Fluid Mechanics: Fundamentals and Applications, 2nd Edition
Yunus A. Cengel, John M. Cimbala
McGraw-Hill, 2010

Chapter 6

MOMENTUM ANALYSIS OF FLOW SYSTEMS

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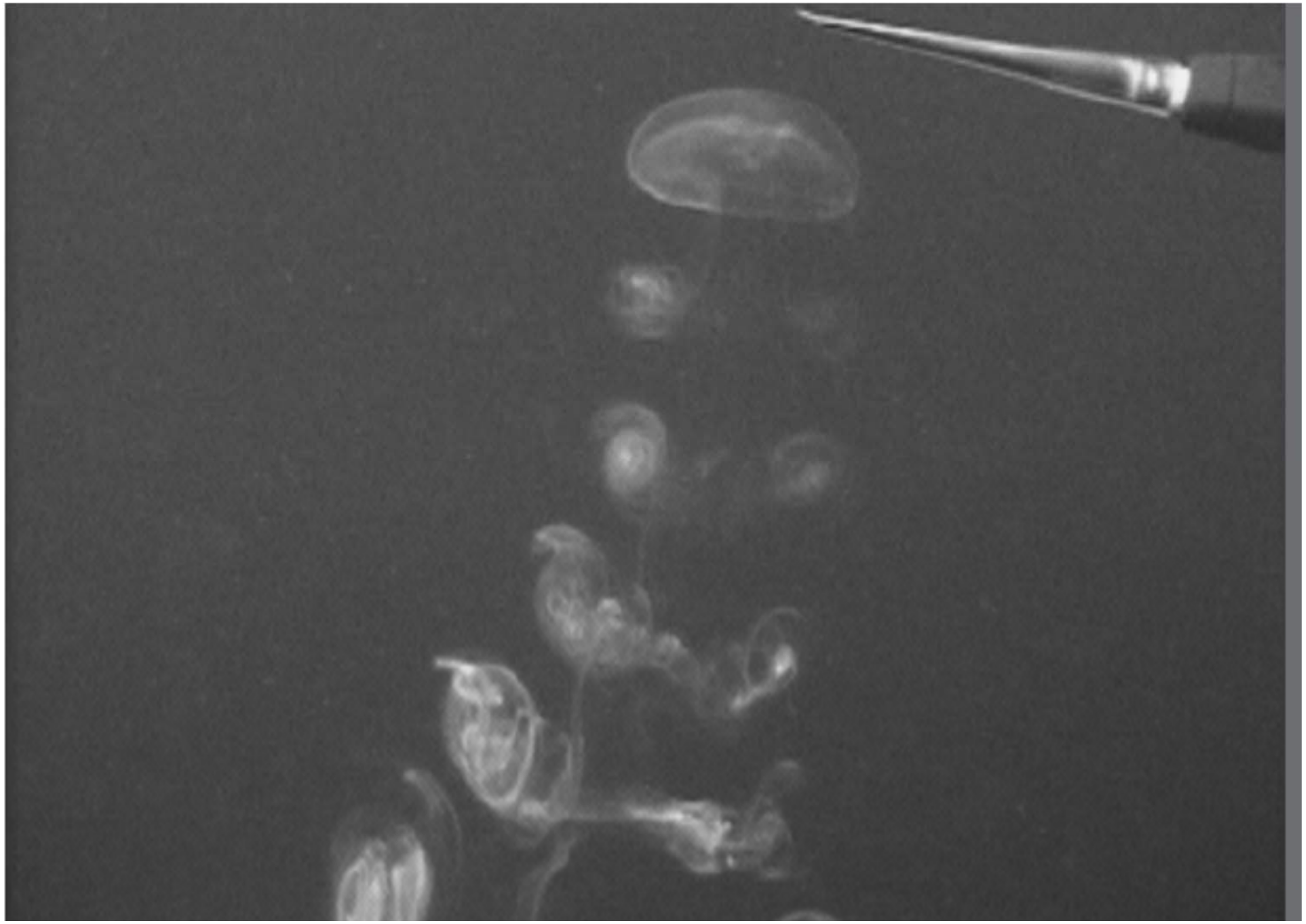
MOMENTUM ANALYSIS OF FLOW SYSTEMS



Steady swimming of the jellyfish Aurelia aurita. Fluorescent dye placed directly upstream of the animal is drawn underneath the bell as the body relaxes and forms vortex rings below the animal as the body contracts and ejects fluid. The vortex rings simultaneously induce flows for both feeding and propulsion.

Objectives

- Identify the various kinds of forces and moments acting on a control volume
- Use control volume analysis to determine the forces associated with fluid flow
- Use control volume analysis to determine the moments caused by fluid flow and the torque transmitted

6–1 NEWTON'S LAWS

Newton's laws: Relations between motions of bodies and the forces acting on them.

Newton's first law: A body at rest remains at rest, and a body in motion remains in motion at the same velocity in a straight path when the net force acting on it is zero.

Therefore, a body tends to preserve its state of inertia.

Newton's second law: The acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass.

Newton's third law: When a body exerts a force on a second body, the second body exerts an equal and opposite force on the first.

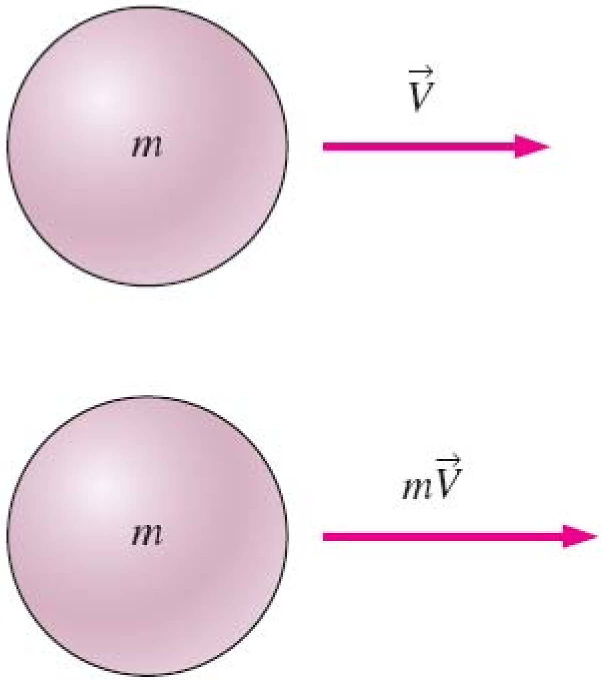
Therefore, the direction of an exposed reaction force depends on the body taken as the system.

Newton's second law: $$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d(m\vec{V})}{dt}$$

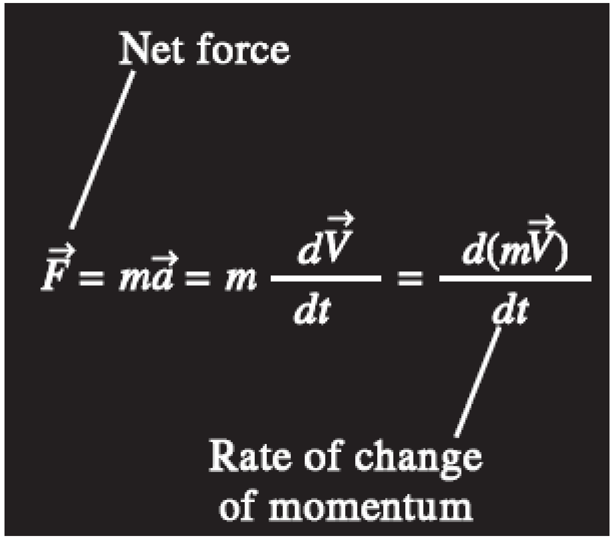
[redacted]

***Linear momentum* or just the *momentum* of the body:** The product of the mass and the velocity of a body.

Newton's second law is usually referred to as the ***linear momentum equation*.**



Conservation of momentum principle: The momentum of a system remains constant only when the net force acting on it is zero.



Linear momentum is the product of mass and velocity, and its direction is the direction of velocity.

Newton's second law is also expressed as *the rate of change of the momentum of a body is equal to the net force acting on it.*

Conservation of Linear Momentum

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The counterpart of Newton's second law for rotating rigid bodies is expressed as $\vec{M} = I\vec{\alpha}$, where $\vec{M}$ is the net moment or torque applied on the body, I is the moment of inertia of the body about the axis of rotation, and $\vec{\alpha}$ is the angular acceleration. It can also be expressed in terms of the rate of change of angular momentum $d\vec{H}/dt$ as

Angular momentum equation: $$\vec{M} = I\vec{\alpha} = I\frac{d\vec{\omega}}{dt} = \frac{d(I\vec{\omega})}{dt} = \frac{d\vec{H}}{dt}$$

Net torque

$$\vec{M} = I\vec{\alpha} = I\frac{d\vec{\omega}}{dt} = \frac{d(I\vec{\omega})}{dt} = \frac{d\vec{H}}{dt}$$

Rate of change of angular momentum

The rate of change of the angular momentum of a body is equal to the net torque acting on it.

The conservation of angular momentum Principle:

The total angular momentum of a rotating body remains constant when the net torque acting on it is zero, and thus the angular momentum of such systems is conserved.

Conservation of Angular Momentum

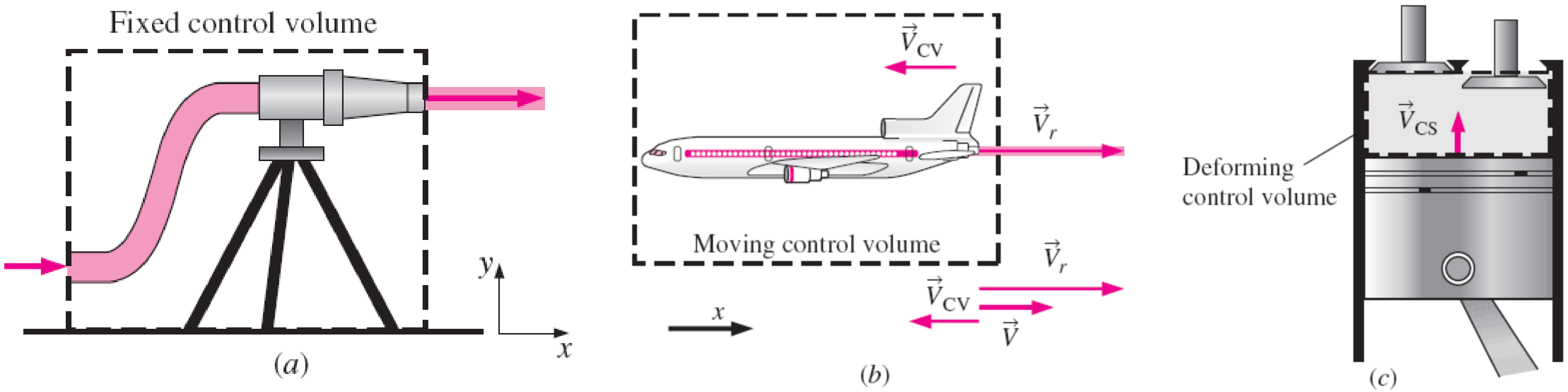
6–2 CHOOSING A CONTROL VOLUME

A control volume can be selected as any arbitrary region in space through which fluid flows, and its bounding control surface can be fixed, moving, and even deforming during flow.

Many flow systems involve stationary hardware firmly fixed to a stationary surface, and such systems are best analyzed using ***fixed* control volumes**.

