Heat and Mass Transfer, 3rd Edition
Yunus A. Cengel
McGraw-Hill, New York, 2007

CHAPTER 3
STEADY HEAT CONDUCTİON

Prof. Dr. Ali PINARBAŞI
Yildiz Technical University
Mechanical Engineering Department
Yildiz, ISTANBUL

OUTLINE

Steady Heat Conduction in Plane Walls

Thermal Contact Resistance

Generalized Thermal Resistance Networks

Heat Conduction in Cylinders and Spheres

Critical Radius of Insulation

Heat Transfer from Finned Surfaces

Heat Transfer in Common Configurations

Conclusions

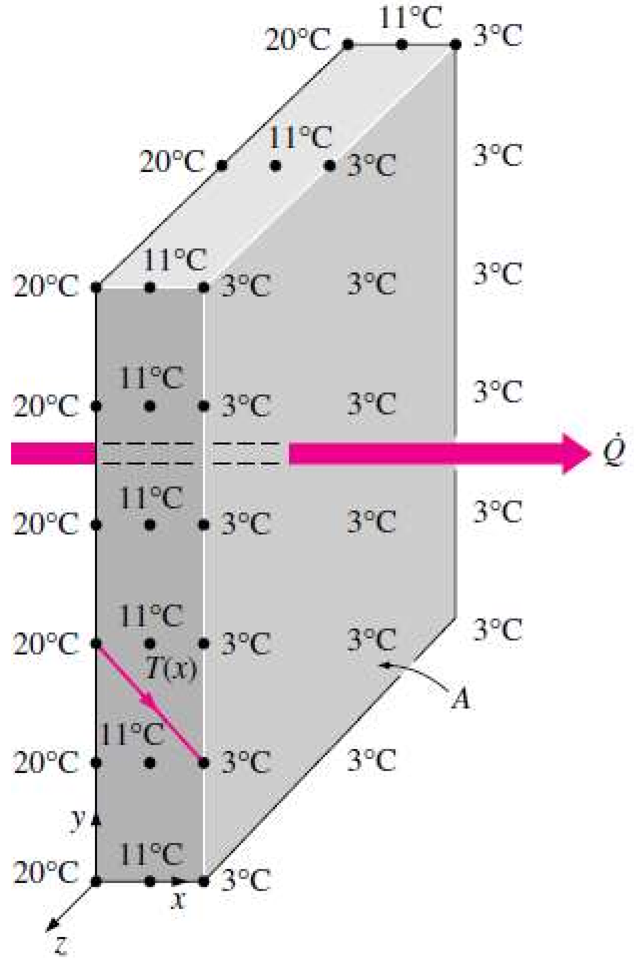
Steady Heat Conduction In Plane Walls

Heat transfer through the wall is in the *normal direction* to the wall surface, and no significant heat transfer takes place in the wall in other directions.

Heat transfer in a certain direction is driven by the *temperature gradient* in that direction.

There will be no heat transfer in a direction in which there is no change in temperature.

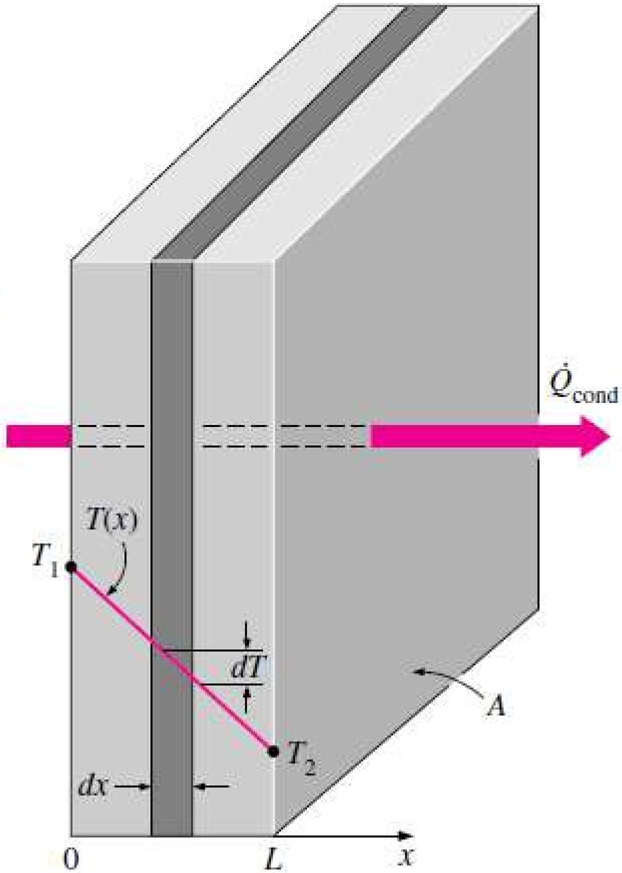
If the air temperatures in and outside the house remain constant, then heat transfer through the wall of a house can be modeled *as steady* and *one-dimensional.*



$$\int_{x=0}^{L} \dot{Q}_{\text{cond, wall}}\, dx = -\int_{T=T_1}^{T_2} kA\, dT$$

Integrating and rearranging

$$\dot{Q}_{\text{cond, wall}} = kA\frac{T_1 - T_2}{L} \qquad \text{(W)}$$



- Energy balance:

$$\begin{pmatrix}\text{Rate of}\\ \text{heat transfer}\\ \text{into the wall}\end{pmatrix} - \begin{pmatrix}\text{Rate of}\\ \text{heat transfer}\\ \text{out of the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{of the wall}\end{pmatrix}$$

$$\boxed{\dot{Q}_{\text{in}} - \dot{Q}_{\text{out}} = \frac{dE_{\text{wall}}}{dt}}$$

$\frac{dE_{wall}}{dt} = 0$ for *steady* operation and $\dot{Q}_{\text{cond,wall}}$=constant

- The Fourier's law of heat conduction for the wall:

$$\dot{Q}_{\text{cond, wall}} = -kA\frac{dT}{dx} \qquad \text{(W)}$$

where *dT/dx*= constant and *T varies linearly with x.*

The Thermal Resistance Concept

Heat conduction through a plane wall is

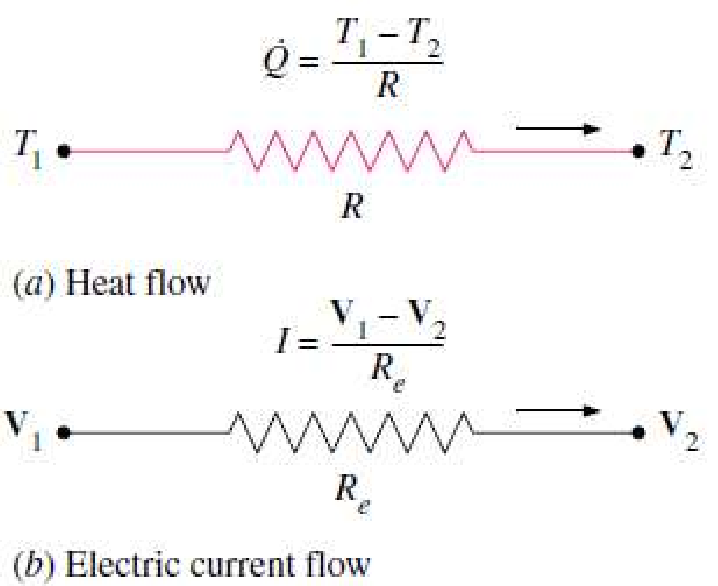
$$\dot{Q}_{\text{cond, wall}} = \frac{T_1 - T_2}{R_{\text{wall}}} \quad \text{(W)} \qquad \text{where} \qquad R_{\text{wall}} = \frac{L}{kA} \quad (^\circ\text{C/W})$$

is the *thermal resistance* of the wall against heat conduction (**conduction resistance**). The thermal resistance of a medium depends on the *geometry* and the *thermal properties* of the medium.

Taking into account analogous to the relation for *electric current flow I:*

$$I = \frac{\mathbf{V}_1 - \mathbf{V}_2}{R_e}$$

$R_e = L/\sigma_e A$: the *electric resistance*
$\mathbf{V}_1 - \mathbf{V}_2$: the *voltage difference* across the resistance
σ_e : the electrical conductivity).



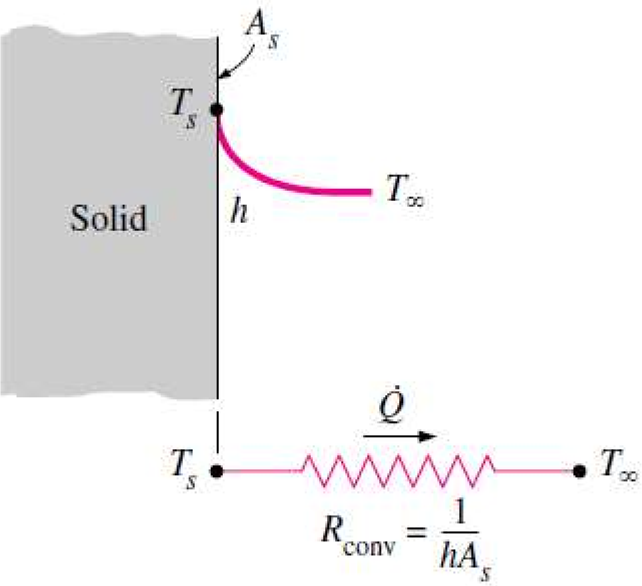
Newton's law of cooling for convection heat transfer rate:

$$\dot{Q}_{conv} = hA_s(T_s - T_\infty)$$

can be rearranged as

$$\dot{Q}_{conv} = \frac{T_s - T_\infty}{R_{conv}} \quad \text{(W)} \quad \text{with} \quad R_{conv} = \frac{1}{hA_s} \quad (^\circ\text{C/W})$$

which is the *thermal resistance* of the surface against heat convection, or simply the **convection resistance** of the surface.



When the convection heat transfer coefficient is very large $(h \to \infty)$, the convection resistance becomes *zero* and $T_s \approx T_\infty$. That is, the surface offers *no resistance to convection*, and thus it does not slow down the heat transfer process.

The rate of radiation heat transfer between a surface of emissivity ε and area A_s at temperature T_s and the surrounding surfaces at some average temperature T_{surr} can be expressed as

$$\dot{Q}_{rad} = \varepsilon\sigma A_s(T_s^4 - T_{surr}^4) = h_{rad} A_s(T_s - T_{surr}) = \frac{T_s - T_{surr}}{R_{rad}} \qquad \text{(W)}$$

with $$R_{rad} = \frac{1}{h_{rad} A_s} \qquad \text{(K/W)}$$

which is the *radiation resistance.*

$$h_{rad} = \frac{\dot{Q}_{rad}}{A_s(T_s - T_{surr})} = \varepsilon\sigma(T_s^2 + T_{surr}^2)(T_s + T_{surr}) \qquad (\text{W/m}^2 \cdot \text{K})$$

is the radiation heat transfer coefficient.

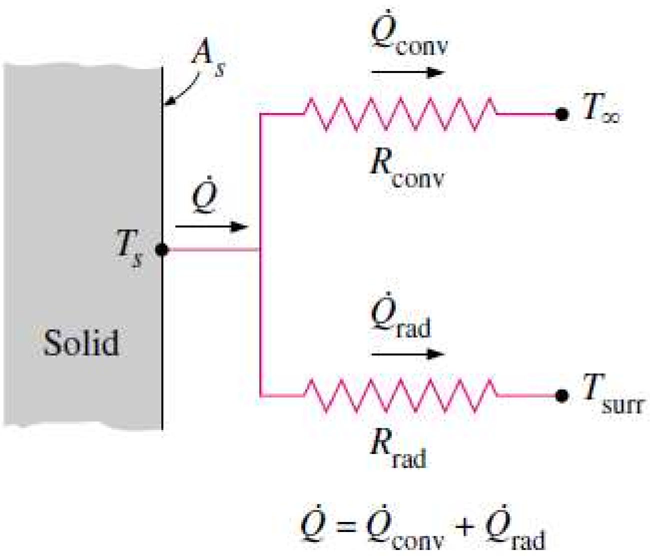
Both T_s and T_{surr} *must* be in K in the evaluation of h_{rad}.

The convection and radiation resistances are parallel to each other, and may cause some complication in the thermal resistance network.

