Leibnitz Kuralı:

İnt. denk. çözmek için kullanılacak tekniklerden biri

int. denklemin dif. denkleme dönüştürülmesidir.

Dönüşüm integrallerinin türevinin alınması için iyi bilinen Leibnitz kuralı kullanılır.

$a \le x \le b$, $t_0 \le t \le t_1$ dikdörtgen bölgede

$f(x,t)$ ve $\dfrac{\partial f}{\partial t}$ sürekli fonk. olsun.

$$\boxed{F(x) = \int_{g(x)}^{h(x)} f(x,t)\,dt}$$ olmak üzere

$$F'(x) = \frac{dF}{dx} = \int_{g(x)}^{h(x)} \frac{\partial f(x,t)}{\partial x}\,dt + f(x,h(x))\cdot\frac{dh}{dx} - f(x,g(x))\frac{dg}{dx}$$

İntegral işareti altında türev

Örnek $$F(x) = \int_{\sin x}^{\cos x} \sqrt{1+t^3}\,dt$$

$$F'(x) = \underbrace{\int_{\sin x}^{\cos x} \frac{\partial}{\partial x}\left(\sqrt{1+t^3}\right)dt}_{0} + \sqrt{1+\cos^3 x}\,(-\sin x) - \sqrt{1+\sin^3 x}\cdot(\cos x)$$

$$F'(x) = -\sin x\sqrt{1+\cos^3 x} - \cos x\sqrt{1+\sin^3 x}$$

Örnek $$F(x) = \int_{x}^{x^2} (x-t)\cos t\,dt$$

$$F'(x) = \int_{x}^{x^2} \frac{\partial}{\partial x}\underbrace{\left[(x-t)\cos t\right]}_{x\cos t - t\cos t}dt + (x-x^2)\cos x^2(2x) - \cancel{\underbrace{(x-x)}_{0}\cos x \cdot 1}$$

$$F'(x) = \int_x^{x^2} \underbrace{\cos t\,dt}_{\sin t\big|_x^{x^2}} + (x - x^2)\, 2x \cos x^2$$

$$= \sin x^2 - \sin x + (2x^2 - 2x^3)\cos x^2$$

Örnek $F(x) = \int_0^1 e^{xt}\,dt \to F'(x) = \int_0^1 t e^{xt}\,dt$

$$F(x) = \int_0^x \underbrace{K(x,t)\,u(t)}_{f(x,t)}\,dt \quad \text{Homojen Volterra int. denk}$$

$$F'(x) = K(x,x)\,u(x) + \int_0^x \frac{\partial}{\partial x}\left[K(x,t)\,u(t)\right] dt$$

Örnek $F(x) = \int_0^x x t\, u(t)\,dt \Rightarrow F'(x),\ F''(x)$

$$F'(x) = x^2 . u(x) + \int_0^x t\,u(t)\,dt$$

$$F''(x) = 2x\,u(x) + x^2 u'(x) + x u(x) = 3x u(x) + x^2 u'(x)$$

Çoklu İntegrallerin tekli integrale indirgenmesi

Başlangıç değer problemlerinin ve buna benzer diğer problemlerin int. denklere dönüştürülebileceği gösterilecektir. Bunun için çok katlı int. tek katlı integrale indirgenebilir. Şimdi 2 katlı integralin tek int. integrale dönüşümünü gösterelim:

$$\int_0^x \int_0^{x_1} F(t)\,dt\,dx_1 = \int_0^x (x-t)\,F(t)\,dt$$

$$G(x)=\int_0^x \underbrace{(x-t).F(t)}\,dt \rightarrow G(0)=0$$

$$G'(x)=\int_0^x F(t)\,dt \rightarrow$$ Her iki tarafını 0 dan x'e integre edelim.

$$\int_0^x \underbrace{G'(x)\,dx} = \int_0^x \left(\int_0^x F(t)\,dt\right)dx$$

$$G(x)\Big|_0^x$$

$$G(x) = \int_0^x \int_0^x F(t)\,dt\,dx$$

$$\int_0^x (x-t)F(t)\,dt = \int_0^x \int_0^{x_1} F(t)\,dt\,dx_1$$

$$\int_0^x \Big(\underbrace{\int_0^{x_1} F(t)\,dt}_{u}\Big)\underbrace{dx}_{dv} = uv - \int v\,du$$