When analyzing flow systems that are moving or deforming, it is usually more convenient to allow the control volume to ***move*** or ***deform***.

In ***deforming* control volume**, part of the control surface moves relative to other parts.



6–3 FORCES ACTING ON A CONTROL VOLUME

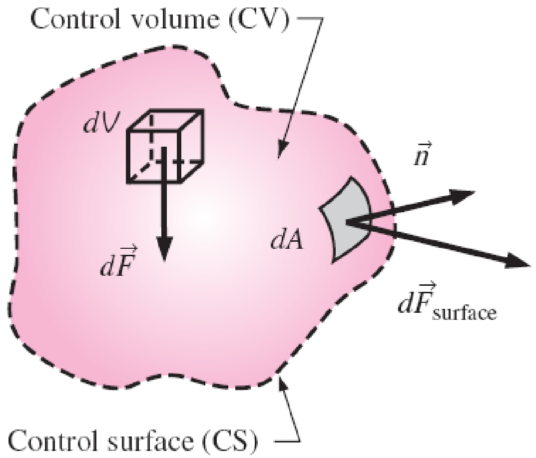
The forces acting on a control volume consist of

Body forces that act throughout the entire body of the control volume (such as gravity, electric, and magnetic forces) and

Surface forces that act on the control surface (such as pressure and viscous forces and reaction forces at points of contact).

Only external forces are considered in the analysis.

Total force acting on control volume: $$\sum \vec{F} = \sum \vec{F}_{\text{body}} + \sum \vec{F}_{\text{surface}}$$



The total force acting on a control volume is composed of body forces and surface forces; body force is shown on a differential volume element, and surface force is shown on a differential surface element.

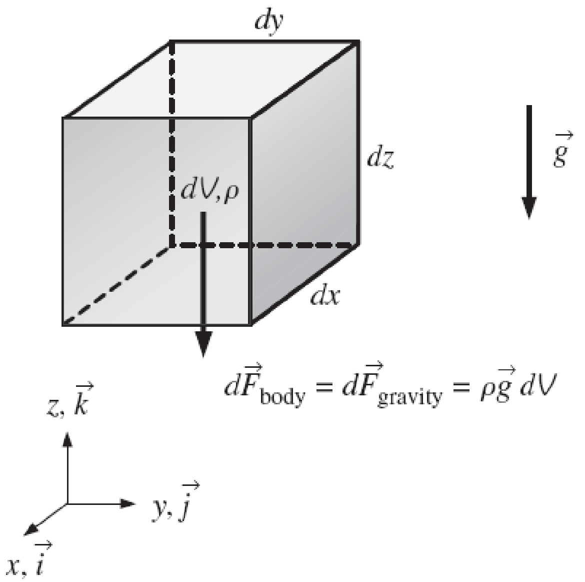
[redacted]

The most common body force is that of gravity, which exerts a downward force on every differential element of the control volume.

Gravitational force acting on a fluid element: $d\vec{F}_{\text{gravity}} = \rho\vec{g}\, dV$

Gravitational vector in Cartesian coordinates: $\vec{g} = -g\vec{k}$

Total body force acting on control volume: $\sum \vec{F}_{\text{body}} = \int_{CV} \rho\vec{g}\, dV = m_{CV}\vec{g}$



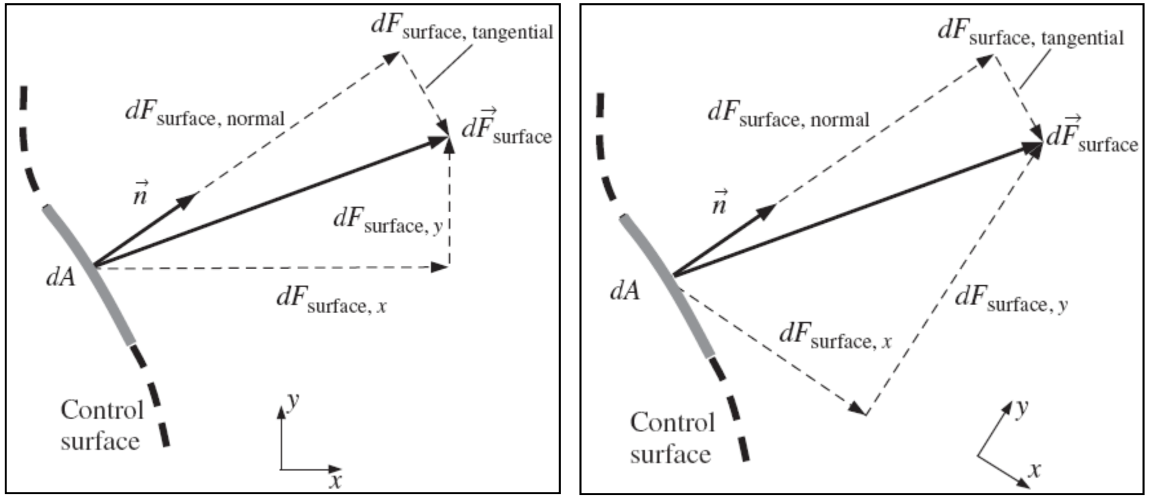
Stress tensor in Cartesian coordinates:

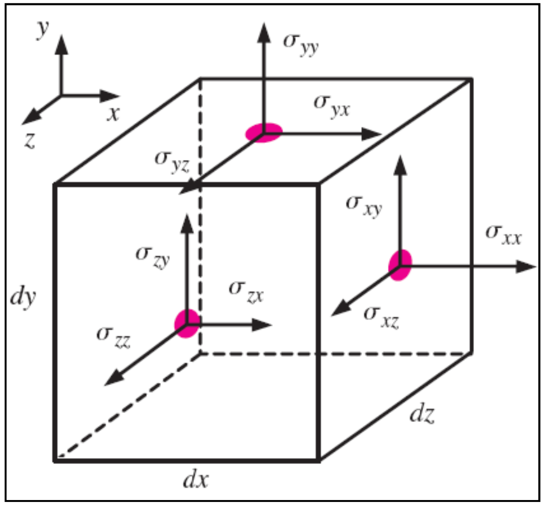
$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

Surface forces are not as simple to analyze since they consist of both *normal* and *tangential* components.
Normal stresses are composed of pressure (which always acts inwardly normal) and viscous stresses.
Shear stresses are composed entirely of viscous stresses.

The gravitational force acting on a differential volume element of fluid is equal to its weight; the axes have been rotated so that the gravity vector acts *downward* in the negative *z*-direction.

[redacted]





When coordinate axes are rotated (*a*) to (*b*), the components of the surface force change, even though the force itself remains the same; only two dimensions are shown here.

Surface force acting on a differential surface element:

$$d\vec{F}_{\text{surface}} = \sigma_{ij} \cdot \vec{n}\, dA$$

Total surface force acting on control surface:

$$\sum \vec{F}_{\text{surface}} = \int_{\text{CS}} \sigma_{ij} \cdot \vec{n}\, dA$$

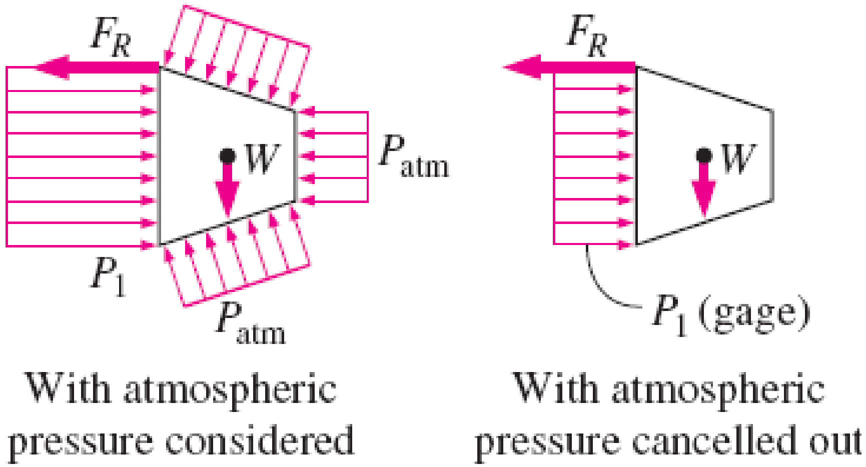
$$\boxed{\sum \vec{F} = \sum \vec{F}_{\text{body}} + \sum \vec{F}_{\text{surface}} = \int_{\text{CV}} \rho \vec{g}\, dV + \int_{\text{CS}} \sigma_{ij} \cdot \vec{n}\, dA}$$

$$\text{Total force:} \quad \underbrace{\sum \vec{F}}_{\text{total force}} = \underbrace{\sum \vec{F}_{\text{gravity}}}_{\text{body force}} + \underbrace{\sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}}}_{\text{surface forces}}$$

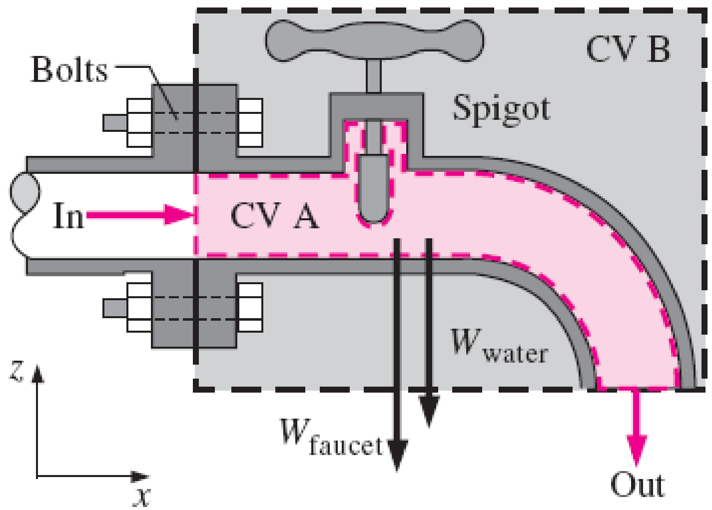
A common simplification in the application of Newton’s laws of motion is to subtract the *atmospheric pressure* and work with gage pressures.

This is because atmospheric pressure acts in all directions, and its effect cancels out in every direction.

This means we can also ignore the pressure forces at outlet sections where the fluid is discharged to the atmosphere since the discharge pressure in such cases is very near atmospheric pressure at subsonic velocities.



Atmospheric pressure acts in all directions, and thus it can be ignored when performing force balances since its effect cancels out in every direction.



Cross section through a faucet assembly, illustrating the importance of choosing a control volume wisely; CV B is much easier to work with than CV A.

$$\sum \vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}(m\vec{V})$$

$$\sum \vec{F} = \frac{d}{dt}\int_{\text{sys}} \rho\vec{V}\, dV$$

Newton's second law can be stated as

The sum of all external forces acting on a system is equal to the time rate of change of linear momentum of the system.

This statement is valid for a coordinate system that is at rest or moves with a constant velocity, called *an inertial coordinate system or inertial reference frame.*

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b\, dV + \int_{\text{CS}} \rho b(\vec{V}_r \cdot \vec{n})\, dA$$

$$B = m\vec{V} \qquad b = \vec{V} \qquad b = \vec{V}$$

$$\frac{d(m\vec{V})_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho\vec{V}\, dV + \int_{\text{CS}} \rho\vec{V}(\vec{V}_r \cdot \vec{n})\, dA$$

The linear momentum equation is obtained by replacing B in the Reynolds transport theorem by the momentum $m\vec{V}$, and b by the momentum per unit mass $\vec{V}$.