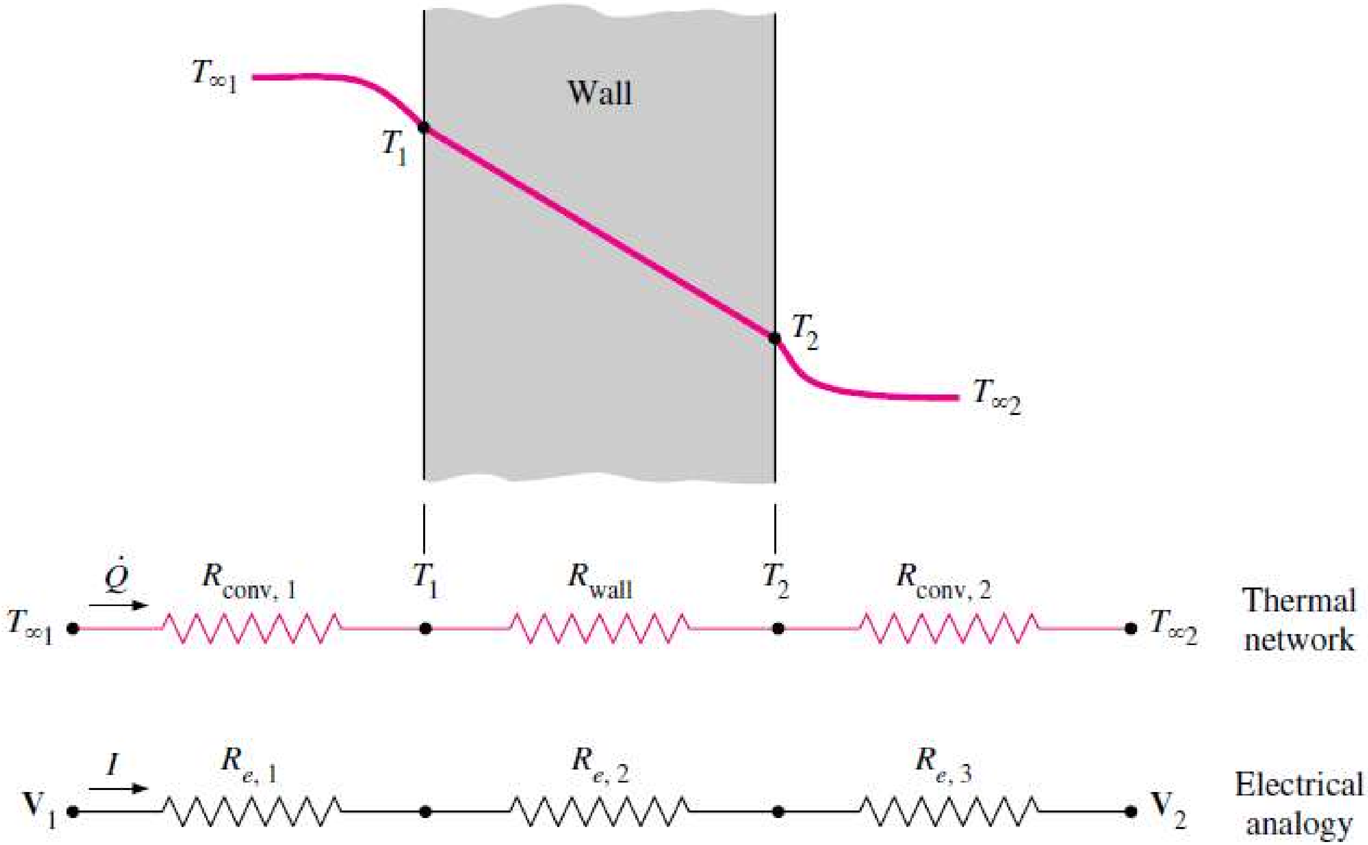
When $T_{surr} \approx T_\infty$, the radiation effect can properly be accounted for by replacing h in the convection resistance relation by

$$h_{\text{combined}} = h_{\text{conv}} + h_{\text{rad}} \qquad (\text{W/m}^2 \cdot \text{K})$$

where h_{combined} is the combined heat transfer coefficient.



Thermal Resistance Network



$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{conv,\,1} + R_{wall} + R_{conv,\,2}}$$

$$I = \frac{V_1 - V_2}{R_{e,\,1} + R_{e,\,2} + R_{e,\,3}}$$

The thermal resistance network for heat transfer through a plane wall subjected to convection on both sides, and the electrical analogy.

Under steady conditions

$$\begin{pmatrix}\text{Rate of} \\ \textit{heat convection} \\ \text{into the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of} \\ \textit{heat conduction} \\ \text{through the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of} \\ \textit{heat convection} \\ \text{from the wall}\end{pmatrix}$$

$$\dot{Q} = h_1 A(T_{\infty 1} - T_1) = kA \frac{T_1 - T_2}{L} = h_2 A(T_2 - T_{\infty 2})$$

which can be rearranged as

$$\dot{Q} = \frac{T_{\infty 1} - T_1}{1/h_1 A} = \frac{T_1 - T_2}{L/kA} = \frac{T_2 - T_{\infty 2}}{1/h_2 A}$$

$$= \frac{T_{\infty 1} - T_1}{R_{\text{conv, 1}}} = \frac{T_1 - T_2}{R_{\text{wall}}} = \frac{T_2 - T_{\infty 2}}{R_{\text{conv, 2}}}$$

Adding the numerators and denominators yields

$$\dot{Q} = \frac{T_{\infty} - T_{\infty 2}}{R_{\text{total}}} \qquad \text{(W)}$$

$$R_{\text{total}} = R_{\text{conv, 1}} + R_{\text{wall}} + R_{\text{conv, 2}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A} \qquad (^{\circ}\text{C/W})$$

The thermal resistances are in *series,* and the equivalent thermal resistance is determined by simply *adding* the individual resistances, just like the electrical resistances connected in series.

The equation $\dot{Q} = \Delta T/R$ can be rearranged as $\Delta T = \dot{Q}R \qquad (°\text{C})$

Here, the *temperature drop* across any layer is equal to the *rate of heat transfer* times the *thermal resistance* across that layer.

If

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n} = c$$

then

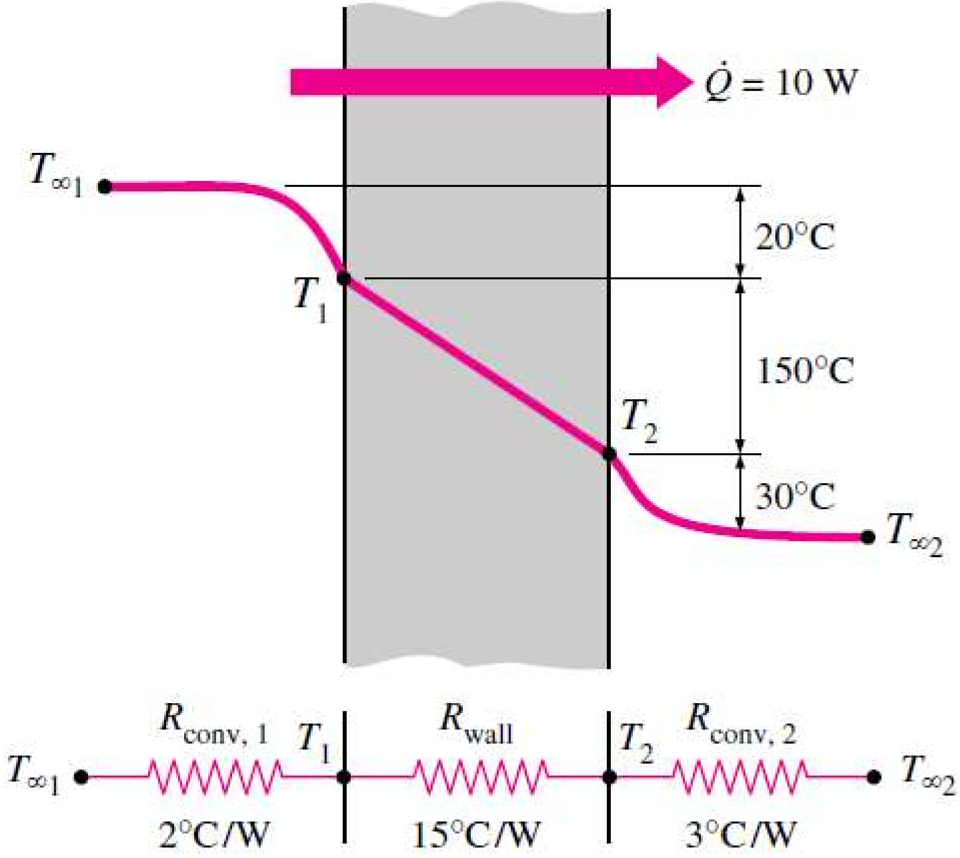
$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} = c$$

For example,

$$\frac{1}{4} = \frac{2}{8} = \frac{5}{20} = 0.25$$

and

$$\frac{1 + 2 + 5}{4 + 8 + 20} = 0.25$$



$\Delta T = \dot{Q}R$

Analogous to Newton's law of cooling as

$$\dot{Q} = UA\,\Delta T \qquad \text{(W)}$$

U: the overall heat transfer coefficient

$$UA = \frac{1}{R_{\text{total}}}$$

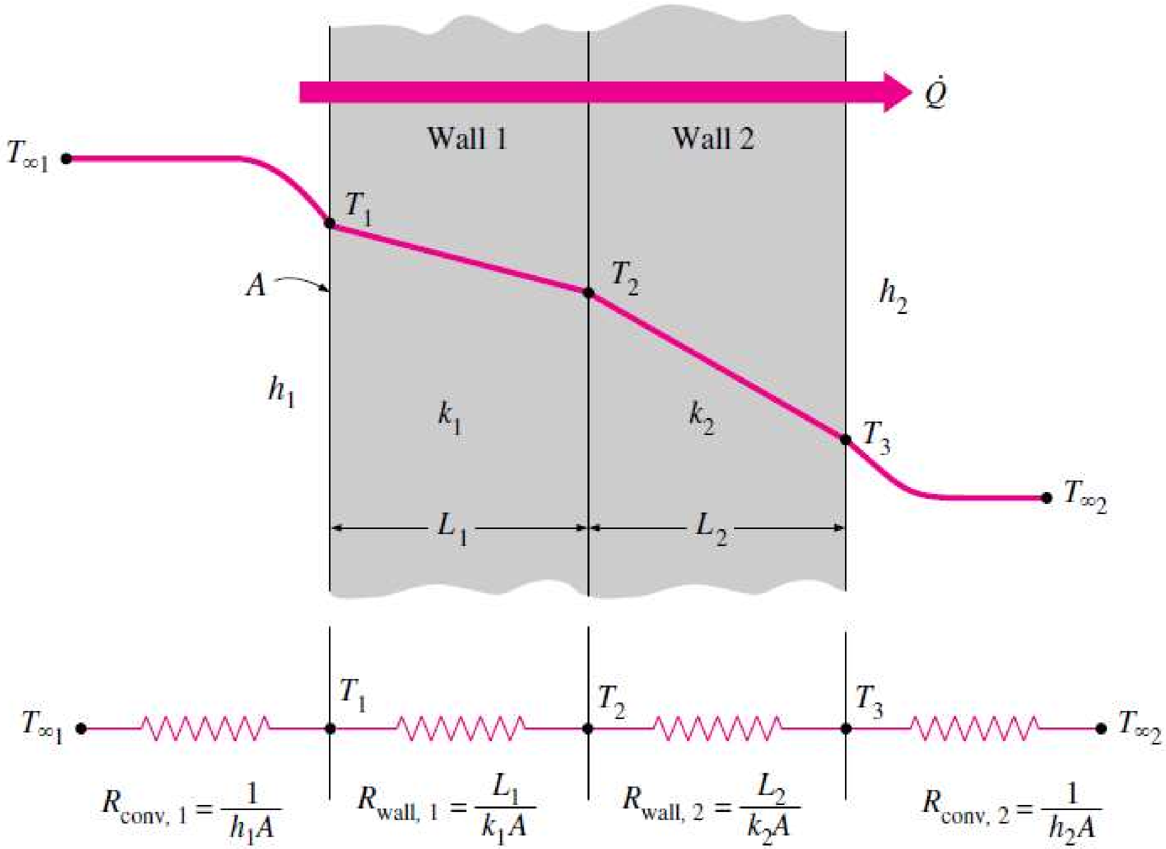
Multilayer Plane Walls

The rate of steady heat transfer through a plane wall consisting of two layers

R_{total}: the *total thermal resistance*

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{total}}$$

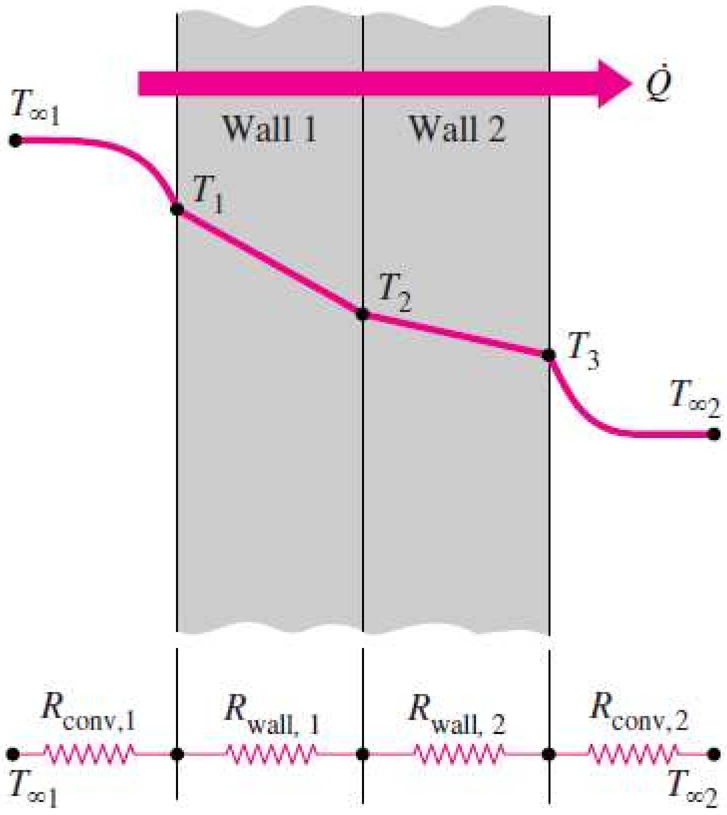
$$R_{total} = R_{conv,1} + R_{wall,1} + R_{wall,2} + R_{conv,2} = \frac{1}{h_1 A} + \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{1}{h_2 A}$$



for the resistances *in series*.

$$\dot{Q} = \frac{T_{\infty 1} - T_2}{R_{\text{conv, 1}} + R_{\text{wall, 1}}} = \frac{T_{\infty 1} - T_2}{\dfrac{1}{h_1 A} + \dfrac{L_1}{k_1 A}}$$

☞ It is limited to systems involving *steady* heat transfer with *no heat generation*.



