Laplace Dönüşümüne bir bakış:

$x \geq 0$ için $f(x)$'in Laplace dön.

$$L\{f(x)\} = \int_0^\infty e^{-sx} f(x)\,dx = F(s)$$

Temel Laplace Dönüşümleri:

$f(x)$		$F(s)$
$c \in \mathbb{R}$	....	$\frac{c}{s}$
x	....	$\frac{1}{s^2}$
$(n \in \mathbb{Z}^+)$ x^n	....	$\frac{n!}{s^{n+1}}$ $(s>0)$
$(a \in \mathbb{R})$ e^{ax}	....	$\frac{1}{s-a}$ $(s>a)$
$\sin(ax)$	....	$\frac{a}{s^2+a^2}$
$\cos(ax)$	....	$\frac{s}{s^2+a^2}$
$\sinh(ax)$	....	$\frac{a}{s^2-a^2}$
$\cosh(ax)$	....	$\frac{s}{s^2-a^2}$

Özellikleri.

1) Lineerlik özl: $a, b \in \mathbb{R}$

$$L\{af(x) \mp bg(x)\} = aL\{f(x)\} \mp bL\{g(x)\}$$

Örnek

$$L\{4x + 3x^2 - 5e^{-2x}\} = 4L\{x\} + 3L\{x^2\} - 8L\{e^{-2x}\}$$

$$= 4\,\frac{1}{s^2} + 3\,\frac{2!}{s^3} - 8 \cdot \frac{1}{s+2}$$

2) Kaydırma özl:

$$L\{f(x)\} = F(s) \Rightarrow L\{e^{ax}f(x)\} = F(s-a)$$

örnek

$$L\{x^2 e^{3x}\} = ?$$

$$L\{x^2 e^{3x}\} = \frac{2}{(s-3)^3}$$

$$L\{x^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$$

3) x^n ile çarpma özl:

$$L\{f(x)\} = F(s) \Rightarrow L\{x^n f(x)\} = (-1)^n \frac{d^n F(s)}{ds^n}$$

örnek

$$L\{x^2 e^{3x}\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right) = \frac{2}{(s-3)^3}$$

$$L\{e^{3x}\} = \frac{1}{s-3}$$

Ters Laplace Dönüşümleri

$x \leftrightarrow s$

$$L\{f(x)\} = F(s) \Leftrightarrow f(x) = L^{-1}\{F(s)\}$$

$a \in \mathbb{R}$

$$L^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$L^{-1}\left\{\frac{1}{s^2}\right\} = x$$

$$L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{x^2}{2!}$$

$$L^{-1}\left\{\frac{1}{s^4}\right\} = \frac{x^3}{3!}$$

$\vdots$

$$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{x^n}{n!}$$

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{ax}$$

$$L^{-1}\left\{\frac{1}{(s-a)^2}\right\} = e^{ax}x$$

$$L^{-1}\left\{\frac{1}{(s-a)^3}\right\} = e^{ax}\frac{x^2}{2!}$$

$\vdots$

$$L^{-1}\left\{\frac{1}{(s-a)^{n+1}}\right\} = e^{ax}\cdot\frac{x^n}{n!}$$

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin ax}{a}$$

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos ax$$

Lineerlik özl:

$a, b \in \mathbb{R}$

$$L^{-1}\{aF(s) \mp bG(s)\} = a L^{-1}\{F(s)\} \mp b L^{-1}\{G(s)\}$$

Kaydırma özl:

$$L^{-1}\{F(s)\} = f(x)$$

$$\Rightarrow L^{-1}\{F(s-a)\} = e^{ax}\cdot f(x)$$

Örnek: $L^{-1}\left\{\frac{1}{s^{10}}\right\} = \frac{x^9}{9!}$

Örnek: $L^{-1}\left\{\frac{1}{s^2+16}\right\} = \frac{\sin 4x}{4}$

$L^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos 4x$

Örnek: $L^{-1}\left\{\frac{1}{\underbrace{s^2+2s+5}_{1+4}}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = e^{-x}\cdot\frac{\sin 2x}{2}$

Örnek: $L^{-1}\left\{\frac{2s+1}{s^2+s}\right\} = L^{-1}\left\{\frac{2s+1}{s(s+1)}\right\} = L^{-1}\left\{\frac{1}{s}+\frac{1}{s+1}\right\} = 1+e^{-x}$

$$\frac{2s+1}{\underbrace{s(s+1)}} = \frac{A}{\underset{(s+1)}{s}} + \frac{B}{\underset{s}{s+1}}$$

$2s+1 = As+A+Bs$

$2s+1 = (A+B)s+A$

$A+B=2$

$A=1 \quad B=1$

$$\frac{2s+1}{s(s+1)} = \frac{1}{s} + \frac{1}{s+1}$$

Örnek: $L^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\} = 4L^{-1}\left\{\frac{s+3^{+1-1}}{(s+4)^2}\right\} = 4L^{-1}\left\{\frac{\cancel{s+4}}{(s+4)^{\cancel{2}}} - \frac{1}{(s+4)^2}\right\}$

$= 4L^{-1}\left\{\frac{1}{s+4} - \frac{1}{(s+4)^2}\right\} = 4\left(e^{-4x} - xe^{-4x}\right)$

Konvolüsyon Teoremi: $L\{\underbrace{f_1(x)}\} = F_1(s)$ ve $L\{\underbrace{f_2(x)}\} = F_2(s)$ olsun.