[redacted]

$$\frac{d(m\vec{V})_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho\vec{V}\, dV + \int_{\text{CS}} \rho\vec{V}\,(\vec{V}_r \cdot \vec{n})\, dA$$

General: $$\sum \vec{F} = \frac{d}{dt}\int_{\text{CV}} \rho\vec{V}\, dV + \int_{\text{CS}} \rho\vec{V}(\vec{V}_r \cdot \vec{n})\, dA$$

$$\vec{V}_r = \vec{V} - \vec{V}_{\text{CS}}$$

$$\begin{pmatrix}\text{The sum of all}\\ \text{external forces}\\ \text{acting on a CV}\end{pmatrix} = \begin{pmatrix}\text{The time rate of change}\\ \text{of the linear momentum}\\ \text{of the contents of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of}\\ \text{linear momentum out of the}\\ \text{control surface by mass flow}\end{pmatrix}$$

Fixed CV: $$\sum \vec{F} = \frac{d}{dt}\int_{\text{CV}} \rho\vec{V}\, dV + \int_{\text{CS}} \rho\vec{V}(\vec{V} \cdot \vec{n})\, dA$$

Note that the momentum equation is a *vector equation*, and thus each term should be treated as a vector. Also, the components of this equation can be resolved along orthogonal coordinates (such as *x*, *y*, and *z* in the Cartesian coordinate system) for convenience. The force *F* in most cases consists of weights, pressure forces, and reaction forces. The momentum equation is commonly used to calculate the forces (usually on support systems or connectors) induced by the flow.

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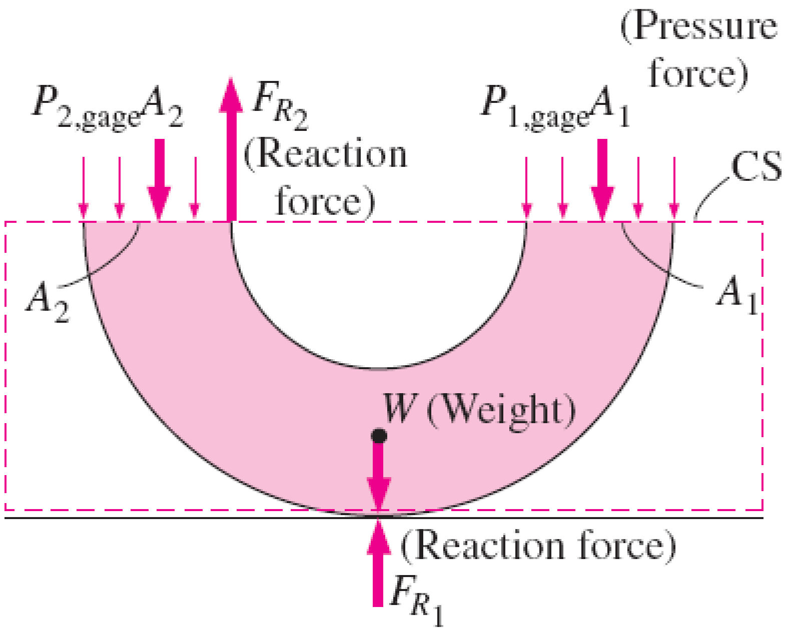


An 180° elbow supported by the ground

The momentum equation is commonly used to calculate the forces (usually on support systems or connectors) induced by the flow.

In most flow systems, the force *F* consists of weights, pressure forces, and reaction forces. Gage pressures are used here since atmospheric pressure cancels out on all sides of the control surface.

Special Cases

Steady flow:

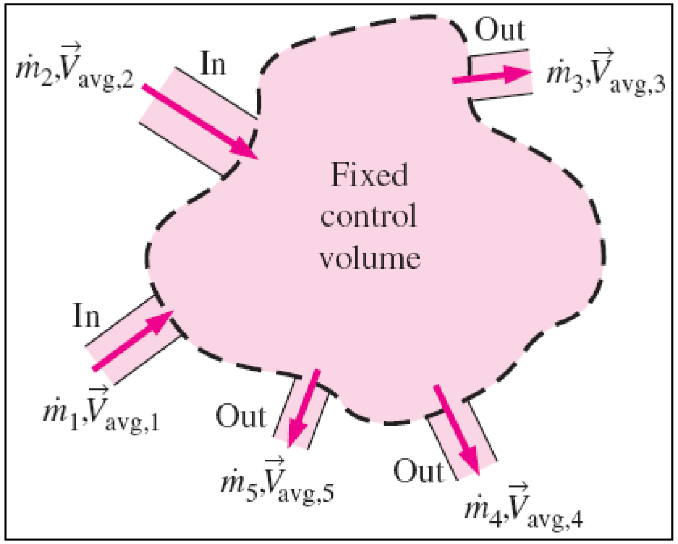
$$\sum \vec{F} = \int_{CS} \rho \vec{V} \, (\vec{V}_r \cdot \vec{n}) \, dA$$

Mass flow rate across an inlet or outlet

$$\dot{m} = \int_{A_c} \rho(\vec{V} \cdot \vec{n}) \, dA_c = \rho V_{avg} A_c$$

Momentum flow rate across a uniform inlet or outlet:

$$\int_{A_c} \rho \vec{V}(\vec{V} \cdot \vec{n}) \, dA_c = \rho V_{avg} \, A_c \vec{V}_{avg} = \dot{m} \vec{V}_{avg}$$



In a typical engineering problem, the control volume may contain many inlets and outlets; at each inlet or outlet we define the mass flow rate and the average velocity.

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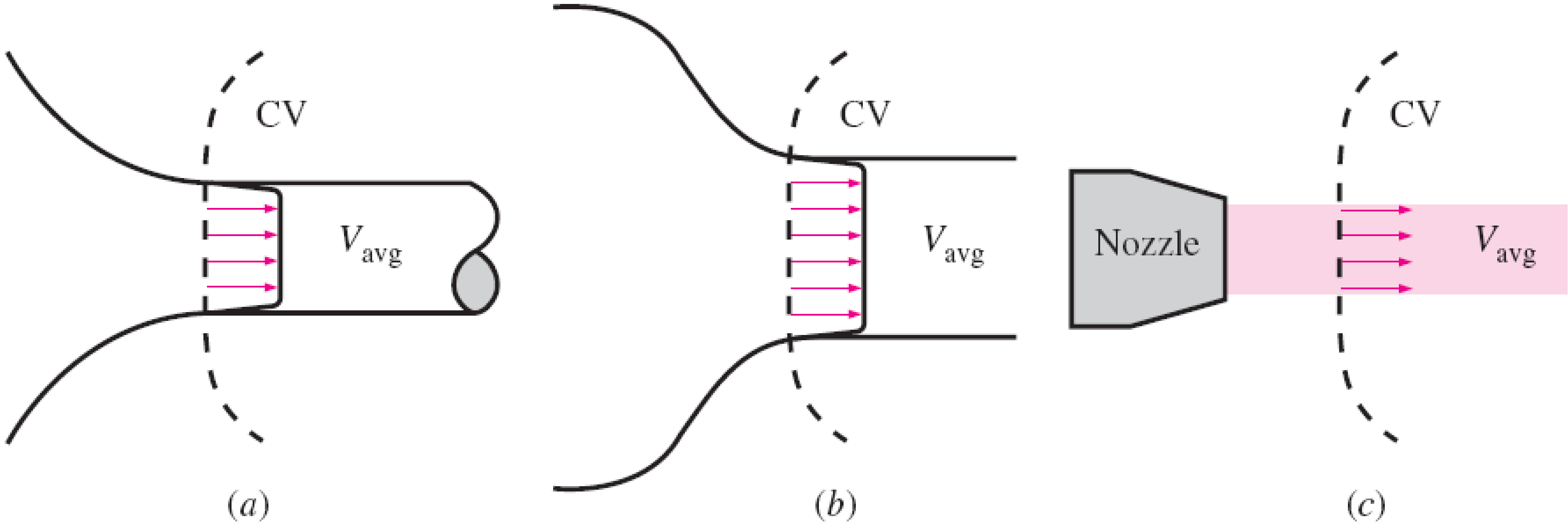


Examples of inlets or outlets in which the uniform flow approximation is reasonable:

(*a*) the well-rounded entrance to a pipe,
(*b*) the entrance to a wind tunnel test section, and
(*c*) a slice through a free water jet in air.

Momentum-Flux Correction Factor, β

The velocity across most inlets and outlets is *not* uniform.
The control surface integral may be converted into algebraic form using a dimensionless correction factor β, called the momentum-flux correction factor.

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \int_{CS} \rho \vec{V}(\vec{V}\cdot\vec{n})\, dA$$

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho \vec{V}\, dV + \sum_{out} \beta \dot{m} \vec{V}_{avg} - \sum_{in} \beta \dot{m} \vec{V}_{avg}$$

For turbulent flow β may have an insignificant effect at inlets and outlets, but for laminar flow β may be important and should not be neglected. It is wise to include β in all momentum control volume problems.

Momentum flux across an inlet or outlet: $\displaystyle \int_{A_c} \rho \vec{V}(\vec{V}\cdot\vec{n})\, dA_c = \beta \dot{m} \vec{V}_{avg}$

$$\beta = \frac{\int_{A_c} \rho V(\vec{V}\cdot\vec{n})\, dA_c}{\dot{m} V_{avg}} = \frac{\int_{A_c} \rho V(\vec{V}\cdot\vec{n})\, dA_c}{\rho V_{avg} A_c V_{avg}}$$

β is always greater than or equal to 1.

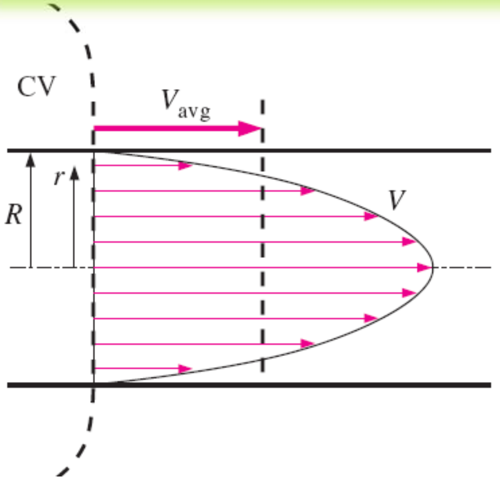
β is close to 1 for turbulent flow and not very close to 1 for fully developed laminar flow.

$$(\vec{V}\cdot\vec{n})\, dA_c = V\, dA_c$$

Momentum-flux correction factor: $\displaystyle \beta = \frac{1}{A_c}\int_{A_c}\left(\frac{V}{V_{avg}}\right)^2 dA_c$

EXAMPLE 6–1

Consider laminar flow through a very long straight section of round pipe. The velocity profile through a cross-sectional area of the pipe is parabolic, with the axial velocity component given by $V = 2V_{avg}\left(1 - \frac{r^2}{R^2}\right)$. where R is the radius of the inner wall of the pipe and V_{avg} is the average velocity. Calculate the momentum-flux correction factor through a cross section of the pipe for the case in which the pipe flow represents an outlet of the control volume, as sketched in Figure.



Solution For a given velocity distribution we are to calculate the momentum- flux correction factor.