To find T_1: $\dot{Q} = \dfrac{T_{\infty 1} - T_1}{R_{\text{conv},1}}$

To find T_2: $\dot{Q} = \dfrac{T_{\infty 1} - T_2}{R_{\text{conv},1} + R_{\text{wall, 1}}}$

To find T_3: $\dot{Q} = \dfrac{T_3 - T_{\infty 2}}{R_{\text{conv},2}}$

GENERALIZED THERMAL RESISTANCE NETWORKS

For the composite wall consisting of two parallel layers, the total heat transfer is the sum of the heat transfers through each layer.

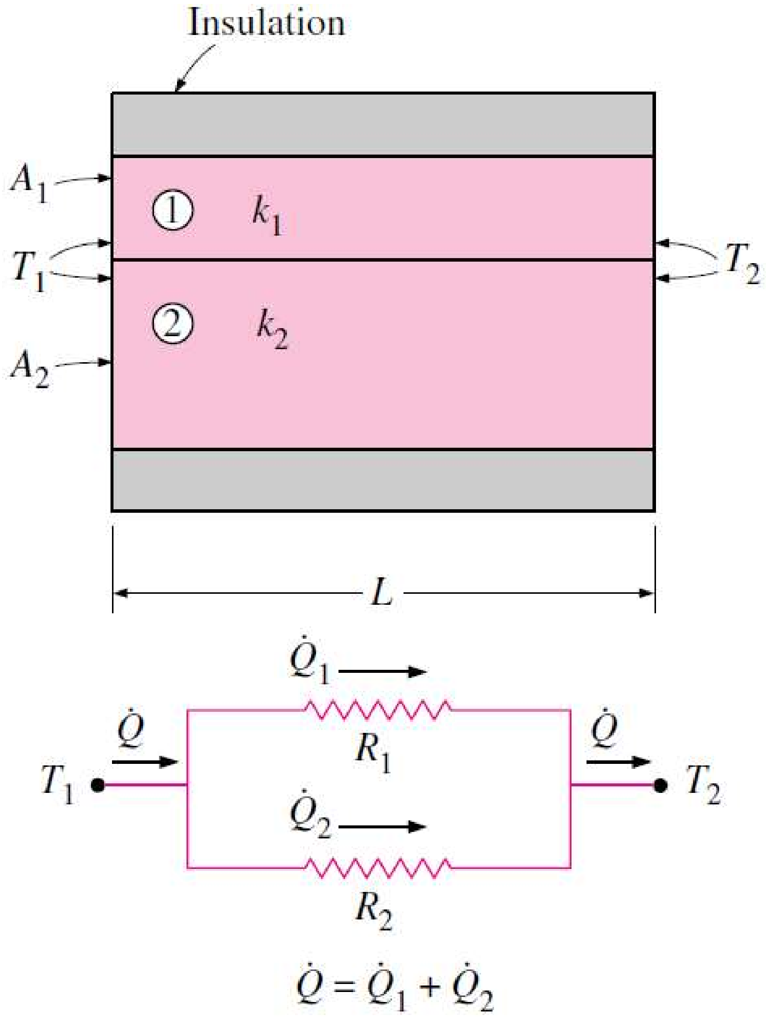
$$\dot{Q} = \dot{Q}_1 + \dot{Q}_2 = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2} = (T_1 - T_2)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

With the electrical analogy

$$\dot{Q} = \frac{T_1 - T_2}{R_{\text{total}}}$$

with

$$\frac{1}{R_{\text{total}}} = \frac{1}{R_1} + \frac{1}{R_2} \longrightarrow R_{\text{total}} = \frac{R_1 R_2}{R_1 + R_2}$$



Thermal resistance network for two parallel layers.

For the combined series-parallel arrangement, the total rate of heat transfer through this composite system is

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{\text{total}}}$$

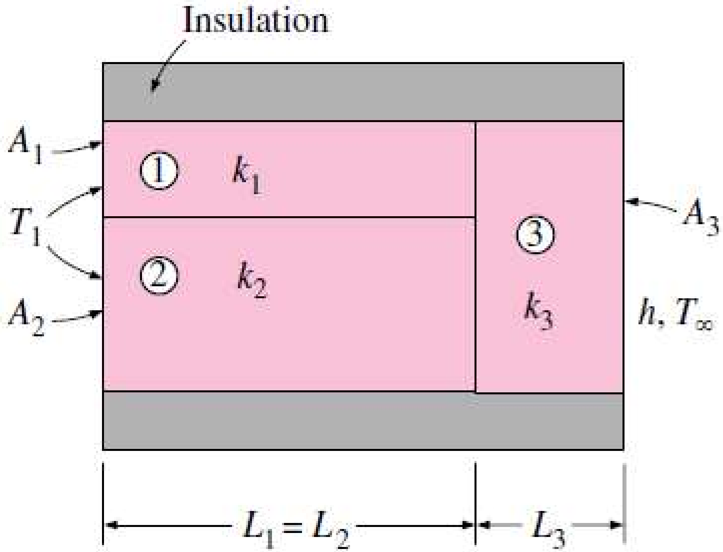
with

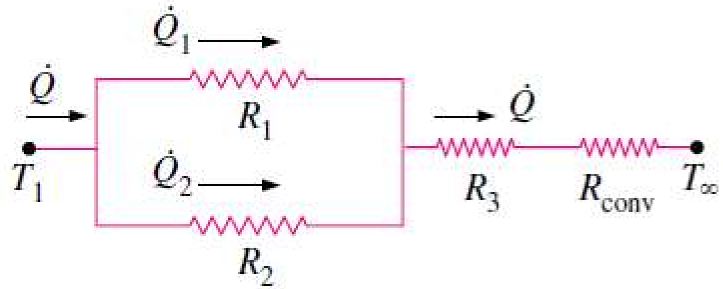
$$\boxed{R_{\text{total}} = R_{12} + R_3 + R_{\text{conv}} = \frac{R_1 R_2}{R_1 + R_2} + R_3 + R_{\text{conv}}}$$

and

$$R_1 = \frac{L_1}{k_1 A_1} \qquad R_2 = \frac{L_2}{k_2 A_2}$$

$$R_3 = \frac{L_3}{k_3 A_3} \qquad R_{\text{conv}} = \frac{1}{hA_3}$$





Two assumptions:

(i) any plane wall normal to the *x*-axis is *isothermal* and

(ii) any plane parallel to the *x*-axis is *adiabatic.*

These assumptions result in different resistance networks, while the actual result lies between two assumptions.

HEAT CONDUCTION IN CYLINDERS AND SPHERES

The Fourier's law of heat conduction for heat transfer through the cylindrical layer is

$$\dot{Q}_{\text{cond, cyl}} = -kA\frac{dT}{dr} \qquad \text{(W)}$$

Here, A=$2\pi rL$ is the heat transfer area at location r

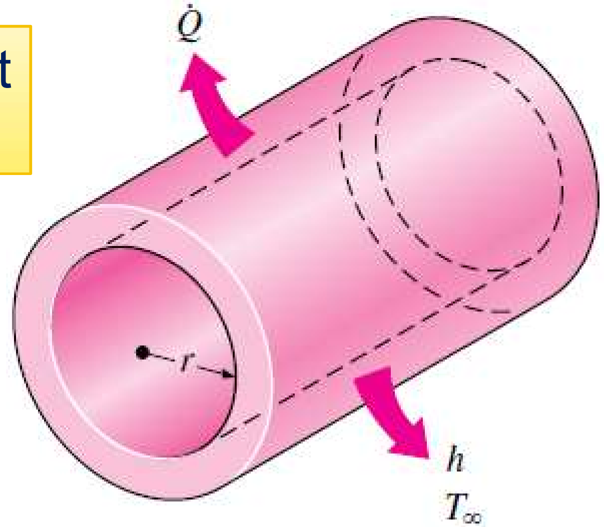
$$\int_{r=r_1}^{r_2} \frac{\dot{Q}_{\text{cond, cyl}}}{A}\, dr = -\int_{T=T_1}^{T_2} k\, dT$$

We obtain

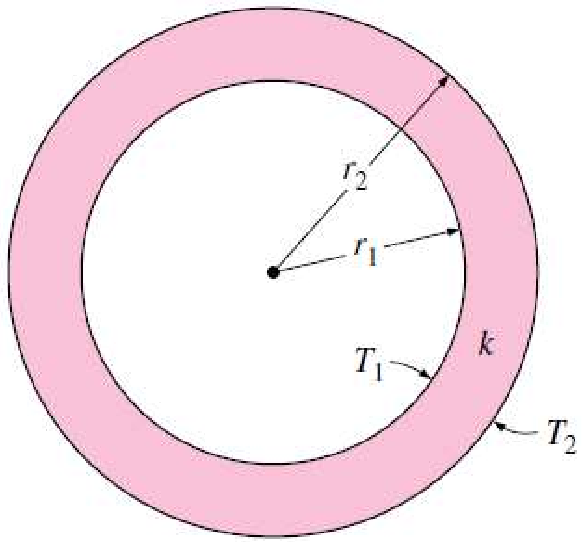
$$\dot{Q}_{\text{cond, cyl}} = 2\pi Lk\frac{T_1 - T_2}{\ln(r_2/r_1)}$$

$$\boxed{\dot{Q}_{\text{cond, cyl}} = \frac{T_1 - T_2}{R_{\text{cyl}}} \qquad \text{(W)}}$$

Since $\dot{Q}_{\text{cond,cyl}}$= constant.



Heat is lost from a hot-water pipe to the air outside in the radial direction, and thus heat transfer from a long pipe is 1-D.



The *thermal resistance* of the cylindrical layer against heat conduction, or simply the **conduction resistance** of the cylinder layer.

$$R_{cyl} = \frac{\ln(r_2/r_1)}{2\pi Lk} = \frac{\ln(\text{Outer radius/Inner radius})}{2\pi \times (\text{Length}) \times (\text{Thermal conductivity})}$$

Repeating the analysis for a *spherical layer* by taking A=4πr^2

$$\dot{Q}_{cond,\,sph} = \frac{T_1 - T_2}{R_{sph}}$$

$$R_{sph} = \frac{r_2 - r_1}{4\pi r_1 r_2 k} = \frac{\text{Outer radius} - \text{Inner radius}}{4\pi(\text{Outer radius})(\text{Inner radius})(\text{Thermal conductivity})}$$

which is the *thermal resistance* of the spherical layer against heat conduction, or simply the **conduction resistance** of the spherical layer.