Genel olarak ;

$$\int_0^x \int_0^{x_1} \cdots \int_0^{x_{n-1}} u(x_n)\,dx_n\,dx_{n-1}\cdots dx_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}u(t)\,dt$$

veya

$$\underbrace{\int_0^x \int_0^x \cdots}_{n \text{ katlı}} \int_0^x (x-t)u(t)\,dt\cdots dt = \frac{1}{n!}\int_0^x (x-t)^n u(t)\,dt$$

örnek

$$\int_0^x \int_0^{x_1} u(x_2)\,dx_1\,dx = \frac{1}{(2-1)!}\int_0^x (x-t)^{2-1}u(t)\,dt$$

$$\int_0^x \int_0^x u(t)\,dt\,dt = \int_0^x (x-t)u(t)\,dt$$

Örnek

$$\int_0^x\int_0^{x_1}\int_0^{x_2} u(x_3)\,dx_2\,dx_1\,dx = \frac{1}{2!}\int_0^x (x-t)^2 u(t)\,dt$$

Örnek

$$\int_0^x\int_0^x\int_0^x (x-t)u(t)\,dt\,dt\,dt = \frac{1}{3!}\int_0^x (x-t)^3 u(t)\,dt$$

Örnek

$$\int_0^x\int_0^{x_1} (x-t)u(x_1)\,dt\,dx_1 = \frac{1}{2!}\int_0^x (x-t)^2 u(t)\,dt$$

İntegral denklemlerin kaynakları (Kökenleri)

Volterra int. denklemleri öngörülen başlangıç değerleriyle başlangıç değer problemlerinin (BDP) dönüştürülmesiyle elde edilebilir. Fredholm int. denklemleri ise verilen sınır koşullarıyla sınır değer problemlerinden (SDP) bulunabilir.

Yaygın Kullanım	Nadiren
Volterra int. denk $\Longleftrightarrow$ BDP	Fredholm int $\Longleftrightarrow$ SDP

BDP nin Volterra int. denklemine Dönüşümü:

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x) \;; \qquad \begin{array}{l} y(0)=\alpha \\ y'(0)=\beta \end{array} \quad (\alpha,\beta\in\mathbb{R})$$

$$\boxed{y''(x) = u(x)} \qquad u(x) \to \text{sürekli fonk.}$$

$$\int_0^x y''(x)\,dx = \int_0^x u(x)\,dx$$

$$y'(x) - \underbrace{y'(0)}_{\beta} = \int_0^x u(t)\,dt \longrightarrow \boxed{y'(x) = \beta + \int_0^x u(t)\,dt}$$

$$\int_0^x y'(x)dx = \int_0^x \left[\beta + \int_0^x u(t)dt\right]dx = \int_0^x \beta dx + \underbrace{\int_0^x \int_0^x u(t)dt\,dx}$$

$$y(x) - \underbrace{y(0)}_{\alpha} = \beta x + \int_0^x (x-t)u(t)dt$$

$$\boxed{y(x) = \alpha + \beta x + \int_0^x (x-t)u(t)dt}$$

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x)$$

$$u(x) + p(x)\left[\beta + \int_0^x u(t)dt\right] + q(x)\left[\alpha + \beta x + \int_0^x (x-t)u(t)dt\right] = g(x)$$

$$u(x) + \beta p(x) + q(x)(\alpha + \beta x) + \int_0^x \left[p(x) + q(x)(x-t)\right]u(t)dt = g(x)$$

$$u(x) = \underbrace{g(x) - \beta p(x) - q(x)(\alpha + \beta x)}_{f(x)} - \int_0^x \underbrace{\left[p(x) + q(x)(x-t)\right]}_{K(x,t)}u(t)dt$$

$$* \quad \left.\begin{aligned} & y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \dots + a_n(x)y(x) = g(x) \\ & y(0) = C_0;\ y'(0) = C_1, \dots, y^{(n-1)}(0) = C_{n-1} \end{aligned}\right\} \qquad \boxed{y^{(n)}(x) = u(x)}$$

Örnek

$$y'(x) - 2xy(x) = e^{x^2} \quad ; \quad y(0) = 1 \qquad \text{BDP}$$

$$\boxed{y'(x) = u(x)}$$

$$\int_0^x y'(x)dx = \int_0^x u(x)dx \longrightarrow y(x)\Big|_0^x = \int_0^x u(t)dt$$

$$y(x) - \underbrace{y(0)}_{1} = \int_0^x u(t)\,dt$$

$$\boxed{y(x) = 1 + \int_0^x u(t)dt}$$

$$u(x) - 2x\left[1+\int_0^x u(t)dt\right] = e^{x^2}$$

$$u(x) - 2x - \int_0^x 2x\,u(t)dt = e^{x^2}$$

$$u(x) = \underbrace{e^{x^2} + 2x}_{} + 2\int_0^x \underbrace{x}_{} u(t)dt$$ Volterra int. denk.