Assumptions **1** The flow is incompressible and steady. **2** The control volume slices through the pipe normal to the pipe axis.

$$\beta = \frac{1}{A_c}\int_{A_c}\left(\frac{V}{V_{avg}}\right)^2 dA_c = \frac{4}{\pi R^2}\int_0^R\left(1 - \frac{r^2}{R^2}\right)^2 2\pi r\,dr$$

$y = 1 - r^2/R^2$

$dy = -2r\,dr/R^2$

y=1 at r= 0,
y=0 at r=R)

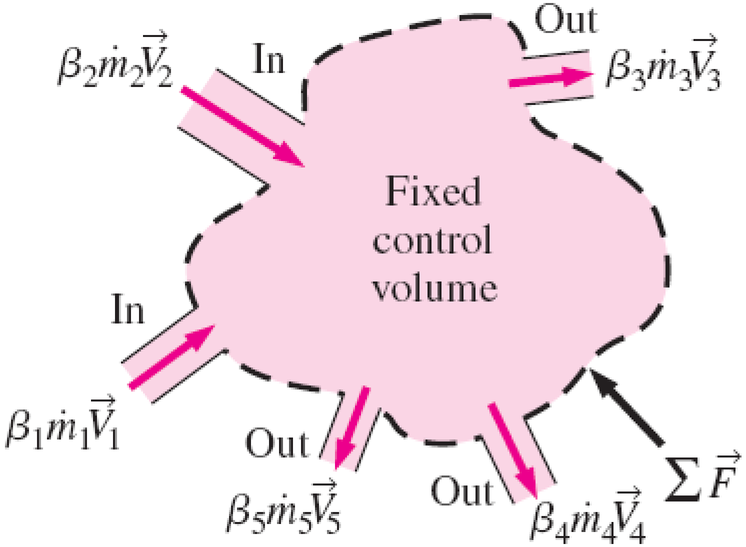
$$\beta = -4\int_1^0 y^2\,dy = -4\left[\frac{y^3}{3}\right]_1^0 = \frac{4}{3}$$

Discussion We have calculated β for an outlet, but the same result would have been obtained if we had considered the cross section of the pipe as an *inlet* to the control volume.

Steady Flow

Steady linear momentum equation: $\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$

The net force acting on the control volume during steady flow is equal to the difference between the rates of outgoing and incoming momentum flows.



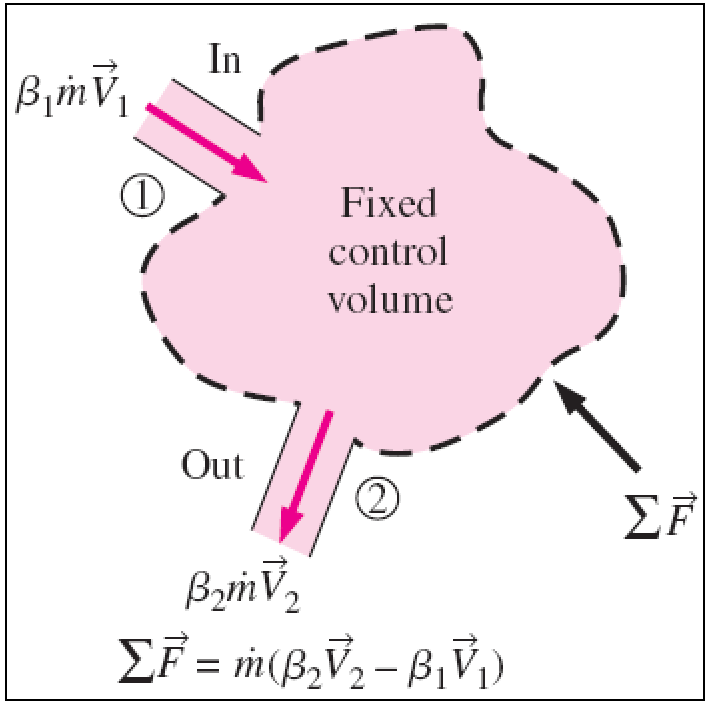
$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

The net force acting on the control volume during steady flow is equal to the difference between the outgoing and the incoming momentum fluxes.

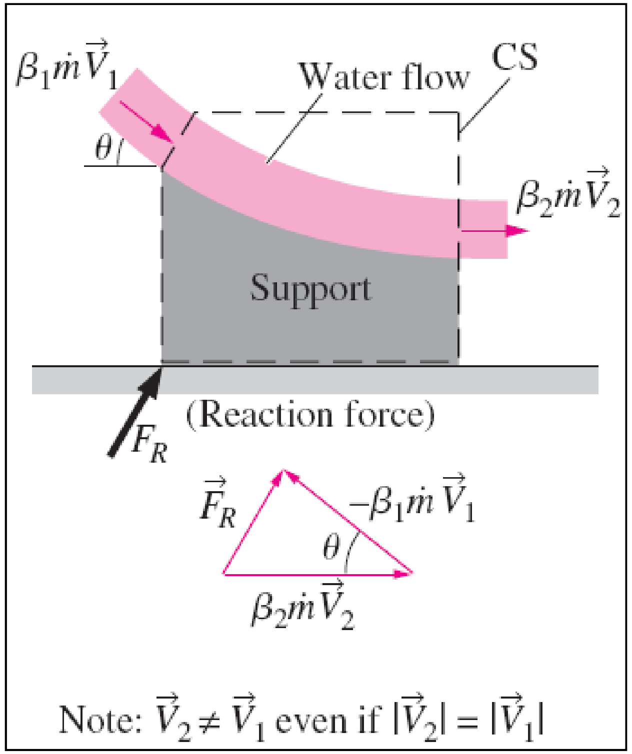
Steady Flow with One Inlet and One Outlet

One inlet and one outlet: $\sum \vec{F} = \dot{m}\,(\beta_2 \vec{V}_2 - \beta_1 \vec{V}_1)$

Along x-coordinate: $\sum F_x = \dot{m}(\beta_2 V_{2,x} - \beta_1 V_{1,x})$



A control volume with only one inlet and one outlet.



The determination by vector addition of the reaction force on the support caused by a change of direction of water.

Flow with No External Forces

No external forces:

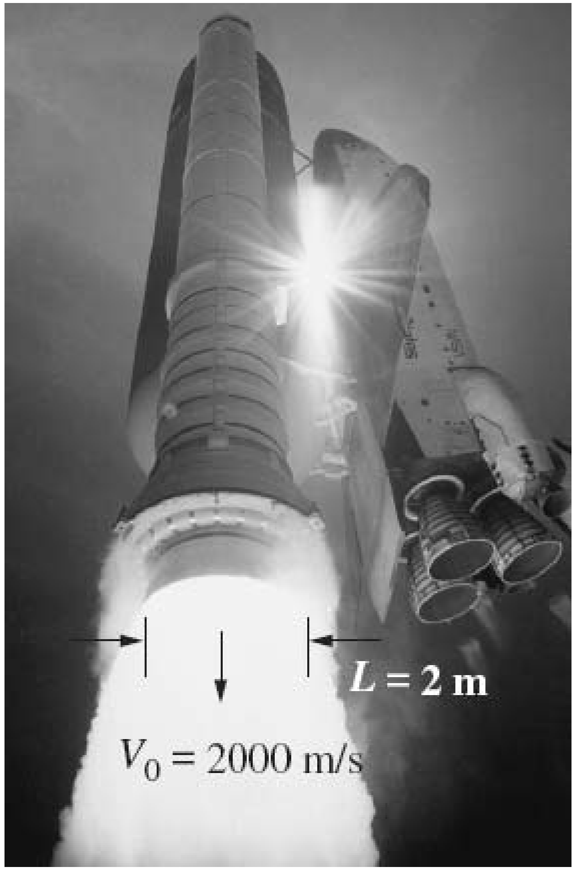
$$0 = \frac{d(m\vec{V})_{CV}}{dt} + \sum_{out} \beta \dot{m}\vec{V} - \sum_{in} \beta \dot{m}\vec{V}$$

In the absence of external forces, the rate of change of the momentum of a control volume is equal to the difference between the rates of incoming and outgoing momentum flow rates.

$$\frac{d(m\vec{V})_{CV}}{dt} = m_{CV}\frac{d\vec{V}_{CV}}{dt} = (m\vec{a})_{CV}$$

Therefore, the control volume in this case can be treated as a solid body, with a net force or **thrust** of acting on the body.

$$\vec{F}_{body} = m_{body}\vec{a} = \sum_{in} \beta \dot{m}\vec{V} - \sum_{out} \beta \dot{m}\vec{V}$$



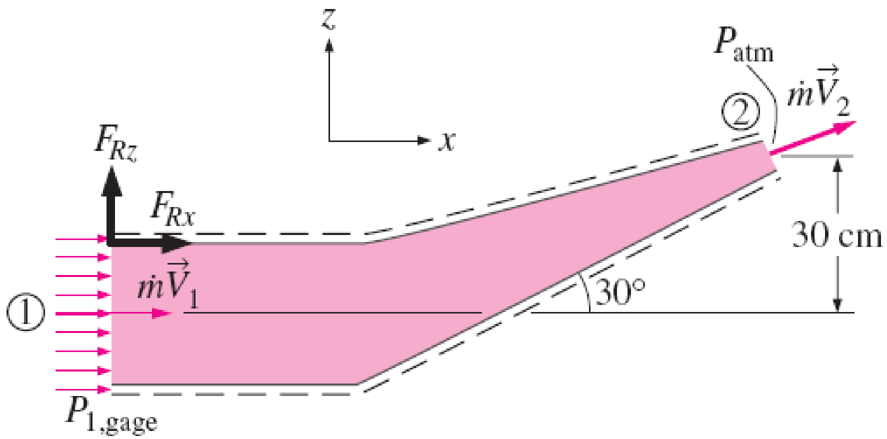
The thrust needed to lift the space shuttle is generated by the rocket engines as a result of momentum change of the fuel as it is accelerated from about zero to an exit speed of about 2000 m/s after combustion.

EXAMPLE 6–2

A reducing elbow is used to deflect water flow at a rate of 14 kg/s in a horizontal pipe upward 30° while accelerating it. The elbow discharges water into the atmosphere. The cross-sectional area of the elbow is 113 cm^2 at the inlet and 7 cm^2 at the outlet. The elevation difference between the centers of the outlet and the inlet is 30 cm. The weight of the elbow and the water in it is considered to be negligible. Determine (*a*) the gage pressure at the center of the inlet of the elbow and (*b*) the anchoring force needed to hold the elbow in place.

Solution A reducing elbow deflects water upward and discharges it to the atmosphere. The pressure at the inlet of the elbow and the force needed to hold the elbow in place are to be determined.

Assumptions **1** The flow is steady, and the frictional effects are negligible. **2** The weight of the elbow and the water in it is negligible. **3** The water is discharged to the atmosphere, and thus the gage pressure at the outlet is zero. **4** The flow is turbulent and fully developed at both the inlet and outlet of the control volume, and we take the momentum-flux correction factor to be β=1.03.