The rate of heat transfer through a cylindrical or spherical layer under steady conditions:

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}}$$

for a *cylindrical* layer:

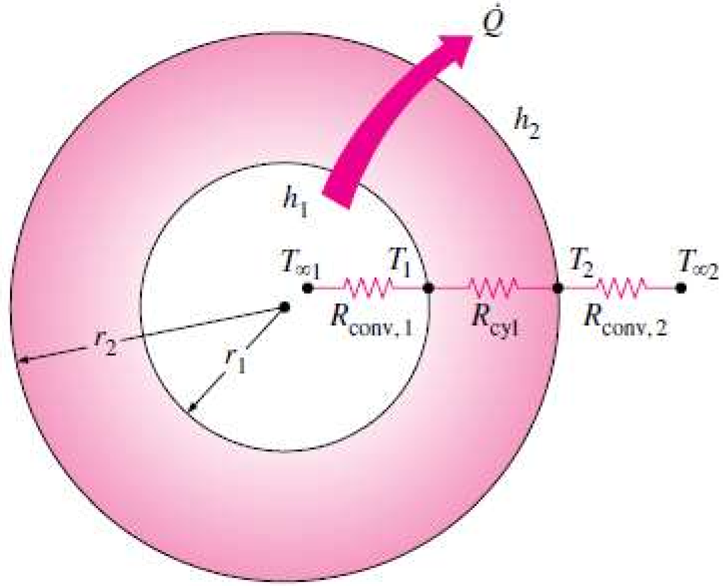
$$R_{\text{total}} = R_{\text{conv},1} + R_{\text{cyl}} + R_{\text{conv},2} = \frac{1}{(2\pi r_1 L)h_1} + \frac{\ln(r_2/r_1)}{2\pi L k} + \frac{1}{(2\pi r_2 L)h_2}$$

for *a spherical* layer:

$$R_{\text{total}} = R_{\text{conv},1} + R_{\text{sph}} + R_{\text{conv},2} = \frac{1}{(4\pi r_1^2)h_1} + \frac{r_2 - r_1}{4\pi r_1 r_2 k} + \frac{1}{(4\pi r_2^2)h_2}$$

A in the convection resistance relation $R_{conv} = 1/hA$ is the *surface area at which convection occurs.*

It is equal to $A = 2\pi rL$ for a cylindrical surface and $A = 4\pi r^2$ for a spherical surface of radius *r*.



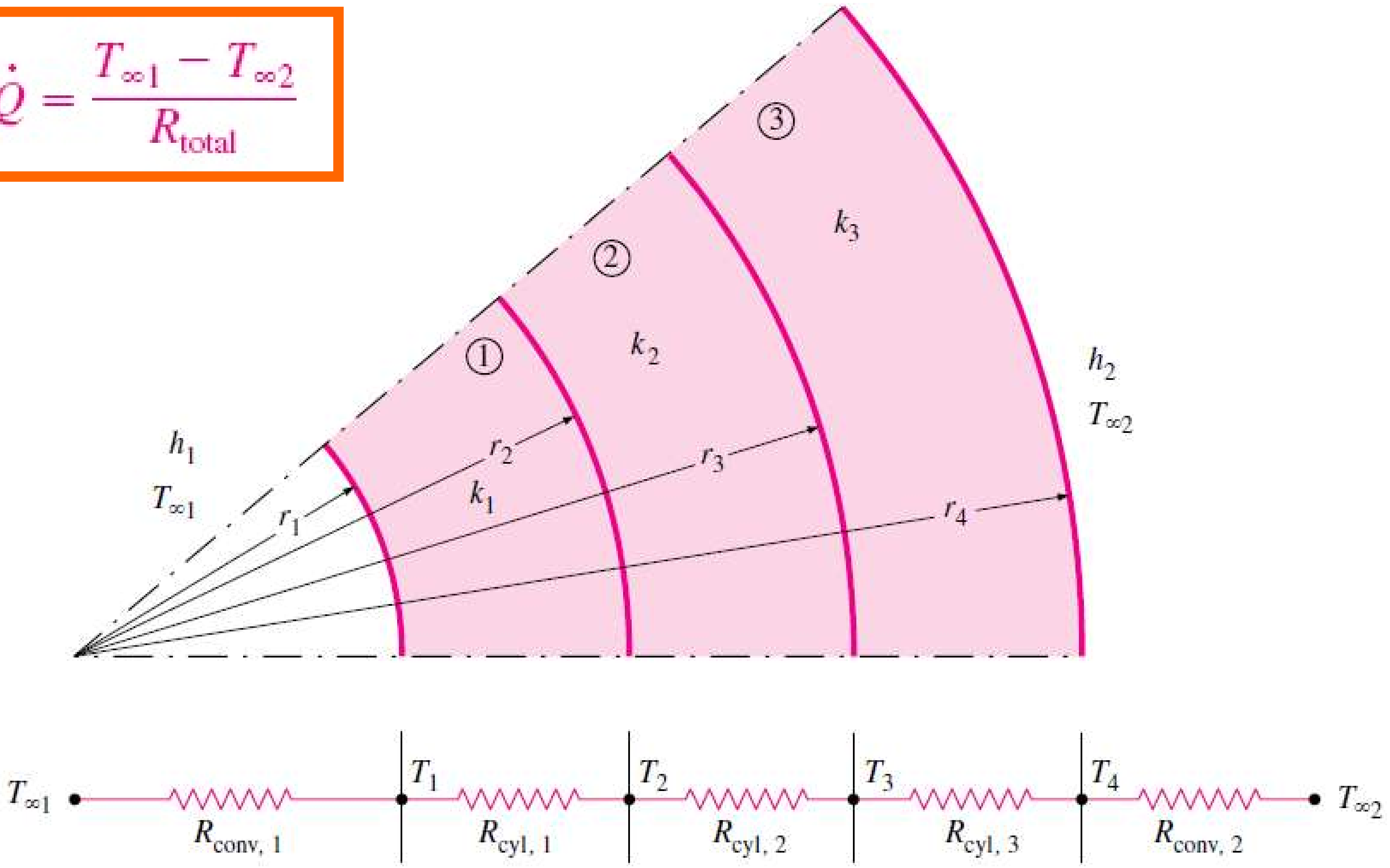
$$R_{\text{total}} = R_{\text{conv},1} + R_{\text{cyl}} + R_{\text{conv},2}$$

The thermal resistance network for a cylindrical (or spherical) shell subjected to convection from both the inner and the outer sides.

Multilayered Cylinders and Spheres

Steady heat transfer through multilayered cylindrical or spherical shells is treated like multilayered plane walls.

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{\text{total}}}$$



three-layered composite cylinder

R_{total} is the *total thermal resistance,* expressed as

$$R_{\text{total}} = R_{\text{conv, 1}} + R_{\text{cyl, 1}} + R_{\text{cyl, 2}} + R_{\text{cyl, 3}} + R_{\text{conv, 2}}$$
$$= \frac{1}{h_1 A_1} + \frac{\ln(r_2/r_1)}{2\pi L k_1} + \frac{\ln(r_3/r_2)}{2\pi L k_2} + \frac{\ln(r_4/r_3)}{2\pi L k_3} + \frac{1}{h_2 A_4}$$

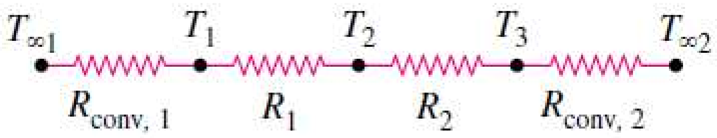
Here, $A_1 = 2\,\pi r_1 L$ and $A_4 = 2\,\pi r_4 L$

The total thermal resistance is simply the *arithmetic sum* of the individual thermal resistances in the path of heat flow

$$\dot{Q} = \frac{T_{\infty 1} - T_2}{R_{\text{conv, 1}} + R_{\text{cyl, 1}}} = \frac{T_{\infty 1} - T_2}{\dfrac{1}{h_1(2\pi r_1 L)} + \dfrac{\ln(r_2/r_1)}{2\pi L k_1}}$$

We can also calculate T_2 from

$$\dot{Q} = \frac{T_2 - T_{\infty 2}}{R_2 + R_3 + R_{\text{conv, 2}}} = \frac{T_2 - T_{\infty 2}}{\dfrac{\ln(r_3/r_2)}{2\pi L k_2} + \dfrac{\ln(r_4/r_3)}{2\pi L k_3} + \dfrac{1}{h_o(2\pi r_4 L)}}$$



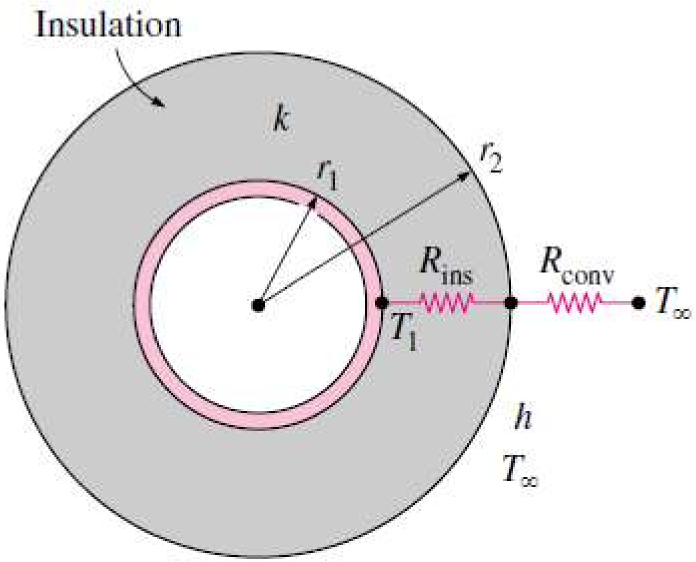
$$\dot{Q} = \frac{T_{\infty 1} - T_1}{R_{\text{conv,1}}}$$
$$= \frac{T_{\infty 1} - T_2}{R_{\text{conv,1}} + R_1}$$
$$= \frac{T_1 - T_3}{R_1 + R_2}$$
$$= \frac{T_2 - T_3}{R_2}$$
$$= \frac{T_2 - T_{\infty 2}}{R_2 + R_{\text{conv,2}}}$$
$$= \cdots$$

The ratio $\Delta T/R$ across any layer is equal to $\dot{Q}$ which remains constant in 1-D steady conduction.

CRITICAL RADIUS OF INSULATION

The rate of heat transfer from the insulated pipe to the surrounding air is

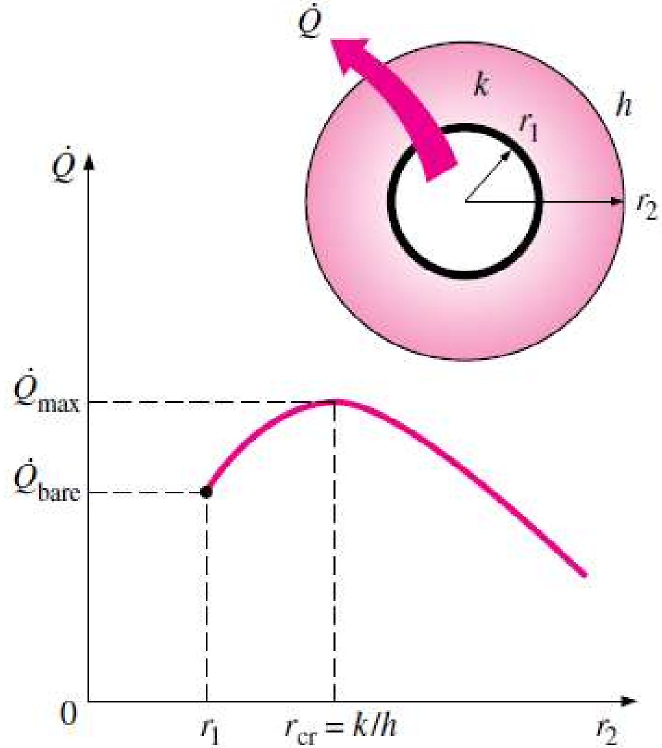
$$\dot{Q} = \frac{T_1 - T_\infty}{R_{ins} + R_{conv}} = \frac{T_1 - T_\infty}{\dfrac{\ln(r_2/r_1)}{2\pi L k} + \dfrac{1}{h(2\pi r_2 L)}}$$



Performing the differentiation and solving for r_2 yields the **critical radius of insulation:** for a to be

cylindrical body $$r_{cr,\ cylinder} = \frac{k}{h} \quad \text{(m)}$$

spherical body $$r_{cr,\ sphere} = \frac{2k}{h}$$



k : the thermal conductivity of the insulation
h : the convection heat transfer coefficient on the outer surface

HEAT TRANSFER FROM FINNED SURFACES

The rate of heat transfer from a surface at a temperature T_s to the surrounding medium at T is given by Newton's law of cooling as

$$\dot{Q}_{conv} = hA_s(T_s - T_\infty)$$

As : the heat transfer surface area
h : the convection heat transfer coefficient

There are *two ways* to increase the rate of heat transfer:

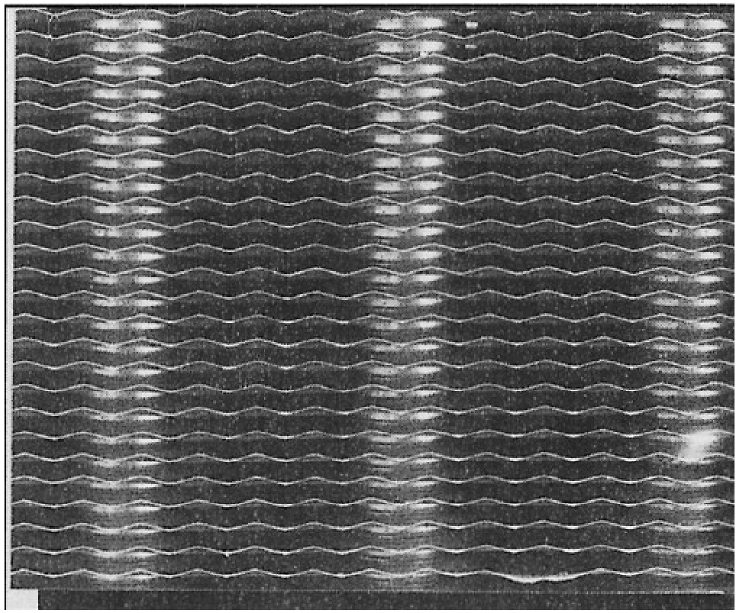
1) to increase the *convection heat transfer coefficient h*
2) to increase the *surface area* A_s