soru $y'''(x) - y''(x) - y'(x) + y(x) = 0$ $\left.\begin{array}{l} y(0)=1 \\ y'(0)=2 \\ y''(0)=3 \end{array}\right\}$ BDP

Cevap: $u(x) = 4 + x - \frac{3}{2}x^2 + \int_0^x \left[1+(x-t)-\frac{1}{2}(x-t)^2\right]u(t)dt$

Volterra int. denkleminin BDP ye dönüşümü:

Leibnitz kuralı kullanarak dif. denklem oluşturulur.

Örnek $u(x) = e^x + \int_0^x u(t)\,dt$ Volterra int. denklemini BDP ye dönüştürelim:

$$u'(x) = e^x + u(x) \Rightarrow u'(x) - u(x) = e^x$$

$u(0) = 1$ → Bulunacak değer!

Örnek

$$u(x) = x^2 + \int_0^x \underbrace{(x-t)u(t)}_{}\,dt$$

$$u'(x) = 2x + \int_0^x \underbrace{u(t)}_{}\,dt$$

$$u''(x) = 2 + u(x) \xrightarrow{BDP} \begin{cases} u''(x) - u(x) = 2 \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$$

SORU $$u(x) = \sin x - \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$$

Cevap: $u'''(x) + u(x) = -\cos x$

$u(0) = 0$

$u'(0) = 1$

$u''(0) = 0$

SDP nin Fredholm int. denklemine dönüşümü:

$$y''(x) + g(x)y'(x) = h(x) \qquad 0<x<1 \qquad \begin{array}{l} y(0)=\alpha \\ y(1)=\beta \end{array} \qquad \alpha,\beta \in \mathbb{R}$$

$\boxed{y''(x) = u(x)}$

$$y'(x) - y'(0) = \int_0^x u(t)\,dt \longrightarrow \boxed{y'(x) = y'(0) + \int_0^x u(t)\,dt}$$

$$y(x) - \underbrace{y(0)}_{\alpha} = x\,y'(0) + \int_0^x\int_0^x u(t)\,dt\,dx$$

$$y(x) = \alpha + x\,y'(0) + \int_0^x (x-t)u(t)\,dt$$

$x=1$ alırsak
$$\underbrace{y(1)}_{\beta} = \alpha + y'(0) + \int_0^1 (1-t)u(t)\,dt$$

$$y'(0) = \beta - \alpha - \int_0^1 (1-t)u(t)\,dt \quad *$$

$$\boxed{y(x) = \alpha + (\beta-\alpha)x - \int_0^1 x(1-t)u(t)\,dt + \int_0^x (x-t)u(t)\,dt} \ *$$

$$y''(x) + g(x)y(x) = h(x)$$

$$u(x) + \alpha g(x) + (\beta-\alpha)xg(x) - \int_0^1 xg(x)(1-t)u(t)\,dt + \int_0^x g(x)(x-t)u(t)\,dt$$

$$= h(x)$$

$$u(x) = h(x) - \alpha g(x) - (\beta-\alpha)xg(x) - \underbrace{\int_0^x g(x)(x-t)u(t)\,dt}_{} + xg(x)\underbrace{\int_0^1 (1-t)u(t)\,dt}_{}$$

$$\int_0^1 (1-t)u(t)\,dt = \underbrace{\int_0^x (1-t)u(t)\,dt + \int_x^1 (1-t)u(t)\,dt}_{}$$

$$u(x) = \text{------} + \int_0^x \underbrace{[xg(x)(1-t) - g(x)(x-t)]}_{\substack{\cancel{xg(x)} - xtg(x) - \cancel{xg(x)} + tg(x) \\ t(1-x)g(x)}} u(t)\,dt + \int_x^1 x(1-t)g(x)u(t)\,dt$$

$$u(x) = \underbrace{h(x) - \alpha g(x) - (\beta-\alpha)xg(x)}_{f(x)} + \underbrace{\int_0^x t(1-x)g(x)u(t)\,dt + \int_x^1 x(1-t)g(x)u(t)\,dt}_{\int_0^1 K(x,t)u(t)\,dt}$$

$$K(x,t) = \begin{cases} t(1-x)g(x); & 0 \le t \le x \\ x(1-t)g(x); & x \le t \le 1 \end{cases}$$