(a) The gage pressure at the center of the inlet of the elbow

$$\dot{m}_1 = \dot{m}_2 = \dot{m} = 14 \text{ kg/s.} \qquad \dot{m} = \rho A V$$

$$V_1 = \frac{\dot{m}}{\rho A_1} = \frac{14 \text{ kg/s}}{(1000 \text{ kg/m}^3)(0.0113 \text{ m}^2)} = 1.24 \text{ m/s}$$

$$V_2 = \frac{\dot{m}}{\rho A_2} = \frac{14 \text{ kg/s}}{(1000 \text{ kg/m}^3)(7 \times 10^{-4} \text{ m}^2)} = 20.0 \text{ m/s}$$

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 \qquad P_1 - P_2 = \rho g\left(\frac{V_2^2 - V_1^2}{2g} + z_2 - z_1\right)$$

$$P_1 - P_{\text{atm}} = (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2) \times \left(\frac{(20 \text{ m/s})^2 - (1.24 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} + 0.3 - 0\right)\left(\frac{1 \text{ kN}}{1000 \text{ kg} \cdot \text{m/s}^2}\right)$$

$$P_{1,\text{ gage}} = 202.2 \text{ kN/m}^2 = \mathbf{202.2 \text{ kPa}} \quad \text{(gage)}$$

(*b*) The momentum equation for steady one-dimensional flow is

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

$$F_{Rx} + P_{1,\,\text{gage}} A_1 = \beta \dot{m} V_2 \cos\theta - \beta \dot{m} V_1$$

$$F_{Rz} = \beta \dot{m} V_2 \sin\theta$$

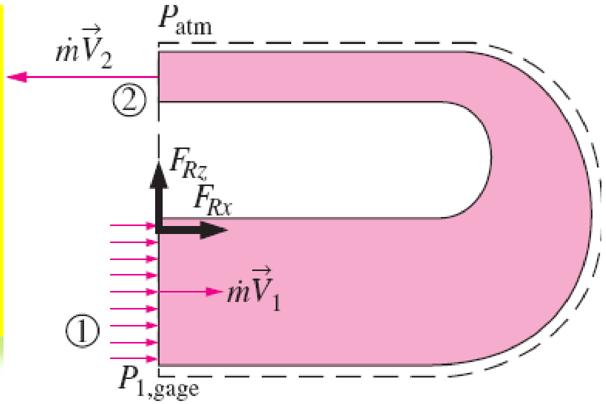
$$\begin{aligned} F_{Rx} &= \beta \dot{m}(V_2 \cos\theta - V_1) - P_{1,\,\text{gage}} A_1 \\ &= 1.03(14\ \text{kg/s})[(20 \cos 30^\circ - 1.24)\ \text{m/s}]\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) \\ &\quad - (202{,}200\ \text{N/m}^2)(0.0113\ \text{m}^2) \\ &= 232 - 2285 = \mathbf{-2053\ N} \end{aligned}$$

$$F_{Rz} = \beta \dot{m} V_2 \sin\theta = (1.03)(14\ \text{kg/s})(20 \sin 30^\circ\ \text{m/s})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) = \mathbf{144\ N}$$

Discussion There is a nonzero pressure distribution along the inside walls of the elbow, but since the control volume is outside the elbow, these pressures do not appear in our analysis. The actual value of *P*1, gage will be higher than that calculated here because of frictional and other irreversible losses in the elbow.

EXAMPLE 6–3

The deflector elbow in Exp. 6–2 is replaced by a reversing elbow such that the fluid makes a 180° U-turn before it is discharged, as shown in Figure. The elevation difference between the centers of the inlet and the exit sections is still 0.3 m. Determine the anchoring force needed to hold the elbow in place.



Solution The inlet and the outlet velocities and the pressure at the inlet of the elbow remain the same, but the vertical component of the anchoring force at the connection of the elbow to the pipe is zero in this case (F_{Rz}=0) since there is no other force or momentum flux in the vertical direction (we are neglecting the weight of the elbow and the water). The horizontal component of the anchoring force is determined from the momentum equation written in the *x*-direction. Noting that the outlet velocity is negative since it is in the negative *x*-direction, we have

$$F_{Rx} + P_{1,\text{ gage}}A_1 = \beta_2\dot{m}(-V_2) - \beta_1\dot{m}V_1 = -\beta\dot{m}(V_2 + V_1)$$

Solving for F_{Rx} and substituting the known values,

$$F_{Rx} = -\beta\dot{m}(V_2 + V_1) - P_{1,\text{ gage}}A_1$$

$$= -(1.03)(14\text{ kg/s})[(20 + 1.24)\text{ m/s}]\left(\frac{1\text{ N}}{1\text{ kg}\cdot\text{m/s}^2}\right) - (202{,}200\text{ N/m}^2)(0.0113\text{ m}^2)$$

$$= -306 - 2285 = \mathbf{-2591\ N}$$

Therefore, the horizontal force on the flange is 2591 N acting in the negative *x*-direction (the elbow is trying to separate from the pipe). This force is equivalent to the weight of about 260 kg mass, and thus the connectors (such as bolts) used must be strong enough to withstand this force.

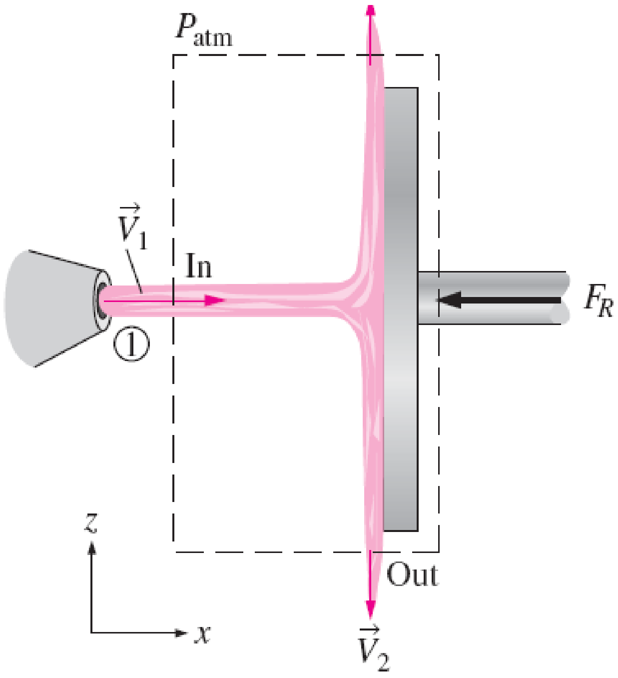
Discussion The reaction force in the *x*-direction is larger than that of Exp. 6–2 since the walls turn the water over a much greater angle. If the reversing elbow is replaced by a straight nozzle (like one used by firefighters) such that water is discharged in the positive *x*-direction, the momentum equation in the *x*-direction becomes

$$F_{Rx} + P_{1,\,\text{gage}}A_1 = \beta\dot{m}V_2 - \beta\dot{m}V_1 \quad \rightarrow \quad F_{Rx} = \beta\dot{m}(V_2 - V_1) - P_{1,\,\text{gage}}A_1$$

since both V_1 and V_2 are in the positive *x*-direction. This shows the importance of using the correct sign (positive if in the positive direction and negative if in the opposite direction) for velocities and forces.

EXAMPLE 6–4

Water is accelerated by a nozzle to an average speed of 20 m/s, and strikes a stationary vertical plate at a rate of 10 kg/s with a normal velocity of 20 m/s. After the strike, the water stream splatters off in all directions in the plane of the plate. Determine the force needed to prevent the plate from moving horizontally due to the water stream.



Solution A water jet strikes a vertical stationary plate normally. The force needed to hold the plate in place is to be determined.

Assumptions **1** The flow of water at nozzle outlet is steady. **2** The water splatters in directions normal to the approach direction of the water jet. **3** The water jet is exposed to the atmosphere, and thus the pressure of the water jet and the splattered water leaving the control volume is atmospheric pressure, which is disregarded since it acts on the entire system. **4** The vertical forces and momentum fluxes are not considered since they have no effect on the horizontal reaction force. **5** The effect of the momentum-flux correction factor is negligible, and thus β=1.

[redacted]

The momentum equation for steady one-dimensional flow

$$\sum \vec{F} = \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

$V_{1,x} = V_1$ and $V_{2,x} = 0$

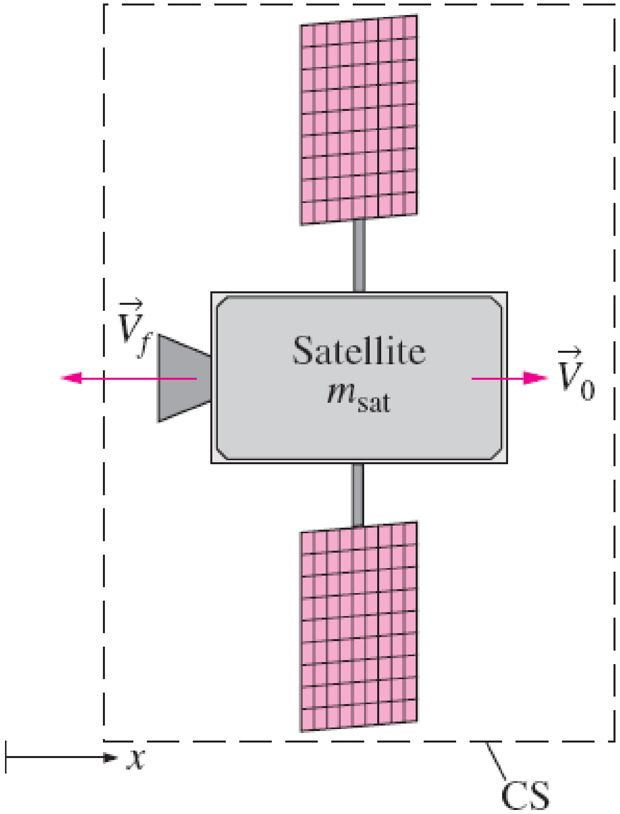
$$-F_R = 0 - \beta \dot{m} \vec{V}_1$$

$$F_R = \beta \dot{m} \vec{V}_1 = (1)(10\ \text{kg/s})(20\ \text{m/s})\left(\frac{1\ \text{N}}{1\ \text{kg} \cdot \text{m/s}^2}\right) = \mathbf{200\ N}$$

Discussion The plate absorbs the full brunt of the momentum of the water jet since the *x*-direction momentum at the outlet of the control volume is zero. If the control volume were drawn instead along the interface between the water and the plate, there would be additional (unknown) pressure forces in the analysis. By cutting the control volume through the support, we avoid having to deal with this additional complexity. This is an example of a “wise” choice of control volume.

EXAMPLE 6–5

An orbiting satellite has a mass of m_{sat} = 5000 kg and is traveling at a constant velocity of V_0. To alter its orbit, an attached rocket discharges m_f =100 kg of gases from the reaction of solid fuel at a velocity V_f =3000 m/s relative to the satellite in a direction opposite to V_0. The fuel discharge rate is constant for 2 s. Determine (*a*) the acceleration of the satellite during this 2-s period, (*b*) the change of velocity of the satellite during this time period, and (*c*) the thrust exerted on the satellite.



SOLUTION The rocket of a satellite is fired in the opposite direction to motion. The acceleration, the velocity change, and the thrust are to be determined.

Assumptions **1** The flow of combustion gases is steady and one-dimensional during the firing period. **2** There are no external forces acting on the satellite, and the effect of the pressure force at the nozzle exit is negligible. **3** The mass of discharged fuel is negligible relative to the mass of the satellite,

and thus the satellite may be treated as a solid body with a constant mass. **4** The nozzle is well-designed such that the effect of the momentum flux correction factor is negligible, and thus β = 1.