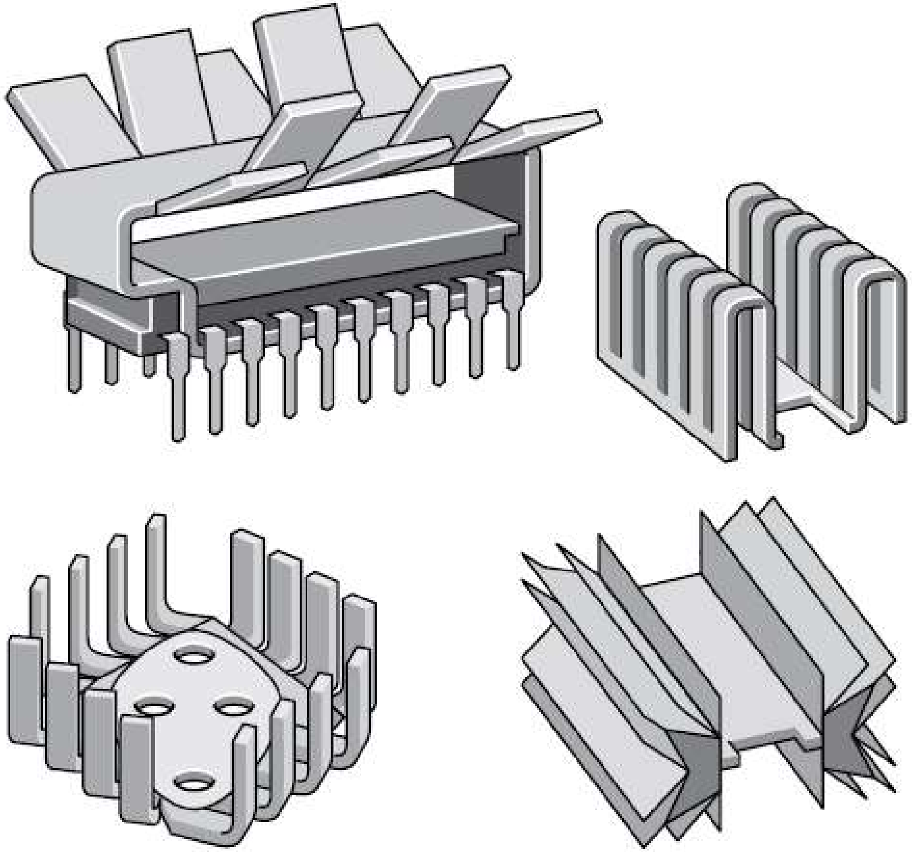
Increasing *h* may require the installation of a pump or fan, or replacing the existing one with a larger one, but this approach may or may not be practical. Besides, it may not be adequate.

The alternative is to increase the surface area by attaching to the surface ***extended surfaces* called *fins*** made of highly conductive materials such as aluminum.

Consider *steady* operation with *no heat generation* in the fin with the following assumptions:

• The thermal **conductivity** *k* of the material remains constant.
• The convection heat transfer coefficient *h is constant* and *uniform* over the entire surface of the fin for convenience in the analysis.





Some innovative fin designs

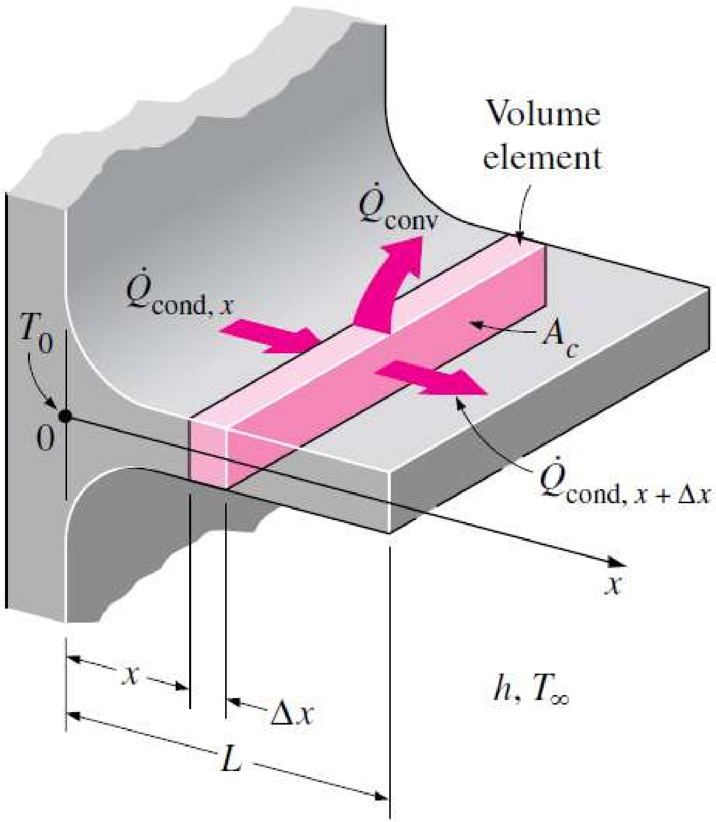
Fin Equation

Under steady conditions, the energy balance on this volume element can be expressed as

$$\begin{pmatrix} \text{Rate of } \textit{heat} \\ \textit{conduction} \text{ into} \\ \text{the element at } x \end{pmatrix} = \begin{pmatrix} \text{Rate of } \textit{heat} \\ \textit{conduction} \text{ from the} \\ \text{element at } x + \Delta x \end{pmatrix} + \begin{pmatrix} \text{Rate of } \textit{heat} \\ \textit{convection} \text{ from} \\ \text{the element} \end{pmatrix}$$

$$\dot{Q}_{\text{cond}, x} = \dot{Q}_{\text{cond}, x + \Delta x} + \dot{Q}_{\text{conv}}$$

$$\dot{Q}_{\text{conv}} = h(p\,\Delta x)(T - T_\infty)$$



Substituting and dividing by Δ*x*, we obtain

$$\frac{\dot{Q}_{\text{cond}, x + \Delta x} - \dot{Q}_{\text{cond}, x}}{\Delta x} + hp(T - T_\infty) = 0$$

Taking the limit as Δ*x* → 0 gives

$$\frac{d\dot{Q}_{\text{cond}}}{dx} + hp(T - T_\infty) = 0$$

From Fourier's law of heat conduction we have

$$\dot{Q}_{\text{cond}} = -kA_c \frac{dT}{dx}$$

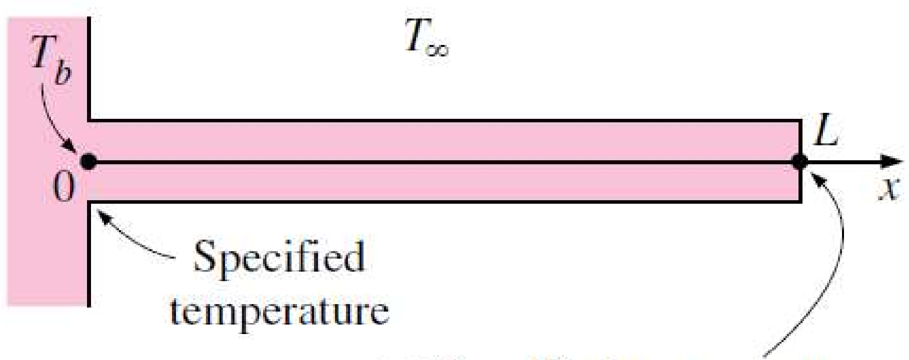
A_c: the cross-sectional area of the fin at location x

$$\frac{d}{dx}\left(kA_c \frac{dT}{dx}\right) - hp(T - T_\infty) = 0$$

In the special case (with *constant cross section* and *thermal conductivity)*:

$$\frac{d^2\theta}{dx^2} - a^2\theta = 0 \qquad a^2 = \frac{hp}{kA_c}$$

and $\theta = T - T_\infty$ is the *temperature excess*. At the fin base we have $\theta_b = T_b - T_\infty$.



(a) Specified temperature
(b) Negligible heat loss
(c) Convection
(d) Convection and radiation

The function u and its second derivative must be *constant multiples* of each other.

$$\theta(x) = C_1 e^{ax} + C_2 e^{-ax}$$

where C_1 and C_2 are arbitrary constants.

Boundary condition at fin base: $\qquad \theta(0) = \theta_b = T_b - T_\infty$

Infinitely Long Fin ($T_{\text{fin tip}} = T_\infty$)

For a sufficiently long fin of *uniform* cross section (A_c= Constant):

Boundary condition at fin tip: $\theta(L) = T(L) - T_\infty = 0$ as $L \rightarrow \infty$

Very long fin:

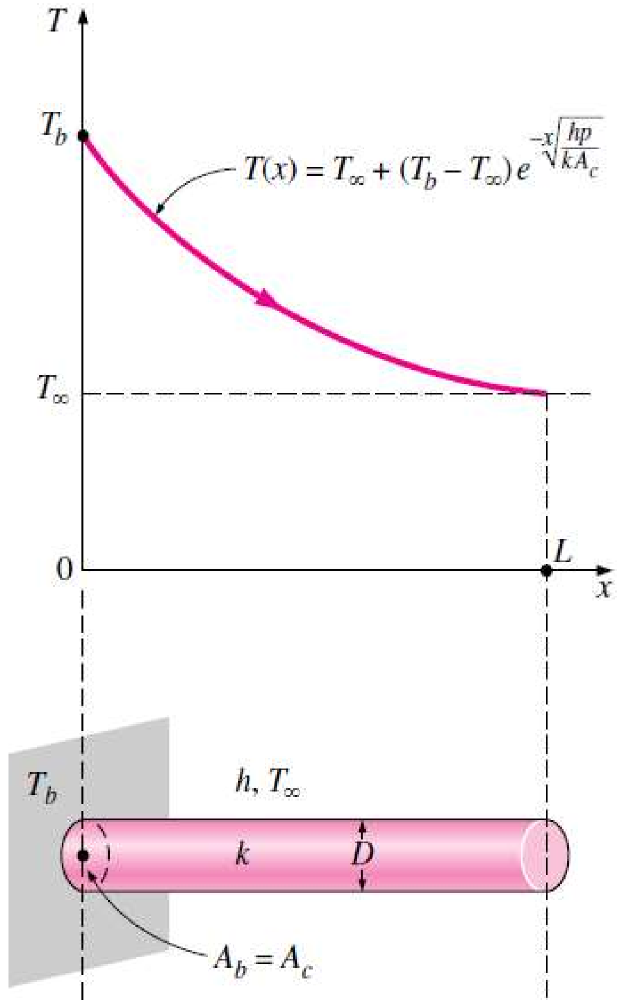
$$\frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-ax} = e^{-x\sqrt{hp/kA_c}}$$

Very long fin:

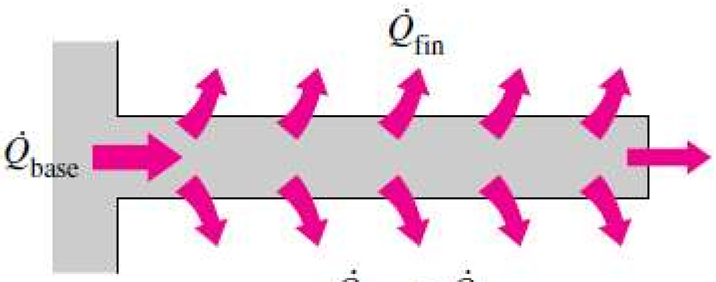
$$\dot{Q}_{\text{long fin}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = \sqrt{hpkA_c}\,(T_b - T_\infty)$$

p : the perimeter
A_c : the cross-sectional area of the fin
x : the distance from the fin base

$$\dot{Q}_{\text{fin}} = \int_{A_{\text{fin}}} h[T(x) - T_\infty]\, dA_{\text{fin}} = \int_{A_{\text{fin}}} h\theta(x)\, dA_{\text{fin}}$$



($p = \pi D$, $A_c = \pi D^2/4$ for a cylindrical fin)



$\dot{Q}_{\text{base}} = \dot{Q}_{\text{fin}}$

Negligible Heat Loss from the Fin Tip (Insulated fin tip, $\dot{Q}_{\text{fin tip}}$ =0)

The fin tip can be assumed to be insulated, and the condition at the fin tip can be expressed as

Boundary condition at fin tip: $$\left.\frac{d\theta}{dx}\right|_{x=L} = 0$$

Adiabatic fin tip: $$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh a(L - x)}{\cosh aL}$$

