 0$ (Euler or y missing)

$$\left.\begin{aligned} y' &= p \\ y'' &= \frac{dp}{dx} \end{aligned}\right\} \quad x^2 \frac{dp}{dx} + 2xp = 1 \Rightarrow \frac{dp}{dx} + \frac{2p}{x} = \frac{1}{x^2} \quad \text{Linear}$$

$$\lambda(x) = e^{\int 2\frac{dx}{x}} = e^{2\ln x} = x^2$$

$$p = \frac{1}{x^2}\left[\int x^2 \cdot \frac{1}{x^2}\,dx + c_1\right] = \frac{1}{x^2}(x + c_1) = \frac{1}{x} + \frac{c_1}{x^2}$$

$$\int dy = \int \left(\frac{1}{x} + \frac{c_1}{x^2}\right) dx$$

$$y = \ln x - \frac{c_1}{x} + c_2$$

E. $yy'' + 4y^2 - \frac{1}{2}(y')^2 = 0$, $x = 0 \Rightarrow y = 1$, $y' = -\sqrt{8}$ (x missing)

$$\left.\begin{aligned} y' &= p \\ y'' &= p\frac{dp}{dy} \end{aligned}\right\} \quad yp\frac{dp}{dy} + 4y^2 - \frac{p^2}{2} = 0 \Rightarrow \frac{dp}{dy} - \frac{p}{2y} = -4yp^{-1} \quad \text{Bernoulli}$$

$$\left.\begin{aligned} p^{1-(-1)} &= p^2 = u \\ 2pp' &= u' \end{aligned}\right\} \quad \begin{aligned} & p'p - \frac{p^2}{2y} = -4y \\ & \frac{u'}{2} - \frac{u}{2y} = -4y \Rightarrow u' - \frac{u}{y} = -8y \quad \text{linear} \end{aligned}$$

$$\lambda(y) = e^{-\int \frac{1}{y} dy} = e^{-\ln y} = \frac{1}{y}$$

$$u = y\left[\int \frac{1}{y}(-8y)\,dy + c_1\right] = y(-8y + c_1)$$

$$p = \sqrt{u} = \sqrt{y(-8y + c_1)} \Rightarrow -\sqrt{8} = \sqrt{-8 + c_1} \Rightarrow 8 = -8 + c_1 \Rightarrow c_1 = 16$$

$$dy = \sqrt{16y - 8y^2}\,dx \Rightarrow \int \frac{dy}{\sqrt{16y - 8y^2}} = \int dx$$

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