(*a*) the acceleration of the satellite during this 2-s period

$$0 = \frac{d(m\vec{V})_{CV}}{dt} + \sum_{\text{out}} \beta \dot{m}\vec{V} - \sum_{\text{in}} \beta \dot{m}\vec{V} \quad \rightarrow \quad m_{\text{sat}}\frac{d\vec{V}_{\text{sat}}}{dt} = -\dot{m}_f\vec{V}_f$$

$$m_{\text{sat}}\frac{dV_{\text{sat}}}{dt} = \dot{m}_f V_f \quad \rightarrow \quad \frac{dV_{\text{sat}}}{dt} = \frac{\dot{m}_f}{m_{\text{sat}}}V_f = \frac{m_f/\Delta t}{m_{\text{sat}}}V_f$$

$$a_{\text{sat}} = \frac{dV_{\text{sat}}}{dt} = \frac{m_f/\Delta t}{m_{\text{sat}}}V_f = \frac{(100\text{ kg})/(2\text{ s})}{5000\text{ kg}}(3000\text{ m/s}) = \mathbf{30\ m/s^2}$$

(*b*) Knowing acceleration, which is constant, the velocity change of the satellite during the first 2 s is determined from the definition of acceleration

$$dV_{\text{sat}} = a_{\text{sat}}\,dt \quad \rightarrow \quad \Delta V_{\text{sat}} = a_{\text{sat}}\,\Delta t = (30\text{ m/s}^2)(2\text{ s}) = \mathbf{60\ m/s}$$

(*c*) The thrust exerted on the satellite is,

$$F_{\text{sat}} = 0 - \dot{m}_f(-V_f) = -(100/2\text{ kg/s})(-3000\text{ m/s})\left(\frac{1\text{ kN}}{1000\text{ kg}\cdot\text{m/s}^2}\right) = \mathbf{150\ kN}$$

Discussion Note that if this satellite were attached somewhere, it would exert a force of 150 kN (equivalent to the weight of 15 tons of mass) to its support. This can be verified by taking the satellite as the system and applying the momentum equation.

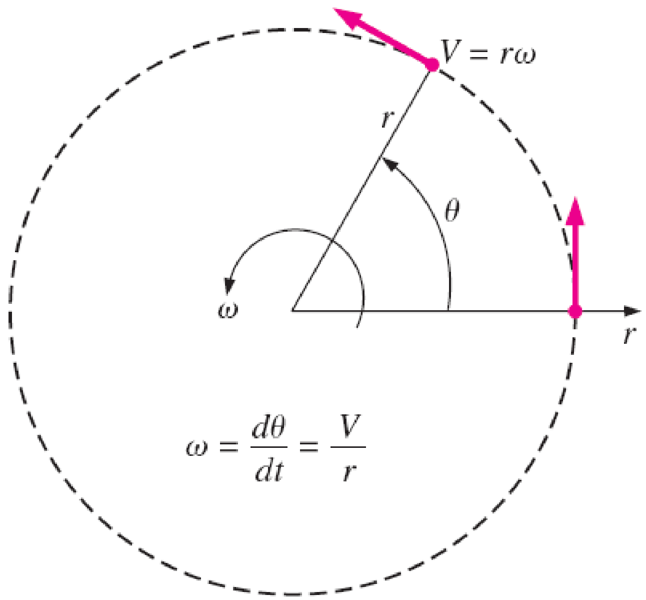
6–5 REVIEW OF ROTATIONAL MOTION AND ANGULAR MOMENTUM

Rotational motion: A motion during which all points in the body move in circles about the axis of rotation.

Rotational motion is described with angular quantities such as the angular distance θ, angular velocity ω, and angular acceleration α.

Angular velocity: The angular distance traveled per unit time.

Angular acceleration: The rate of change of angular velocity.



$$\omega = \frac{d\theta}{dt} = \frac{d(l/r)}{dt} = \frac{1}{r}\frac{dl}{dt} = \frac{V}{r}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{1}{r}\frac{dV}{dt} = \frac{a_t}{r}$$

$V = r\omega$ and $a_t = r\alpha$

The relations between angular distance θ, angular velocity ω, and linear velocity V.

[redacted]

- Newton's second law requires that there must be a force acting in the tangential direction to cause angular acceleration.
- The strength of the rotating effect, called the ***moment or torque***, is proportional to the magnitude of the force and its distance from the axis of rotation.
- The perpendicular distance from the axis of rotation to the line of action of the force is called the ***moment arm***, and the torque *M* acting on a point mass *m* at a normal distance *r* from the axis of rotation is expressed as

$$M = rF_t = rma_t = mr^2\alpha$$

Torque

$$M = \int_{\text{mass}} r^2\alpha \, dm = \left[\int_{\text{mass}} r^2 \, dm\right]\alpha = I\alpha$$

I is the *moment of inertia* of the body about the axis of rotation, which is a measure of the inertia of a body against rotation.

Unlike mass, the rotational inertia of a body also depends on the distribution of the mass of the body with respect to the axis of rotation.

Mass, m	$\longleftrightarrow$	Moment of inertia, I
Linear acceleration, $\vec{a}$	$\longleftrightarrow$	Angular acceleration, $\vec{\alpha}$
Linear velocity, $\vec{V}$	$\longleftrightarrow$	Angular velocity, $\vec{\omega}$
Linear momentum	$\longleftrightarrow$	Angular momentum
$m\vec{V}$	$\longleftrightarrow$	$I\vec{\omega}$
Force, $\vec{F}$	$\longleftrightarrow$	Torque, M
$\vec{F} = m\vec{a}$	$\longleftrightarrow$	$\vec{M} = I\vec{\alpha}$
Moment of force, $\vec{M}$	$\longleftrightarrow$	Moment of momentum, $\vec{H}$
$\vec{M} = \vec{r} \times \vec{F}$	$\longleftrightarrow$	$\vec{H} = \vec{r} \times m\vec{V}$

Analogy between corresponding linear and angular quantities.

[redacted]

$$H = \int_{\text{mass}} r^2\omega \, dm = \left[\int_{\text{mass}} r^2 \, dm\right]\omega = I\omega$$

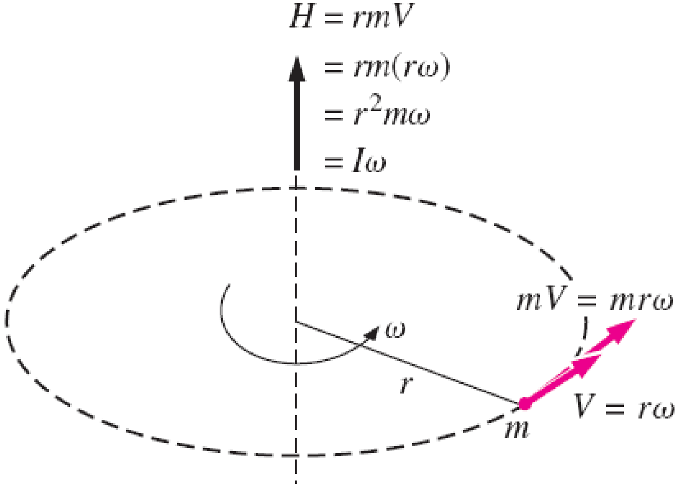
Angular momentum $\vec{H} = I\vec{\omega}$

$$\vec{M} = I\vec{\alpha} = I\frac{d\vec{\omega}}{dt} = \frac{d(I\vec{\omega})}{dt} = \frac{d\vec{H}}{dt}$$

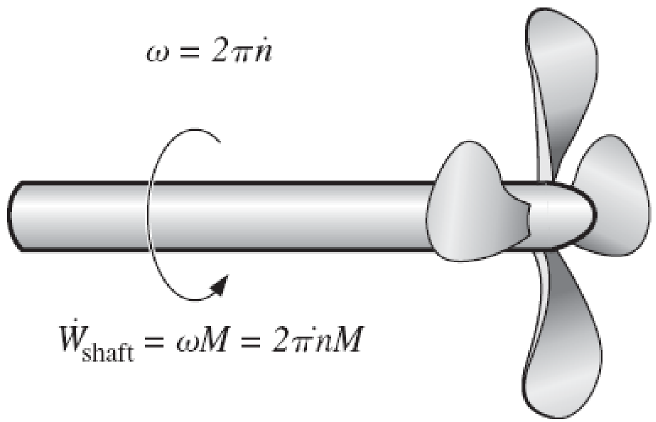
Angular momentum equation

$$\omega = \frac{2\pi\dot{n}}{60} \qquad \text{(rad/s)}$$

Angular velocity versus rpm



Angular momentum of point mass *m* rotating at angular velocity ω at distance *r* from the axis of rotation.



The relations between angular velocity, rpm, and the power transmitted through a shaft.

[redacted]

$$\dot{W}_{\text{shaft}} = FV = Fr\omega = M\omega$$

Shaft power: $$\dot{W}_{\text{shaft}} = \omega M = 2\pi \dot{n} M \quad \text{(W)}$$

Rotational kinetic energy: $$\text{KE}_r = \tfrac{1}{2} I \omega^2$$

During rotational motion, the direction of velocity changes even when its magnitude remains constant. Velocity is a vector quantity, and thus a change in direction constitutes a change in velocity with time, and thus acceleration. This is called centripetal acceleration.

$$a_r = \frac{V^2}{r} = r\omega^2$$

Centripetal acceleration is directed toward the axis of rotation (opposite direction of radial acceleration), and thus the radial acceleration is negative. Centripetal acceleration is the result of a force acting on an element of the body toward the axis of rotation, known as the **centripetal force**, whose magnitude is $F_r = mV^2/r$.

Tangential and radial accelerations are perpendicular to each other, and the total linear acceleration is determined by their vector sum:

$$\vec{a} = \vec{a}_t + \vec{a}_r$$

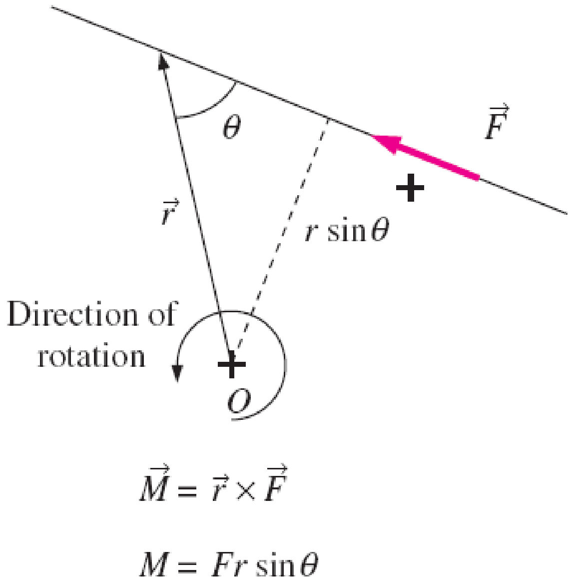
6–6 THE ANGULAR MOMENTUM EQUATION

Many engineering problems involve the moment of the linear momentum of flow streams, and the rotational effects caused by them.