The rate of heat transfer from the fin can be determined again from Fourier's law of heat conduction:

Adiabatic fin tip: $$\dot{Q}_{\text{insulated tip}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = \sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL$$

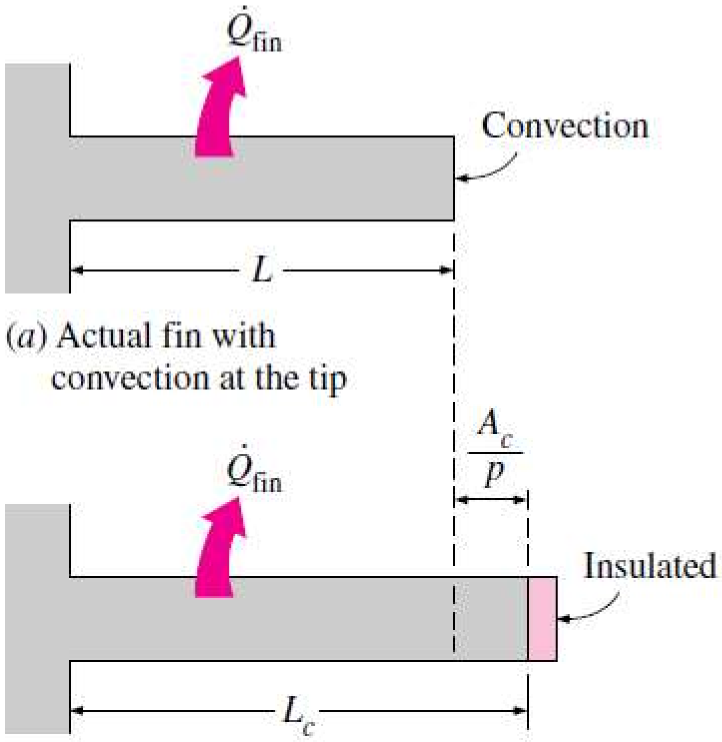
The heat transfer relations for the very long fin and the fin with negligible heat loss at the tip differ by the factor tanh aL, which approaches 1 as L becomes very large.

Convection (or Combined Convection and Radiation) from Fin Tip

A practical way of accounting for the heat loss from the fin tip is to replace the *fin length L* in the relation for the *insulated tip* case by a **corrected length** defined as

$$L_c = L + \frac{A_c}{p}$$

t: the thickness of the rectangular fins
D: the diameter of the cylindrical fins.



(*a*) Actual fin with convection at the tip
(*b*) Equivalent fin with insulated tip

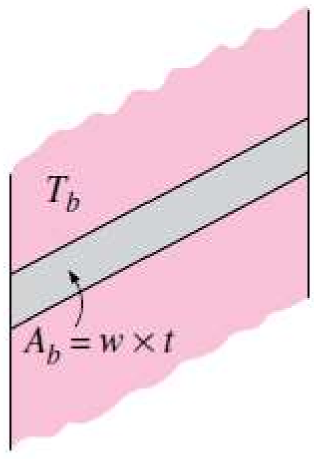
Corrected fin length L_c is defined such that heat transfer from a fin of length L_c with insulated tip is equal to heat transfer from the actual fin of length *L* with convection at the fin tip.

Fin Efficiency

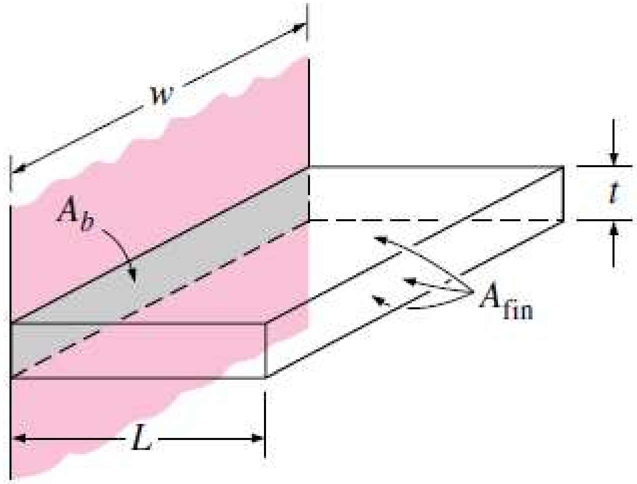
In the limiting case of *zero thermal resistance* or *infinite thermal conductivity*, ($k \to \infty$) the temperature of the fin will be uniform at the base value of T_b.

The heat transfer from the fin will be *maximum* in this case and can be expressed as

$$\dot{Q}_{\text{fin, max}} = hA_{\text{fin}}(T_b - T_\infty)$$



(*a*) Surface without fins



(*b*) Surface with a fin

$$A_{\text{fin}} = 2 \times w \times L + w \times t$$
$$\cong 2 \times w \times L$$

Fin efficiency:

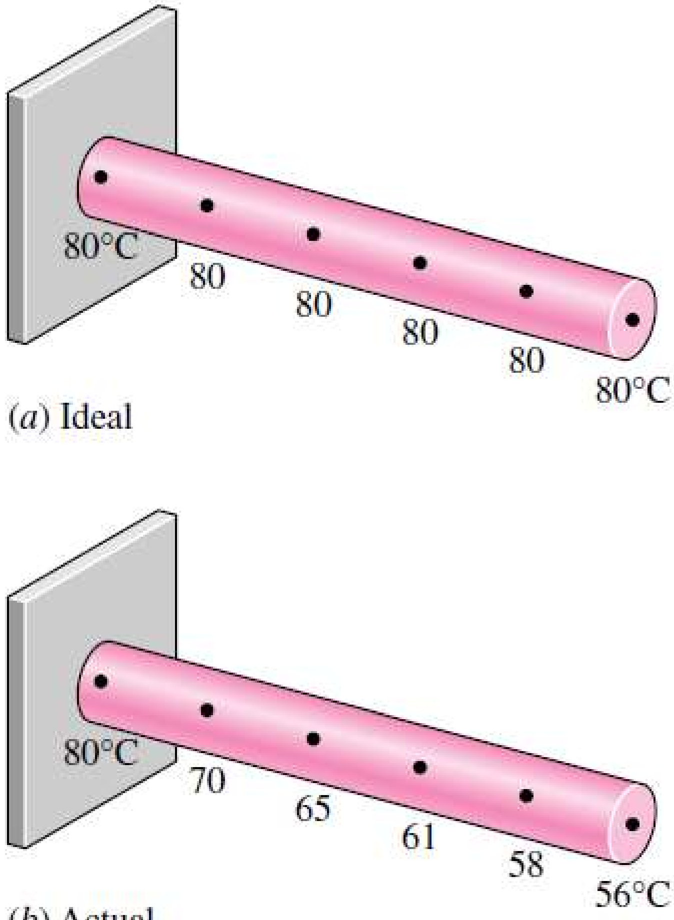
$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\text{Actual heat transfer rate from the fin}}{\begin{array}{c}\text{Ideal heat transfer rate from the fin}\\ \text{if the entire fin were at base temperature}\end{array}}$$

$$\dot{Q}_{\text{fin}} = \eta_{\text{fin}}\,\dot{Q}_{\text{fin, max}} = \eta_{\text{fin}}\,hA_{\text{fin}}\,(T_b - T_\infty)$$

For the cases of constant cross section of ***very long fins and fins with insulated tips***, the fin efficiency can be expressed as

$$\eta_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_{\text{fin}}\,(T_b - T_\infty)} = \frac{1}{L}\sqrt{\frac{kA_c}{hp}} = \frac{1}{aL}$$

$$\eta_{\text{insulated tip}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL}{hA_{\text{fin}}\,(T_b - T_\infty)} = \frac{\tanh aL}{aL}$$



(*a*) Ideal

(*b*) Actual

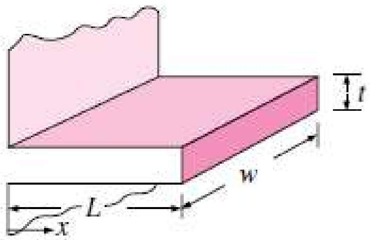
since $A_{\text{fin}} = pL$ for fins with constant cross section.

Efficiency and surface areas of common fin configurations

Straight rectangular fins

$$m = \sqrt{2h/kt}$$
$$L_c = L + t/2$$
$$A_{\text{fin}} = 2wL_c$$

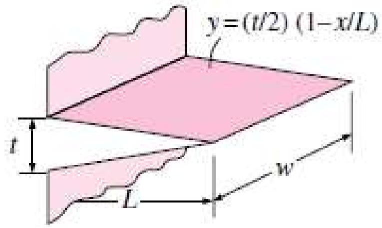
$$\eta_{\text{fin}} = \frac{\tanh mL_c}{mL_c}$$



Straight triangular fins

$$m = \sqrt{2h/kt}$$
$$A_{\text{fin}} = 2w\sqrt{L^2 + (t/2)^2}$$

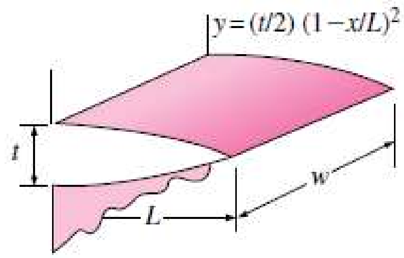
$$\eta_{\text{fin}} = \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)}$$



Straight parabolic fins

$$m = \sqrt{2h/kt}$$
$$A_{\text{fin}} = wL[C_1 + (L/t)\ln(t/L + C_1)]$$
$$C_1 = \sqrt{1 + (t/L)^2}$$

$$\eta_{\text{fin}} = \frac{2}{1 + \sqrt{(2mL)^2 + 1}}$$

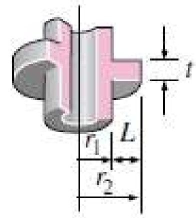


Circular fins of rectangular profile

$$m = \sqrt{2h/kt}$$
$$r_{2c} = r_2 + t/2$$
$$A_{fin} = 2\pi(r_{2c}^2 - r_1^2)$$

$$\eta_{\text{fin}} = C_2\frac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})}$$

$$C_2 = \frac{2r_1/m}{r_{2c}^2 - r_1^2}$$

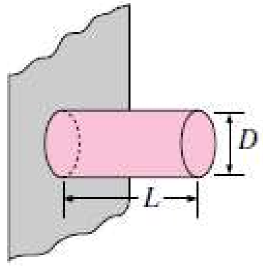


Efficiency and surface areas of common fin configurations

Pin fins of rectangular profile

$$m = \sqrt{4h/kD}$$
$$L_c = L + D/4$$
$$A_{\text{fin}} = \pi D L_c$$

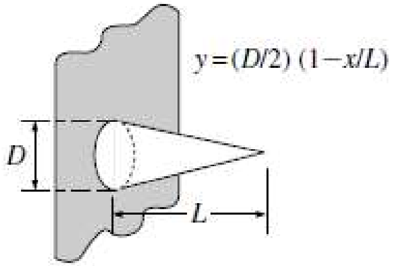
$$\eta_{\text{fin}} = \frac{\tanh mL_c}{mL_c}$$



Pin fins of triangular profile

$$m = \sqrt{4h/kD}$$
$$A_{\text{fin}} = \frac{\pi D}{2}\sqrt{L^2 + (D/2)^2}$$

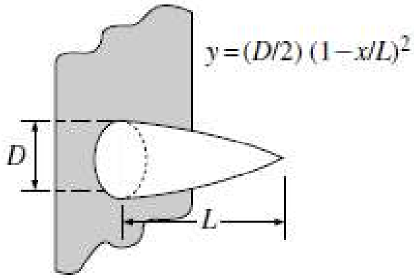
$$\eta_{\text{fin}} = \frac{2}{mL}\frac{I_2(2mL)}{I_1(2mL)}$$



Pin fins of parabolic profile

$$m = \sqrt{4h/kD}$$
$$A_{\text{fin}} = \frac{\pi L^3}{8D}\left[C_3 C_4 - \frac{L}{2D} ln(2DC_4/L + C_3)\right]$$
$$C_3 = 1 + 2(D/L)^2$$
$$C_4 = \sqrt{1 + (D/L)^2}$$

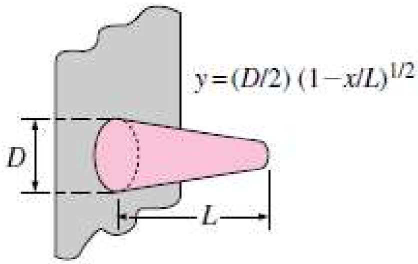
$$\eta_{\text{fin}} = \frac{2}{1 + \sqrt{(2mL/3)^2 + 1}}$$