Bu iki fonk. konvolüsyon çarpımı

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)\,f_2(t)\,dt$$

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)\,f_1(t)\,dt$$

$(f_1*f_2)(x) = (f_2*f_1)(x)$

$$L\{(f_1 * f_2)(x)\} = L\left\{\int_0^x f_1(x-t)\cdot f_2(t)\,dt\right\} = F_1(s)F_2(s)$$

Örnek

$$x + \int_0^x (x-t)\,y(t)\,dt$$ ifadesinin Laplace dön.?

$$L\left\{x+\int_0^x (x-t)\,y(t)\right\} = L\{x\} + L\left\{\int_0^x (x-t)\,y(t)\,dt\right\}$$

$$= L\{x\} + L\{x\}\cdot L\{y(x)\}$$

$$= \frac{1}{s^2} + \frac{1}{s^2}Y(s)$$

Örnek

$$x e^x - \int_0^x e^{x-t}\,y(t)\,dt$$

$f_1(x) = e^x \to F_1(s) = ?$

$f_2(x) = y(x) \to F_2(s) = ?$

$$L\left\{x e^x - \int_0^x e^{x-t}\,y(t)\right\} = L\{x e^x\} - L\left\{\int_0^x e^{x-t}\,y(t)\,dt\right\}$$

$$= L\{x e^x\} - L\{e^x\}\,L\{y(x)\}$$

$$= \frac{1}{(s-1)^2} - \frac{1}{s-1}Y(s)$$

Laplace Dönüşüm Yöntemi:

$$u(x) = f(x) + \lambda\int_0^x K(x,t)\,u(t)\,dt$$

int. denkleminin çekirdeği $x-t$ farkına bağlıysa buna "fark çekirdeği" denir.

$$e^{x-t},\ \sin(x-t),\ (x-t)^3,\ \dots$$

$$u(x) = f(x) + \lambda\int_0^x K(x-t)\,u(t)\,dt$$

$$L\{u(x)\} = L\{f(x)\} + \lambda L\Big\{\int_0^x \underbrace{K(x-t)}_{K(x)}\,\underbrace{u(t)}_{u(x)}\,dt\Big\}$$

$$L\{u(x)\} = U(s) \Leftrightarrow L^{-1}\{U(s)\} = u(x)$$

$$L\{f(x)\} = F(s) \qquad L\{K(x)\} = Q(s)$$

$$U(s) = F(s) + \lambda Q(s) U(s)$$

$$U(s) = \frac{F(s)}{1-\lambda Q(s)}$$

$$L^{-1}\{U(s)\} = L^{-1}\Big\{\frac{F(s)}{1-\lambda Q(s)}\Big\} = u(x)$$

$$(\lambda Q(s) \neq 1)$$

Örnek

$u(x) = 1 + \int_0^x u(t)\,dt$ Laplace dön. çözünüz.

$$K(x-t) = 1 \longrightarrow f_1(x) = 1 \qquad f_2(x) = u(x)$$

$$L\{u(x)\} = L\{1\} + L\{1 * u(x)\}$$

$$U(s) = \frac{1}{s} + \frac{1}{s} U(s)$$

$$\left(1 - \frac{1}{s}\right) U(s) = \frac{1}{s}$$

$$\frac{s-1}{\cancel{s}} U(s) = \frac{1}{\cancel{s}} \Rightarrow U(s) = \frac{1}{s-1}$$

$$u(x) = L^{-1}\{U(s)\} = L^{-1}\Big\{\frac{1}{s-1}\Big\}$$

$$u(x) = e^x$$

örnek $u(x) = 1 - \int_0^x (x-t)u(t)dt$ Laplace?

$K(x-t) = x-t \longrightarrow \begin{array}{l} f_1(x)=x \\ f_2(x) = u(x) \end{array}$

$$L\{u(x)\} = L\{1\} - L\{x * u(x)\}$$

$$U(s) = \frac{1}{s} - \frac{1}{s^2}U(s)$$

$$\left(1+\frac{1}{s^2}\right)U(s) = \frac{1}{s} \rightarrow \left(\frac{s^2+1}{\cancel{s^2}}\right)U(s) = \frac{1}{\cancel{s}}$$

$$U(s) = \frac{s}{s^2+1}$$

$$u(x) = L^{-1}\{U(s)\} = L^{-1}\left\{\frac{s}{s^2+1}\right\} \rightarrow u(x) = \cos x$$