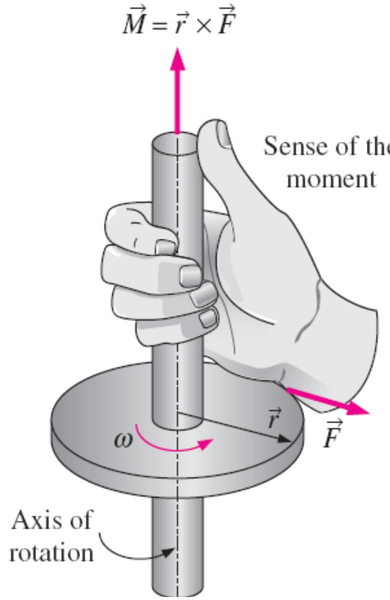
Such problems are best analyzed by the ***angular momentum equation***, also called the *moment of momentum equation.*

An important class of fluid devices, called *turbomachines*, which include centrifugal pumps, turbines, and fans, is analyzed by the angular momentum equation.

A force whose line of action passes through point *O* produces zero moment about point *O*.



The moment of a force *F* about a point *O* is the vector product of the position vector *r* and *F*



The determination of the direction of the moment by the right-hand rule.

[redacted]

Moment of momentum: $\vec{H} = \vec{r} \times m\vec{V}$

Moment of momentum (system): $\vec{H}_{\text{sys}} = \int_{\text{sys}} (\vec{r} \times \vec{V})\rho \, dV$

Rate of change of moment of momentum: $\dfrac{d\vec{H}_{\text{sys}}}{dt} = \dfrac{d}{dt}\int_{\text{sys}} (\vec{r} \times \vec{V})\rho \, dV$

Angular momentum equation for a system

$$\sum \vec{M} = \frac{d\vec{H}_{\text{sys}}}{dt} \qquad \sum \vec{M} = \sum (\vec{r} \times \vec{F})$$

$$\frac{dH_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} (\vec{r} \times \vec{V})\rho \, dV + \int_{\text{CS}} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n}) \, dA$$

General: $\sum \vec{M} = \dfrac{d}{dt}\int_{\text{CV}} (\vec{r} \times \vec{V})\rho \, dV + \int_{\text{CS}} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n}) \, dA$

$$\left(\begin{array}{c}\text{The sum of all}\\ \text{external moments}\\ \text{acting on a CV}\end{array}\right) = \left(\begin{array}{c}\text{The time rate of change}\\ \text{of the angular momentum}\\ \text{of the contents of the CV}\end{array}\right) + \left(\begin{array}{c}\text{The net flow rate of}\\ \text{angular momentum}\\ \text{out of the control}\\ \text{surface by mass flow}\end{array}\right)$$

[redacted]

$$\text{Fixed CV:} \qquad \sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho\, dV + \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V} \cdot \vec{n})\, dA$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b\, dV + \int_{CS} \rho b(V_r \cdot \vec{n})\, dA$$

$$B = \vec{H} \qquad b = \vec{r} \times \vec{V} \qquad b = \vec{r} \times \vec{V}$$

$$\frac{dH_{sys}}{dt} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho\, dV + \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA$$

The angular momentum equation is obtained by replacing *B* in the Reynolds transport theorem by the angular momentum *H*, and *b* by the angular momentum per unit mass *r* x *V*.

Special Cases

During ***steady flow***, the amount of angular momentum within the control volume remains constant, and thus the time rate of change of angular momentum of the contents of the control volume is zero.

$$\text{Steady flow:} \qquad \sum \vec{M} = \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA$$

An approximate form of the angular momentum equation in terms of average properties at inlets and outlets:

$$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho\, dV + \sum_{\text{out}} \vec{r} \times \dot{m}\vec{V} - \sum_{\text{in}} \vec{r} \times \dot{m}\vec{V}$$

$$\text{Steady flow:} \qquad \sum \vec{M} = \sum_{\text{out}} \vec{r} \times \dot{m}\vec{V} - \sum_{\text{in}} \vec{r} \times \dot{m}\vec{V}$$

The net torque acting on the control volume during steady flow is equal to the difference between the outgoing and incoming angular momentum flow rates.

$$\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$$

scalar form of angular momentum equation

Flow with No External Moments

$$\textit{No external moments:} \qquad 0 = \frac{d\vec{H}_{CV}}{dt} + \sum_{out} \vec{r} \times \dot{m}\vec{V} - \sum_{in} \vec{r} \times \dot{m}\vec{V}$$

In the absence of external moments, the rate of change of the angular momentum of a control volume is equal to the difference between the incoming and outgoing angular momentum fluxes.

When the moment of inertia *I* of the control volume remains constant, the irst term on the right side of the above equation becomes simply moment of inertia times angular acceleration. Therefore, the control volume in this case can be treated as a solid body, with a net torque of

$$\vec{M}_{body} = I_{body}\vec{\alpha} = \sum_{in} (\vec{r} \times \dot{m}\vec{V}) - \sum_{out} (\vec{r} \times \dot{m}\vec{V})$$

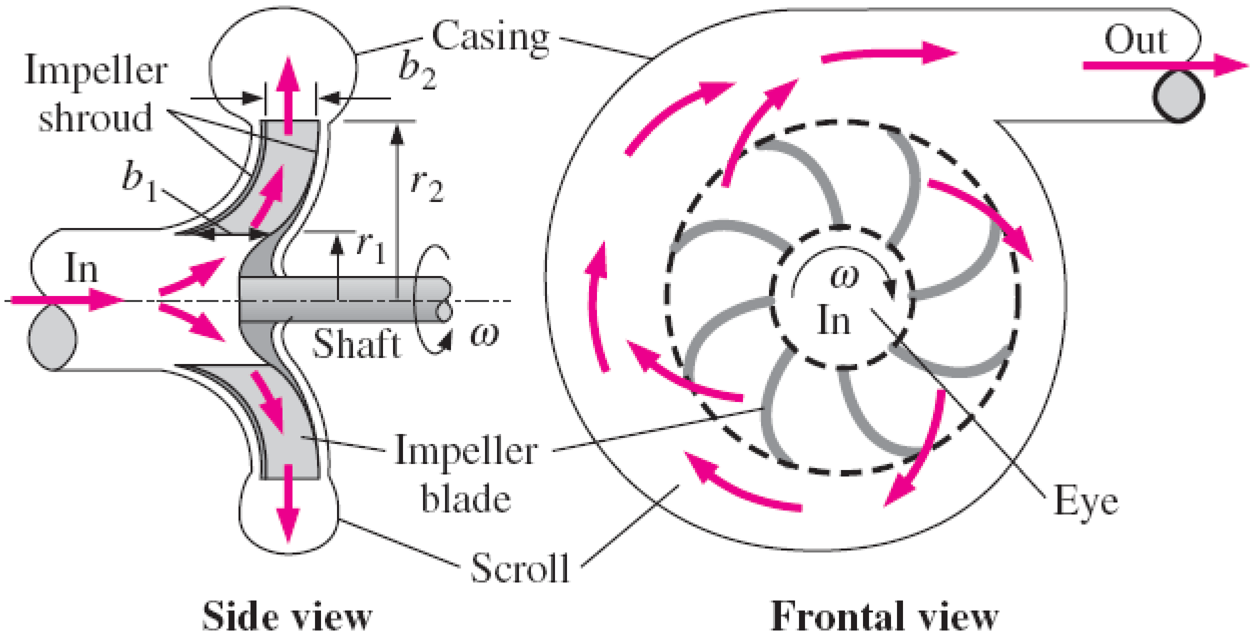
This approach can be used to determine the angular acceleration of space vehicles and aircraft when a rocket is fired in a direction different than the direction of motion.

Radial-Flow Devices

Radial-flow devices: Many rotary-flow devices such as centrifugal pumps and fans involve flow in the radial direction normal to the axis of rotation.

Axial-flow devices are easily analyzed using the **linear momentum equation.**

Radial-flow devices involve large changes in angular momentum of the fluid and are best analyzed with the help of the **angular momentum equation**.

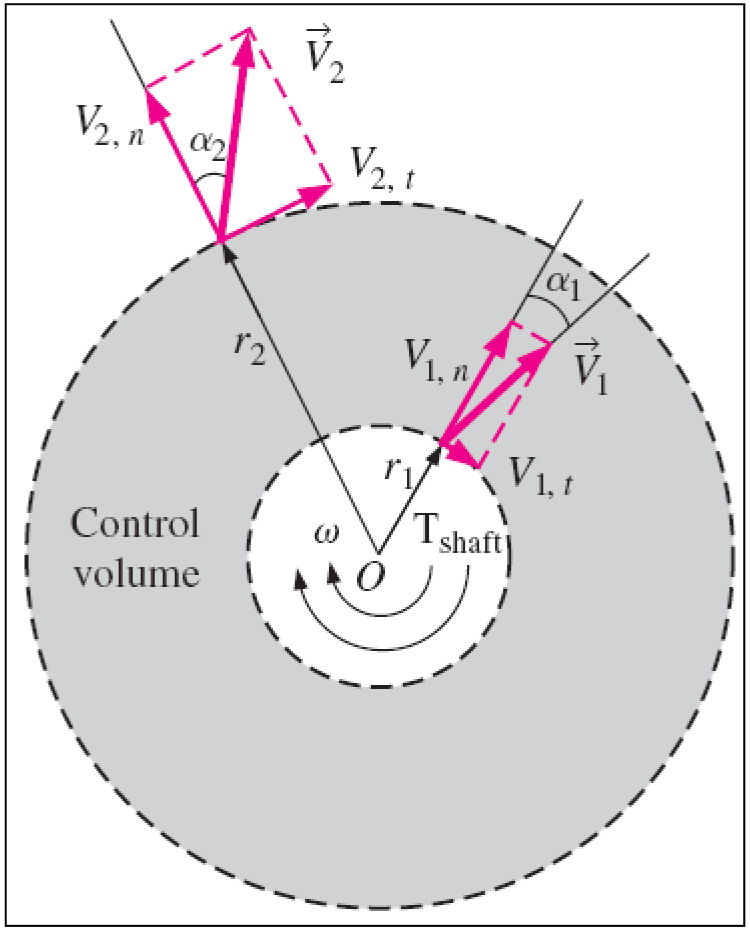


Side and frontal views of a typical centrifugal pump.