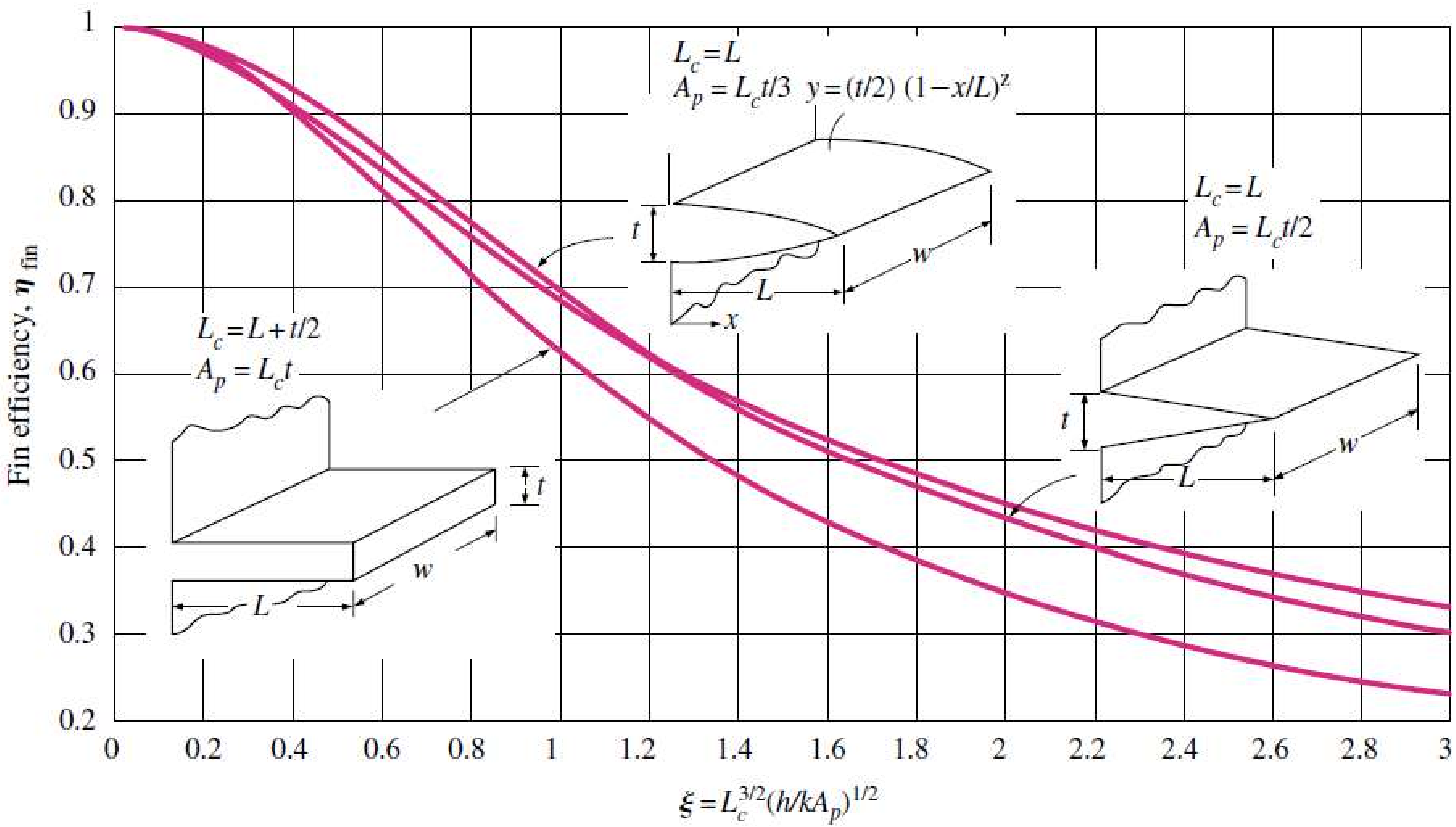


Pin fins of parabolic profile (blunt tip)

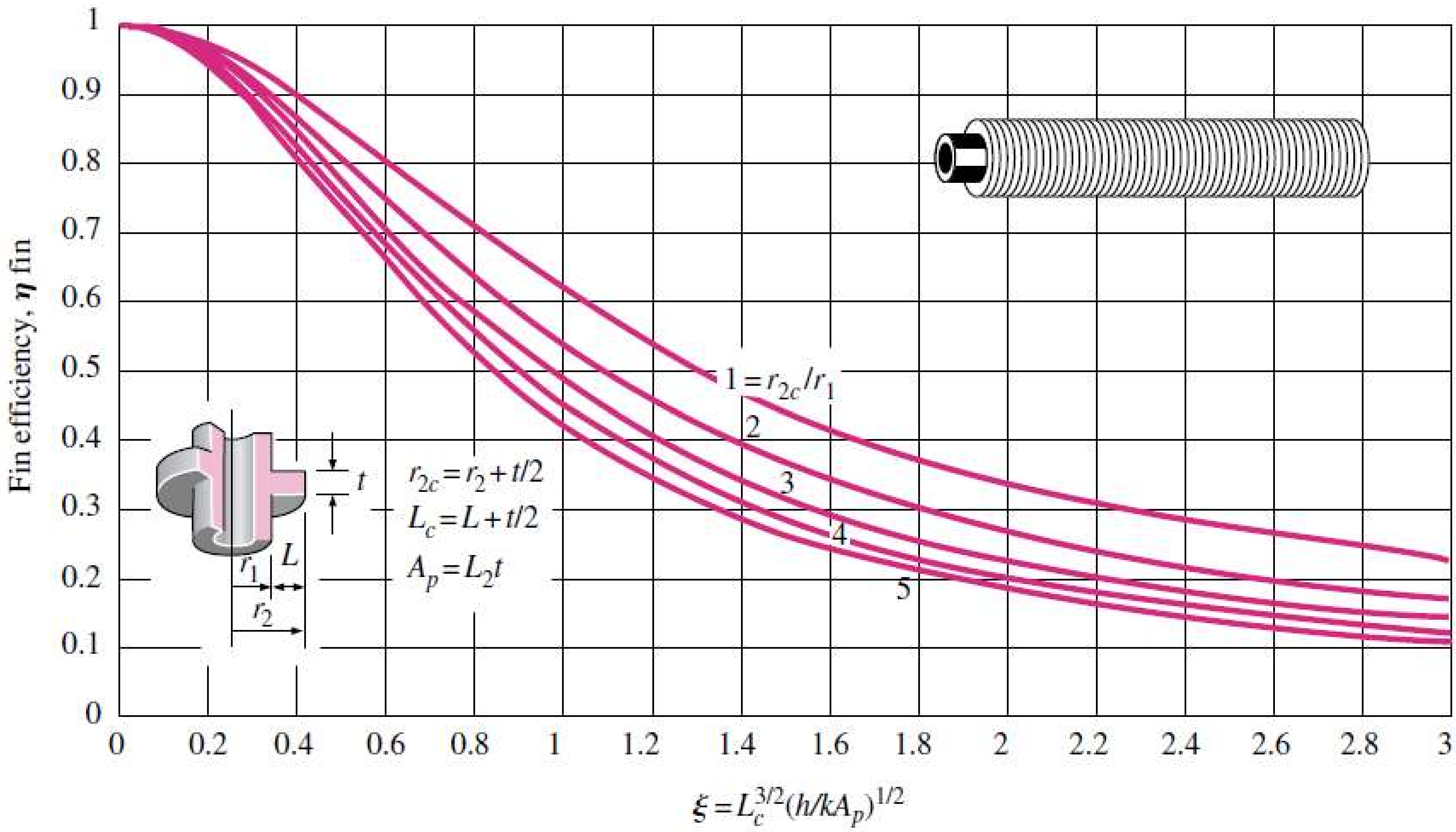
$$m = \sqrt{4h/kD}$$
$$A_{\text{fin}} = \frac{\pi D^4}{96L^2}\left\{[16(L/D)^2 + 1]^{3/2} - 1\right\}$$

$$\eta_{\text{fin}} = \frac{3}{2mL}\frac{I_1(4mL/3)}{I_0(4mL/3)}$$





Efficiency of straight fins of rectangular, triangular, and parabolic profiles.



Efficiency of annular fins of constant thickness *t*.

Fins with triangular and parabolic profiles contain less material and are more efficient than the ones with rectangular profiles, and thus are more suitable for applications requiring minimum weight such as space applications.

An important consideration in the design of finned surfaces is the selection of the proper *fin length L.* Normally the *longer* the fin, the *larger* the heat transfer area and thus the *higher* the rate of heat transfer from the fin.

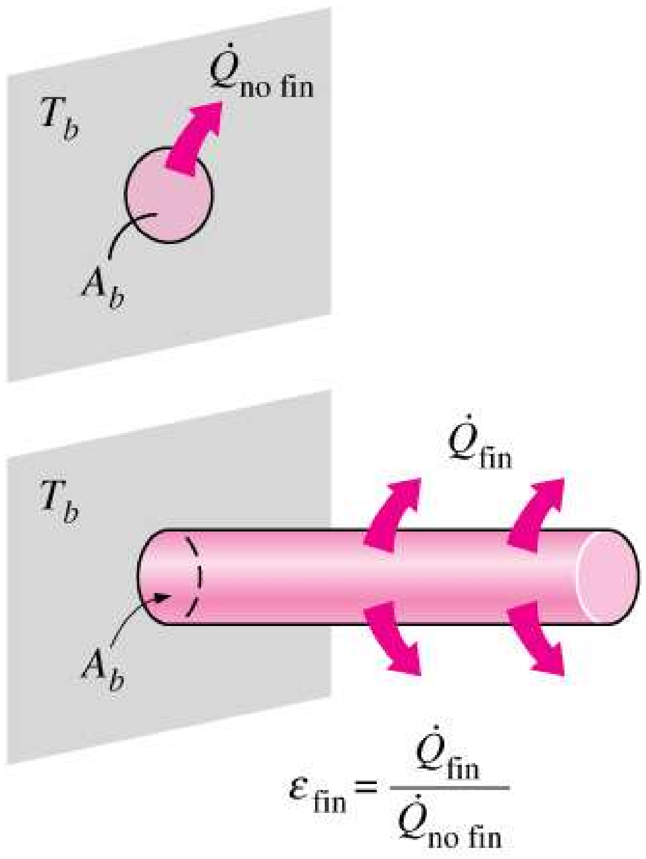
The larger the fin, the bigger the mass, the higher the price, and the larger the fluid friction. Therefore, increasing the length of the fin beyond a certain value cannot be justified unless the added benefits outweigh the added cost.

Fin lengths that cause the fin efficiency to drop below 60% percent usually cannot be justified economically and should be avoided. The efficiency of most fins used in practice is above 90%.

Fin Effectiveness

The performance of fins expressed in terms of the ***fin effectiveness*** ε_{fin} is defined

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b\,(T_b - T_\infty)} = \frac{\text{Heat transfer rate from the fin of } base\ area\ A_b}{\text{Heat transfer rate from the surface of } area\ A_b}$$



A_b : the cross-sectional area of the fin at the base
$\dot{Q}_{\text{no fin}}$: the rate of heat transfer from this area if no fins are attached to the surface.

❖ An effectiveness of $\varepsilon_{fin} = 1$ indicates that the addition of fins to the surface does not affect heat transfer at all.

❖ An effectiveness of $\varepsilon_{fin} < 1$ indicates that the fin actually acts as *insulation,* slowing down the heat transfer from the surface.

❖ An effectiveness of $\varepsilon_{fin} > 1$ indicates that fins are *enhancing* heat transfer from the surface, as they should.

Finned surfaces are designed on the basis of *maximizing* effectiveness for a specified cost or *minimizing* cost for a desired effectiveness.

The fin efficiency and fin effectiveness are related to each other by

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b(T_b - T_\infty)} = \frac{\eta_{\text{fin}} hA_{\text{fin}}(T_b - T_\infty)}{hA_b(T_b - T_\infty)} = \frac{A_{\text{fin}}}{A_b}\eta_{\text{fin}}$$

The effectiveness of a sufficiently *long* fin of *uniform* cross section under steady conditions is determined to be

$$\varepsilon_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_b\,(T_b - T_\infty)} = \sqrt{\frac{kp}{hA_c}}$$

since $A_c = A_b$.

In the design and selection of the fins, the following should be taken into account:

- The *thermal conductivity k* of the fin material should be as high as possible. Thus it is no coincidence that fins are made from metals, with copper, aluminum, and iron being the most common ones. Perhaps the most widely used fins are made of aluminum because of its low cost and weight and its resistance to corrosion.

- The ratio of the *perimeter* to the *cross-sectional area* of the fin p/A_c should be as high as possible. This criterion is satisfied by *thin* plate fins and *slender* pin fins.