$$u(x) = f(x) + \int_0^1 K(x,t)u(t)\,dt$$

Örnek $y''(x) + 9y(x) = \cos x \qquad y(0) = y(1) = 0$

$y''(x) = u(x)$

$y'(x) - y'(0) = \int_0^x u(t)\,dt \rightarrow \quad y'(x) = y'(0) + \int_0^x u(t)\,dt$

$$y(x) - \underbrace{y(0)}_{0} = x\,y'(0) + \int_0^x \int_0^x u(t)\,dt$$

$$y(x) = x\,y'(0) + \int_0^x (x-t)\,u(t)\,dt$$

$x=1 \quad \underbrace{y(1)}_{0} = y'(0) + \int_0^1 (1-t)\,u(t)\,dt \rightarrow \quad y'(0) = -\int_0^1 (1-t)\,u(t)\,dt$

$$y(x) = -x\int_0^1 (1-t)\,u(t)\,dt + \int_0^x (x-t)\,u(t)\,dt$$

$$y''(x) + 9\,y(x) = \cos x$$

$$u(x) - 9x\int_0^1 (1-t)\,u(t)\,dt + 9\int_0^x (x-t)\,u(t)\,dt = \cos x$$

$$u(x) = \cos x + 9x\underbrace{\int_0^1 (1-t)\,u(t)\,dt}_{\int_0^x (1-t)u(t)dt + \int_x^1 (1-t)u(t)dt} - 9\int_0^x (x-t)\,u(t)\,dt$$

$$u(x) = \cos x + 9x\int_0^x (1-t)\,u(t)\,dt + 9x\int_x^1 (1-t)\,u(t)\,dt - 9\int_0^x (x-t)\,u(t)\,dt$$

$$\int_0^x \left[9x(1-t) - 9(x-t)\right]u(t)\,dt$$

$$\cancel{9x} - 9xt - \cancel{9x} + 9t$$

$$9t(1-x)$$

$$u(x) = \cos x + \int_0^x 9t\underbrace{(1-x)}u(t)\,dt + \int_x^1 9x\underbrace{(1-t)}u(t)\,dt$$

$$u(x) = \cos x + 9\int_0^1 K(x,t)u(t)\,dt$$

$$K(x,t) = \begin{cases} t(1-x) & ; 0 \le t \le x \\ x(1-t) & ; x \le t \le 1 \end{cases}$$

Soru

$$y''(x) + xy(x) = 0 \qquad \begin{array}{l} y(0) = 0 \\ y(1) = 2 \end{array}$$

$$u(x) = -2x^2 + \int_0^1 K(x,t)u(t)\,dt$$

$$K(x,t) = \begin{cases} tx(1-x); & 0 \le t \le x \\ x^2(1-t) & ; x \le t \le 1 \end{cases}$$

Fredholm int. denk SDP'ye dönüştürme

$$u(x) = f(x) + \int_0^1 K(x,t)u(t)\,dt$$

$$K(x,t) = \begin{cases} t(1-x)g(x); & 0 \le t \le x \\ x(1-t)g(x); & x \le t \le 1 \end{cases}$$

Kolaylık için $g(x) = \lambda$ sabit olsun

$$* \; u(x) = f(x) + \int_0^x \lambda\underbrace{(1-x)}t\,u(t)\,dt + \int_x^1 \lambda x(1-t)\,u(t)\,dt$$

$$u'(x) = f'(x) + \lambda\cancel{(1-x)x\,u(x)} - \int_0^x \lambda t\,u(t)\,dt - \cancel{\lambda x(1-x)u(x)} + \int_x^1 \lambda(1-t)u(t)\,dt$$

$$u'(x) = f'(x) - \lambda\int_0^x t\,u(t)\,dt + \lambda\int_x^1 (1-t)\,u(t)\,dt$$

$$u''(x) = f''(x) - \lambda\cancel{x\,u(x)} - \lambda(1-x)\,u(x)$$

$$-\lambda u(x) + \lambda\cancel{x u(x)}$$

$$u''(x) = f''(x) - \lambda u(x)$$

$$u''(x) + \lambda u(x) = f''(x)$$

$x=0 \quad u(0) = f(0) = \alpha \qquad x=1 \quad u(1) = f(1) = \beta$

Soru $\quad u(x) = e^x + \int_0^1 K(x,t)u(t)\,dt$

$$K(x,t) = \begin{cases} 9t(1-x) & ,\ 0 \le t \le x \\ 9x(1-t) & ,\ x \le t \le 1 \end{cases}$$

cevap: $\quad u'' + 9u = e^x \qquad u(0) = 1$

$u(1) = e$