* $|\lambda| < \dfrac{1}{M(b-a)}$ ✓

$$|\lambda| < \left(\int_a^b\int_a^b [K(x,t)]^2\,dx\,dt\right)^{-1/2}$$ ✓

Resolventin Tekliği:

$u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)\,dt$ int. denkleminin resolventi $\Gamma(x,t;\lambda)$ tek türlü belirlenir.

İspat: $\Gamma_1(x,t;\lambda)$ ve $\Gamma_2(x,t;\lambda)$ gibi farklı iki resolvent kabul edelim. O halde

$$u(x) = f(x) + \lambda\int_a^b \Gamma_1(x,t;\lambda)f(t)\,dt \quad \text{ve}$$

$$u(x) = f(x) + \lambda\int_a^b \Gamma_2(x,t;\lambda)f(t)\,dt$$

yazılabilir.

Bu iki denklemi taraf tarafa çıkardım:

$$\cancel{\lambda}\int_a^b \underbrace{[\Gamma_1(x,t;\lambda) - \Gamma_2(x,t;\lambda)]}_{=0} f(t)\,dt = 0$$

$f(t) \neq 0$

$\Gamma_1(x,t;\lambda) - \Gamma_2(x,t;\lambda) = 0$ olması ile bu integral sıfıra eşit çıkacaktır.

Buradan

$$\Gamma_1(x,t;\lambda) = \Gamma_2(x,t;\lambda) \text{ olur ki}$$

ispatlanması gereken budur.

Teorem: Eğer $M(x,t)$ ve $N(x,t)$ ortogonal iki fonk. ise ve

$$K(x,t) = M(x,t) + N(x,t)$$

olduğu takdirde

$K(x,t)$ ye ait resolvent : $\Gamma(x,t;\lambda)$
$M(x,t)$ 〃 〃 〃 : $\Gamma_M(x,t;\lambda)$
$N(x,t)$ 〃 〃 〃 : $\Gamma_N(x,t;\lambda)$

olacak şekilde

$$\Gamma(x,t;\lambda) = \Gamma_M(x,t;\lambda) + \Gamma_N(x,t;\lambda)$$

bağıntısı vardır.

$$\int_a^b M(x,s)\,N(s,t)\,ds = 0$$

ortogonallik şartı

İspat: $$u(x) = f(x) + \lambda \int_a^b K(x,t)\,u(t)\,dt$$

$\Gamma(x,t;\lambda)$ * $$u(x) = f(x) + \lambda \int_a^b \Gamma(x,t;\lambda) f(t)\,dt$$

$\Gamma(x,t;\lambda)$ serisi Neumann serisi olarak bilinir ve λnın belli değerleri için yakınsaktır.

$$K(x,t) = M(x,t) + N(x,t)$$

$$u(x) = f(x) + \lambda \int_a^b [M(x,t) + N(x,t)]\, u(t)\, dt$$

$$= \underbrace{f(x) + \lambda \int_a^b M(x,t) u(t) dt}\; + \underbrace{\lambda \int_a^b N(x,t) u(t) dt}$$

$$u(x) + f(x) = \underbrace{f(x) + \lambda \int_a^b M(x,t) u(t) dt}_{1} + \underbrace{f(x) + \lambda \int_a^b N(x,t) u(t) dt}_{2}$$

$$u(x) + \cancel{f(x)} = f(x) + \lambda \int_a^b \Gamma_M(x,t;\lambda) f(t) dt + \cancel{f(x)} + \lambda \int_a^b \Gamma_N(x,t;\lambda) f(t) dt$$

* $u(x) = f(x) + \lambda \int_a^b \underbrace{\left[\Gamma_M(x,t;\lambda) + \Gamma_N(x,t;\lambda)\right]} f(t)\, dt$ olur.

$\Gamma(x,t;\lambda) = \Gamma_M(x,t;\lambda) + \Gamma_N(x,t;\lambda)$ yazılabilir.

$\Gamma(x,t;\lambda)$ nın yakınsak olması

$$|\lambda| < \left[\int_a^b \int_a^b [K(x,t)]^2\, dx\, dt\right]^{-\frac{1}{2}}$$

eşitsizliğindeki λ değerleri ile belirlenir.

$$\int_a^b\int_a^b K^2(x,t)\,dx\,dt = \int_a^b\int_a^b \left[M(x,t)+N(x,t)\right]^2 dx\,dt$$

$$= \int_a^b\int_a^b M^2(x,t)\,dx\,dt + \int_a^b\int_a^b N^2(x,t)\,dx\,dt + 2\int_a^b\int_a^b M(x,t)N(x,t)\,dx\,dt$$

$M(x,t)$ ve $N(x,t)$ ortogonaldır. O halde

$$\int_a^b\int_a^b M(x,s)N(s,t)\,ds = 0 \quad \text{olacağından}$$

$$\int_a^b\int_a^b K^2(x,t)\,dx\,dt = \int_a^b\int_a^b M^2(x,t)\,dx\,dt + \int_a^b\int_a^b N^2(x,t)\,dx\,dt$$