The conservation of mass equation for steady incompressible flow

$$\dot{V}_1 = \dot{V}_2 = \dot{V} \quad \rightarrow \quad (2\pi r_1 b_1)V_{1,n} = (2\pi r_2 b_2)V_{2,n}$$

$$V_{1,n} = \frac{\dot{V}}{2\pi r_1 b_1} \quad \text{and} \quad V_{2,n} = \frac{\dot{V}}{2\pi r_2 b_2}$$



An annular control volume that encloses the impeller section of a centrifugal pump.

angular momentum equation

$$\sum M = \sum_{out} r\dot{m}V - \sum_{in} r\dot{m}V$$

Euler's turbine formula

$$T_{shaft} = \dot{m}(r_2 V_{2,t} - r_1 V_{1,t})$$

When

$$V_{1,t} = \omega r_1 \qquad V_{2,t} = \omega r_2$$

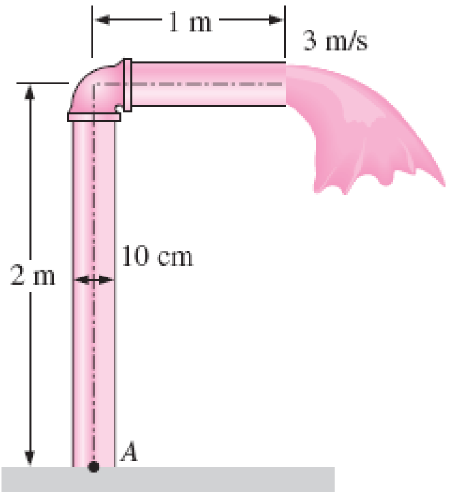
$$T_{shaft} = \dot{m}(r_2 V_2 \sin\alpha_2 - r_1 V_1 \sin\alpha_1)$$

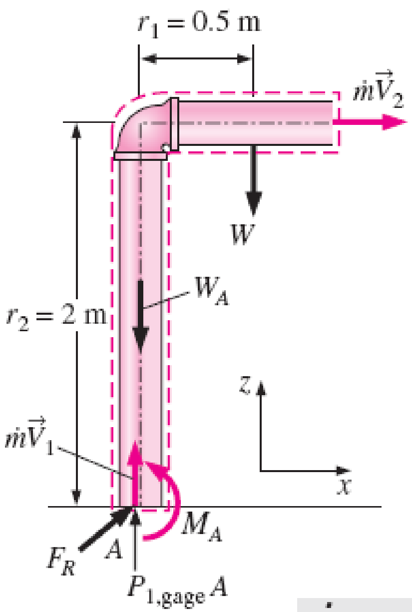
$$T_{shaft,\,ideal} = \dot{m}\omega(r_2^2 - r_1^2)$$

$$\dot{W}_{shaft} = \omega T_{shaft} = 2\pi\dot{n}T_{shaft}$$

EXAMPLE 6-8

Underground water is pumped to a sufficient height through a 10-cmdiameter pipe that consists of a 2-m-long vertical and 1-m-long horizontal section. Water discharges to atmospheric air at an average velocity of 3 m/s, and the mass of the horizontal pipe section when filled with water is 12 kg per meter length. The pipe is anchored on the ground by a concrete base. Determine the bending moment acting at the base of the pipe (point *A*) and the required length of the horizontal section that would make the moment at point *A* zero.





Solution Water is pumped through a piping section. The moment acting at the base and the required length of the horizontal section to make this moment zero is to be determined.

Assumptions **1** The flow is steady. **2** The water is discharged to the atmosphere, and thus the gage pressure at the outlet is zero. **3** The pipe diameter is small compared to the moment arm, and thus we use average values of radius and velocity at the outlet.

$$\dot{m}_1 = \dot{m}_2 = \dot{m}, \text{ and } V_1 = V_2 = V$$

$$\dot{m} = \rho A_c V = (1000 \text{ kg/m}^3)[\pi(0.10 \text{ m})^2/4](3 \text{ m/s}) = 23.56 \text{ kg/s}$$

$$W = mg = (12 \text{ kg/m})(1 \text{ m})(9.81 \text{ m/s}^2)\left(\frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2}\right) = 117.7 \text{ N}$$

$$\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V \qquad M_A - r_1 W = -r_2\dot{m}V_2$$

$$\begin{aligned} M_A &= r_1 W - r_2\dot{m}V_2 \\ &= (0.5\ \text{m})(118\ \text{N}) - (2\ \text{m})(23.56\ \text{kg/s})(3\ \text{m/s})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) \\ &= \mathbf{-82.5\ N\cdot m} \end{aligned}$$

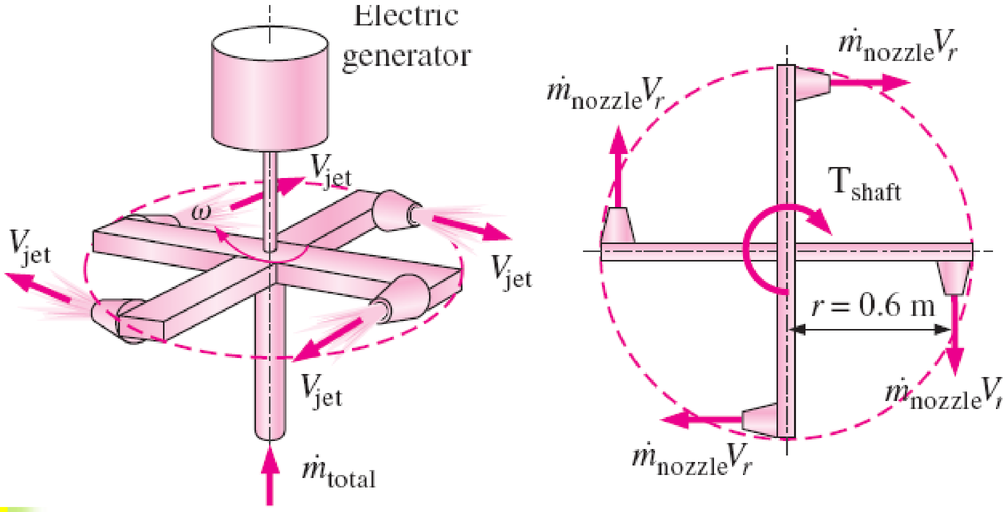
$$w = W/L = 117.7\ \text{N} \qquad 0 = r_1 W - r_2\dot{m}V_2 \quad \rightarrow \quad 0 = (L/2)Lw - r_2\dot{m}V_2$$

$$L = \sqrt{\frac{2r_2\dot{m}V_2}{w}} = \sqrt{\frac{2(2\ \text{m})(23.56\ \text{kg/s})(3\ \text{m/s})}{117.7\ \text{N/m}}\left(\frac{\text{N}}{\text{kg}\cdot\text{m/s}^2}\right)} = \mathbf{1.55\ m}$$

Discussion Note that the pipe weight and the momentum of the exit stream cause opposing moments at point *A*. This example shows the importance of accounting for the moments of momentums of flow streams when performing a dynamic analysis and evaluating the stresses in pipe materials at critical cross sections.

EXAMPLE 6-9

A large lawn sprinkler with four identical arms is to be converted into a turbine to generate electric power by attaching a generator to its rotating head. Water enters the sprinkler from the base along the axis of rotation at a rate of 20 L/s and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 300 rpm in a horizontal plane. The diameter of each jet is 1 cm, and the normal distance between the axis of rotation and the center of each nozzle is 0.6 m. Estimate the electric power produced.



Solution A four-armed sprinkler is used to generate electric power. For a specified flow rate and rotational speed, the power produced is to be determined.

Assumptions **1** The flow is cyclically steady (i.e., steady from a frame of reference rotating with the sprinkler head). **2** The water is discharged to the atmosphere, and thus the gage pressure at the nozzle exit is zero. **3** Generator losses and air drag of rotating components are neglected. **4** The nozzle diameter is small compared to the moment arm, and thus we use average values of radius and velocity at the outlet.

$$V_{jet} = \frac{\dot{V}_{nozzle}}{A_{jet}} = \frac{5\ \text{L/s}}{[\pi(0.01\ \text{m})^2/4]}\left(\frac{1\ \text{m}^3}{1000\ \text{L}}\right) = 63.66\ \text{m/s}$$

[redacted]

$$\omega = 2\pi\dot{n} = 2\pi(300 \text{ rev/min})\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 31.42 \text{ rad/s}$$

$$V_{\text{nozzle}} = r\omega = (0.6 \text{ m})(31.42 \text{ rad/s}) = 18.85 \text{ m/s}$$

$$V_r = V_{\text{jet}} - V_{\text{nozzle}} = 63.66 - 18.85 = 44.81 \text{ m/s}$$

$$\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$$

$$-\text{T}_{\text{shaft}} = -4r\dot{m}_{\text{nozzle}}V_r \qquad \text{or} \qquad \text{T}_{\text{shaft}} = r\dot{m}_{\text{total}}V_r$$

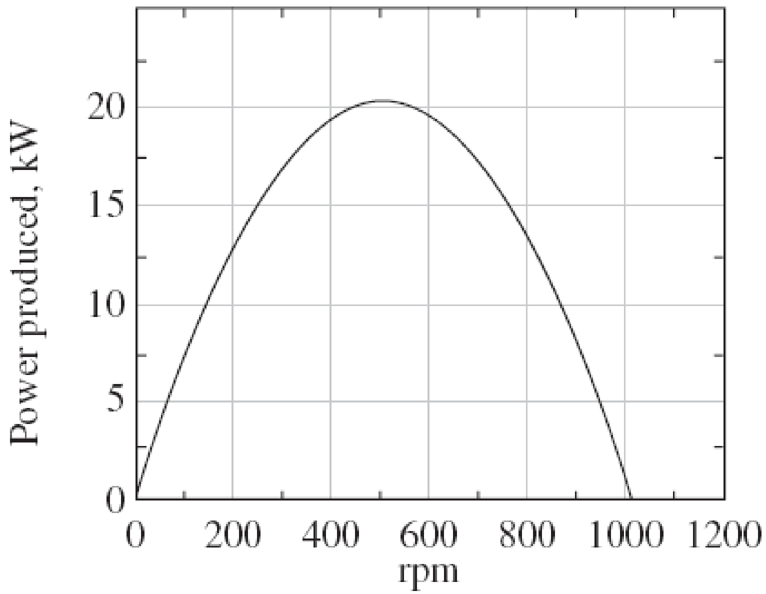
$$\text{T}_{\text{shaft}} = r\dot{m}_{\text{total}}V_r = (0.6 \text{ m})(20 \text{ kg/s})(44.81 \text{ m/s})\left(\frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2}\right) = 537.7 \text{ N}\cdot\text{m}$$

$$\dot{m}_{\text{total}} = \rho\dot{V}_{\text{total}} = (1 \text{ kg/L})(20 \text{ L/s}) = 20 \text{ kg/s}$$

$$\dot{W} = 2\pi\dot{n}\text{T}_{\text{shaft}} = \omega\text{T}_{\text{shaft}} = (31.42 \text{ rad/s})(537.7 \text{ N}\cdot\text{m})\left(\frac{1 \text{ kW}}{1000 \text{ N}\cdot\text{m/s}}\right) = \mathbf{16.9 \text{ kW}}$$

Discussion To put the result obtained in perspective, we consider two limiting cases. In the first limiting case, the sprinkler is stuck and thus the angular velocity is zero. The torque developed will be maximum in this case since V_{nozzle}=0 and thus V_r =V_{jet} = 63.66 m/s, giving $T_{shaft,\ max}$ =764 N · m. But the power generated will be zero since the shaft does not rotate.

In the second limiting case, the shaft is disconnected from the generator (and thus both the torque and power generation are zero) and the shaft accelerates until it reaches an equilibrium velocity. Setting T_{shaft} =0 in the angular momentum equation gives V_r =0 and thus V_{jet} =V_{nozzle} =63.66 m/s. The corresponding angular speed of the sprinkler is



$$\dot{n} = \frac{\omega}{2\pi} = \frac{V_{\text{nozzle}}}{2\pi r} = \frac{63.66 \text{ m/s}}{2\pi(0.6 \text{ m})}\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 1013 \text{ rpm}$$

The variation of power produced with angular speed for the turbine.

At this rpm, the velocity of the jet will be zero relative to an observer on earth (or relative to the fixed disk-shaped control volume selected). The variation of power produced with angular speed is plotted. Note that the power produced increases with increasing rpm, reaches a maximum (at about 500 rpm in this case), and then decreases. The actual power produced will be less than this due to generator inefficiency.

Summary

- Newton's Laws
- Choosing a Control Volume
- Forces Acting on a Control Volume
- The Linear Momentum Equation
  - Special Cases
  - Momentum-Flux Correction Factor, β
  - Steady Flow
  - Flow with No External Forces
- Review of Rotational Motion and Angular Momentum
- The Angular Momentum Equation
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