örnek $u(x) = \frac{1}{3!}x^3 - \int_0^x \underbrace{(x-t)}\, u(t)dt$

$$L\{u(x)\} = \frac{1}{3!}L\{x^3\} - L\{x * u(x)\}$$

$$U(s) = \frac{1}{\cancel{3!}}\frac{\cancel{3!}}{s^4} - \frac{1}{s^2}U(s)$$

$$\left(1+\frac{1}{s^2}\right)U(s) = \frac{1}{s^4} \Rightarrow \left(\frac{s^2+1}{\cancel{s^2}}\right)U(s) = \frac{1}{s^{\cancel{4}\,2}}$$

$$U(s) = \frac{1}{s^2(s^2+1)}$$

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} \Rightarrow \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$u(x) = L^{-1}\{U(s)\} = L^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} \Rightarrow u(x) = x - \sin x$$

SORULAR

1) $u(x) = \sin x + \cos x + 2\int_0^x \sin(x-t)u(t)\,dt$ cevap: $u(x)=e^x$

2) $u(x) = 1 + \int_0^x \sin(x-t)\,u(t)\,dt$ cevap: $u(x) = x + \frac{x^2}{2}$

Seri Çözümü Yöntemi

Bir $u(x)$ reel fonk. tanım aralığı içerisindeki herhangi bir b noktası için $|x-b|<R$ olacak şekilde bir Taylor serisine açılıyorsa, fonk. analitik olarak adlandırılır.

$$u(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!}(x-b)^n \; ; \quad |x-b|<R$$

$$u(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \; ; \quad |x|<R$$

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \; ; \quad |x|<R$$

$$u(x) = \underbrace{f(x)} + \lambda\int_0^x K(x,t)u(t)\,dt$$ analitik fonk. olsun.

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} a_n x^n = T(f(x)) + \lambda\int_0^x K(x,t)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt \qquad a_i \in \mathbb{R}\ ?$$

$$a_0 + a_1 x + a_2 x^2 + \cdots = T(f(x)) + \lambda\int_0^x K(x,t)\left(a_0 + a_1 t + a_2 t^2 + \cdots\right)dt$$

$\equiv$

$a_i\ ?$

Örnek $u(x) = 1 + \int_0^x u(t)dt$ seri çözümü yöntemi ile?

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\int_0^x a_0 dt + \int_0^x a_1 t dt + \int_0^x a_2 t^2 dt$$

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

$$a_0 + a_1 x + a_2 x^2 + \dots = 1 + \int_0^x (a_0 + a_1 t + a_2 t^2 + \dots) dt$$

$$= 1 + a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + \dots$$

$$\left.\begin{aligned} a_0 &= 1 \\ a_1 &= a_0 = 1 \\ a_2 &= \frac{a_1}{2} = \frac{1}{2} \\ a_3 &= \frac{a_2}{3} = \frac{1}{3\cdot 2} \\ a_4 &= \frac{a_3}{4} = \frac{1}{4\cdot3\cdot2} \\ &\vdots \end{aligned}\right\}$$

$$a_n = \frac{1}{n!} \quad (n \geq 0)$$

$$u(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$u(x) = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots$$

$$u(x) = e^x \quad (\forall x \in \mathbb{R})$$

Örnek

$$u(x) = 1 - x(\sin x) + \int_0^x t u(t) dt$$

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} a_n x^n = 1 - x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) + \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

$$\underbrace{a_0}+a_1x+a_2x^2+a_3x^3+\cdots = 1-x^2+\frac{x^4}{3!}-\frac{x^6}{5!}+\cdots+\int_0^x t\,(a_0+a_1t+a_2t^2+a_3t^3+\cdots)dt$$

$a_0t+a_1t^2+a_2t^3$

$$=\underbrace{1}-x^2+\frac{x^4}{3!}-\frac{x^6}{5!}+\cdots+a_0\frac{x^2}{2}+a_1\frac{x^3}{3}+a_2\frac{x^4}{4}+\cdots$$

$$=1+\left(\frac{a_0}{2}-1\right)x^2+\frac{a_1}{3}x^3+\left(\frac{a_2}{4}+\frac{1}{6}\right)x^4+\cdots$$

$a_0=1$

$a_1=0$

$a_2=\frac{a_0}{2}-1=\frac{-1}{2!}$

$a_3=\frac{a_1}{3}=0$

$a_4=\frac{a_2}{4}+\frac{1}{6}=\frac{-1}{8}+\frac{1}{6}=\frac{1}{24}=\frac{1}{4!}$

$a_5=0$

$a_6=\frac{-1}{6!}$?

$$u(x)=\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+a_3x^3+\cdots$$

$$u(x)=\underbrace{1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots}_{\cos x}$$

$$u(x)=\cos x \qquad (\forall x\in\mathbb{R})$$

SORULAR

1) $u(x)=2e^x-2-x+\int_0^x(x-t)\,u(t)\,dt$ Cevap $u(x)=xe^x$

2) $u(x)=x+\int_0^x(x-t)\,u(t)\,dt$ Cevap: $u(x)=\sinh x=\frac{e^x-e^{-x}}{2}$

$$\int(x-t)(a_0+a_1t+a_2t^2+\cdots)=x\int_0^x(a_0+a_1t+a_2t^2+\cdots)dt-\int_0^x(a_0t+\cancel{a_1}t^2+a_2t^3+\cdots)dt$$

$$u(x) = \underbrace{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}} + \cdots = \sinh x$$

$$? = \cosh x$$