$|\lambda|$

Örnek

$$K(x,t) = \underbrace{xt}_{M(x,t)} + \underbrace{x^2t^2}_{N(x,t)} \qquad \begin{matrix} a=-1 \\ b=1 \end{matrix}$$

$$\int_{-1}^{1} M(x,s)N(s,t)\,ds = \int_{-1}^{1}(xs)(s^2t^2)\,ds$$

$$= \int_{-1}^{1} xt^2s^3\,ds = xt^2\left(\frac{s^4}{4}\right)\Big|_{-1}^{1} = 0$$

$$\left.\begin{matrix} M(x,t) = xt \\ N(x,t) = x^2t^2 \end{matrix}\right\} \text{ortogonaldir.}$$

$$\Gamma(x,t;\lambda) = \Gamma_M(x,t;\lambda) + \Gamma_N(x,t;\lambda)$$

$$M(x,t) = xt \qquad \Gamma_M(x,t;\lambda)$$

$$\Gamma_M(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} M_{(n)}(x,t)$$

$$= \underbrace{M_{(1)}(x,t)}_{xt} + \lambda \underbrace{M_{(2)}(x,t)}_{\frac{2}{3}xt} + \lambda^2 \underbrace{M_{(3)}(x,t)}_{\left(\frac{2}{3}\right)^2 xt} + \cdots$$

$$= xt\left[1 + \frac{2}{3}\lambda + \left(\frac{2}{3}\lambda\right)^2 + \left(\frac{2}{3}\lambda\right)^3 + \cdots\right]$$

$$= xt \underbrace{\sum_{n=1}^{\infty} \left(\frac{2}{3}\lambda\right)^{n-1}} \qquad \sum_{n=1}^{\infty} \left(\frac{2}{3}\lambda\right)^{n-1} = \frac{1}{1-\frac{2\lambda}{3}}$$

$$\Gamma_M(x,t;\lambda) = \frac{3xt}{3-2\lambda} \qquad \left|\frac{2}{3}\lambda\right| < 1 \rightarrow |\lambda| < \frac{3}{2}$$

$$N(x,t) = x^2 t^2$$

$$\Gamma_N(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} N_{(n)}(x,t)$$

$$= \underbrace{N_{(1)}(x,t)}_{x^2t^2} + \lambda \underbrace{N_{(2)}(x,t)}_{\frac{2}{5}x^2t^2} + \lambda^2 \underbrace{N_{(3)}(x,t)}_{\left(\frac{2}{5}\right)^2 x^2t^2} + \cdots$$

$$\Gamma_N(x,t;\lambda) = x^2t^2\left(1+\frac{2}{5}\lambda+\left(\frac{2}{5}\lambda\right)^2+\cdots\right)$$

$$= x^2t^2\sum_{n=1}^{\infty}\left(\frac{2}{5}\lambda\right)^{n-1}$$

$$\sum_{n=1}^{\infty}\left(\frac{2}{5}\lambda\right)^{n-1} = \frac{1}{1-\frac{2}{5}\lambda}$$

$$\left|\frac{2}{5}\lambda\right| < 1 \qquad = \frac{5}{5-2\lambda}$$

$$\boxed{|\lambda| < \frac{5}{2}}$$

$$\Gamma_N(x,t;\lambda) = \frac{5x^2t^2}{5-2\lambda}$$

$$\Gamma(x,t;\lambda) = \frac{3xt}{3-2\lambda} + \frac{5x^2t^2}{5-2\lambda}$$

$$\int_a^b\int_a^b K^2(x,t)\,dx\,dt = \underbrace{\int_a^b\int_a^b M^2(x,t)\,dx\,dt}_{} + \underbrace{\int_a^b\int_a^b N^2(x,t)\,dx\,dt}_{}$$

$$\int_{-1}^{1}\int_{-1}^{1} M^2(x,t)\,dx\,dt = \int_{-1}^{1}\int_{-1}^{1} x^2t^2\,dx\,dt = \frac{4}{9}$$

$$\int_{-1}^{1}\int_{-1}^{1} N^2(x,t)\,dx\,dt = \int_{-1}^{1}\int_{-1}^{1} x^4 t^4\,dx\,dt = \frac{4}{25}$$

$$\int_{-1}^{1}\int_{-1}^{1} K^2(x,t)\,dx\,dt = \frac{4}{9} + \frac{4}{25} = \frac{100+36}{9 \cdot 25}$$

$$|\lambda| < \left[\int_{-1}^{1}\int_{-1}^{1} K^2(x,t)\,dx\,dt\right]^{-1/2}$$

$$|\lambda| < \left(\frac{136}{9 \cdot 25}\right)^{-1/2} = \sqrt{\frac{9 \cdot 25}{136}} = \frac{3 \cdot 5}{2\sqrt{34}} < \frac{3 \cdot \cancel{5}}{2 \cdot \cancel{5}}$$