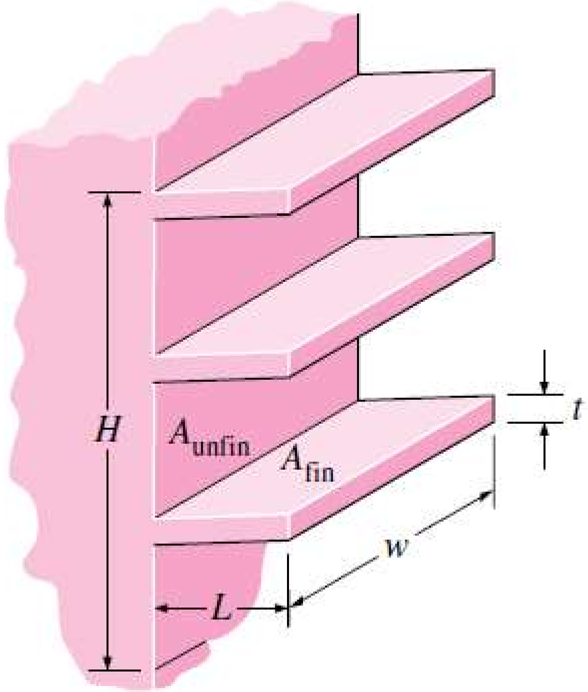
- The use of fins is *most effective* in applications involving a *low convection heat transfer coefficient.*

The rate of heat transfer for a surface containing *n* fins can be expressed as

$$\begin{aligned}\dot{Q}_{\text{total, fin}} &= \dot{Q}_{\text{unfin}} + \dot{Q}_{\text{fin}} \\ &= hA_{\text{unfin}}(T_b - T_\infty) + \eta_{\text{fin}} hA_{\text{fin}}(T_b - T_\infty) \\ &= h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_\infty)\end{aligned}$$

The **overall effectiveness** for a finned surface is defined as the ratio of the total heat transfer from the finned surface to the heat transfer from the same surface if there were no fins.

$$\varepsilon_{\text{fin, overall}} = \frac{\dot{Q}_{\text{total, fin}}}{\dot{Q}_{\text{total, no fin}}} = \frac{h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_\infty)}{hA_{\text{no fin}}(T_b - T_\infty)}$$



$$\begin{aligned}A_{\text{no fin}} &= w \times H \\ A_{\text{unfin}} &= w \times H - 3 \times (t \times w) \\ A_{\text{fin}} &= 2 \times L \times w + t \times w \\ &\cong 2 \times L \times w \text{ (one fin)}\end{aligned}$$

$A_{\text{no fin}}$ **:** the area of the surface when there are no fins

A_{fin} **:** the total surface area of all the fins on the surface

A_{unfin} **:** the area of the unfinned portion of the surface

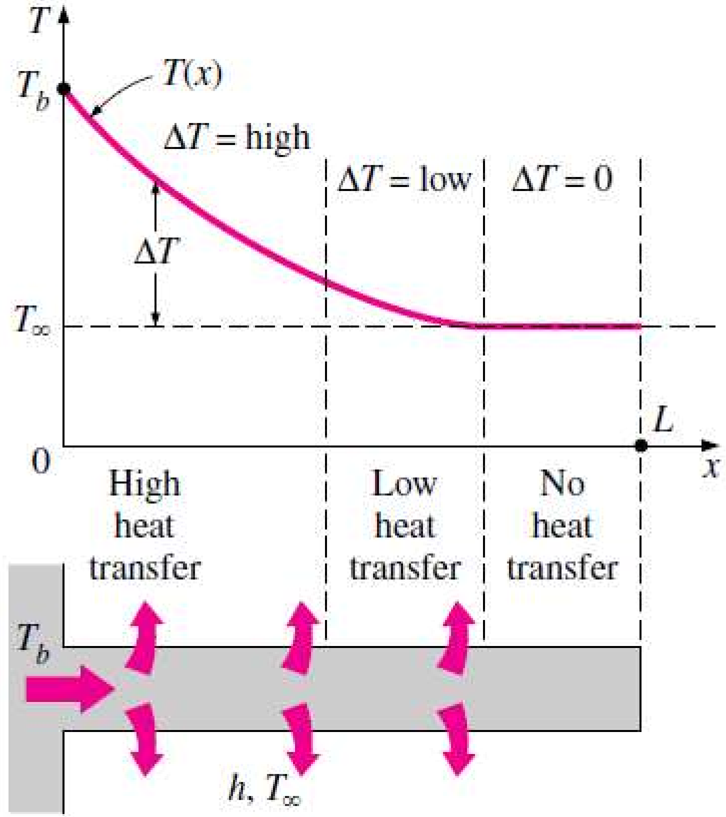
Proper Length of a Fin

To get a sense of the proper length of a fin, we compare heat transfer from a fin of finite length to heat transfer from an infinitely long fin under the same conditions. The ratio of these two heat transfers is

Heat transfer ratio:
$$\frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh mL}{\sqrt{hpkA_c}\,(T_b - T_\infty)} = \tanh mL$$

Studies have shown that the error involved in one-dimensional fin analysis is negligible (less than about 1%) when

$$\frac{h\delta}{k} < 0.2$$



The heat transfer performance of heat sinks is usually expressed in terms of their *thermal resistances R* in °C/W, which is defined as

$$\dot{Q}_{\text{fin}} = \frac{T_b - T_\infty}{R} = hA_{\text{fin}}\,\eta_{\text{fin}}\,(T_b - T_\infty)$$

Combined natural convection and radiation thermal resistance of various heat sinks used in the cooling of electronic devices between the heat sink and the surroundings. All fins are made of aluminum 6063T-5, are black anodized, and are 76 mm (3 in) long.

HS 5030

R = 0.9°C/W (vertical)
R = 1.2°C/W (horizontal)

Dimensions: 76 mm × 105 mm × 44 mm
Surface area: 677 cm²

HS 6065

R = 5°C/W

Dimensions: 76 mm × 38 mm × 24 mm
Surface area: 387 cm²

HS 6071

R = 1.4°C/W (vertical)
R = 1.8°C/W (horizontal)

Dimensions: 76 mm × 92 mm × 26 mm
Surface area: 968 cm²

HS 6105

R = 1.8°C/W (vertical)
R = 2.1°C/W (horizontal)

Dimensions: 76 mm × 127 mm × 91 mm
Surface area: 677 cm²

HS 6115

R = 1.1°C/W (vertical)
R = 1.3°C/W (horizontal)

Dimensions: 76 mm × 102 mm × 25 mm
Surface area: 929 cm²

HS 7030

R = 2.9°C/W (vertical)
R = 3.1°C/W (horizontal)

Dimensions: 76 mm × 97 mm × 19 mm
Surface area: 290 cm²

HEAT TRANSFER IN COMMON CONFIGURATIONS

- We have dealt with 1-D simple geometries.

☞The question:**What happens if we have 2- or 3-D complicated geometries?**

- The steady rate of heat transfer between two surfaces at *constant* temperatures T_1 and T_2 is expressed as

$$Q = Sk(T_1 - T_2)$$

S : the conduction shape factor **(which** has the dimension of *length)*
k : the thermal conductivity of the medium between the surfaces

☞ The conduction shape factor depends on the *geometry* of the system only.

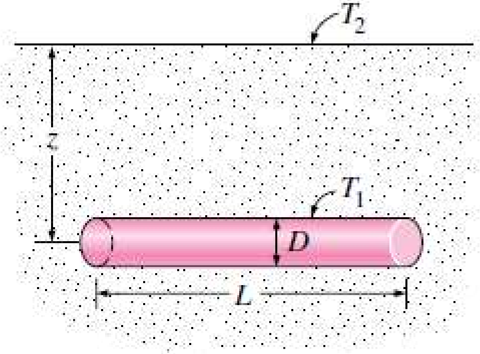
A comparison of the following equations reveals that the conduction shape factor S is related to the thermal resistance R by $R = 1/kS$ or $S = 1/kR$.

$$\dot{Q}_{\text{cond, wall}} = \frac{T_1 - T_2}{R_{\text{wall}}} \quad \text{(W)} \qquad Q = Sk(T_1 - T_2)$$

Conduction shape factors S for several configurations for use in $\dot{Q} = kS(T_1 - T_2)$ to determine the steady rate of heat transfer through a medium of thermal conductivity k between the surfaces at temperatures T_1 and T_2

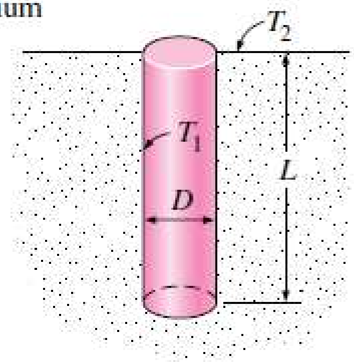
(1) Isothermal cylinder of length L buried in a semi-infinite medium ($L >> D$ and $z > 1.5D$)

$$S = \frac{2\pi L}{\ln(4z/D)}$$



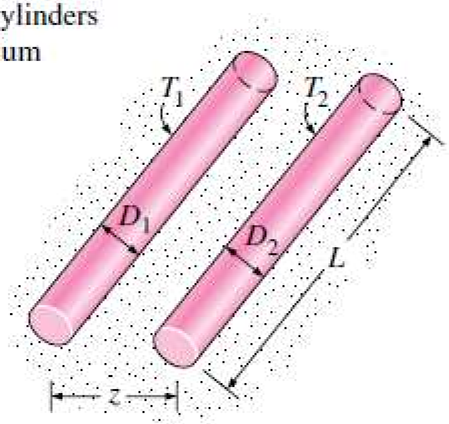
(2) Vertical isothermal cylinder of length L buried in a semi-infinite medium ($L >> D$)

$$S = \frac{2\pi L}{\ln(4L/D)}$$



(3) Two parallel isothermal cylinders placed in an infinite medium ($L >> D_1, D_2, z$)

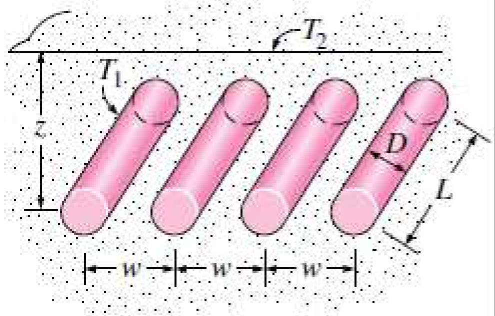
$$S = \frac{2\pi L}{\cosh^{-1}\left(\dfrac{4z^2 - D_1^2 - D_2^2}{2D_1 D_2}\right)}$$



(4) A row of equally spaced parallel isothermal cylinders buried in a semi-infinite medium ($L >> D$, z, and $w > 1.5D$)

$$S = \frac{2\pi L}{\ln\left(\dfrac{2w}{\pi D} \sinh \dfrac{2\pi z}{w}\right)}$$

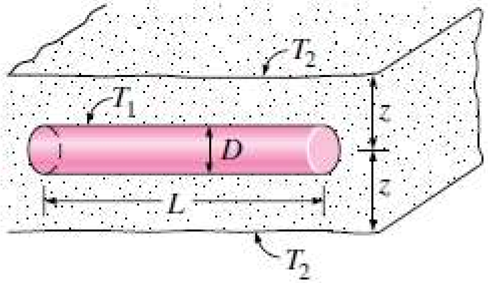
(per cylinder)



(*Continued*)

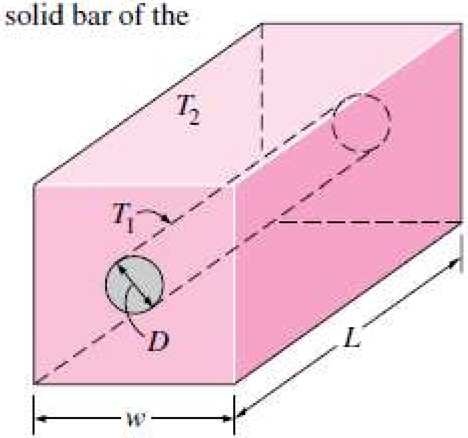
(5) Circular isothermal cylinder of length L in the midplane of an infinite wall ($z > 0.5D$)

$$S = \frac{2\pi L}{\ln(8z/\pi D)}$$



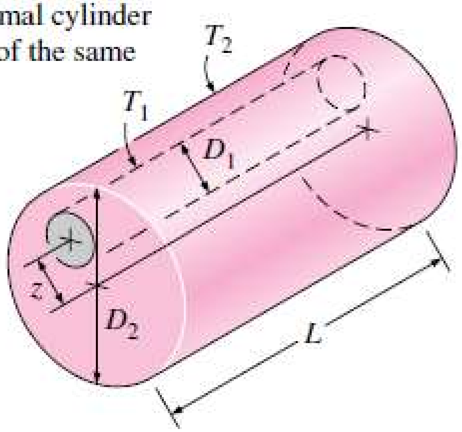
(6) Circular isothermal cylinder of length L at the center of a square solid bar of the same length

$$S = \frac{2\pi L}{\ln(1.08\,w/D)}$$



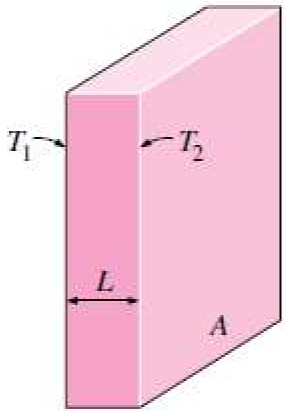
(7) Eccentric circular isothermal cylinder of length L in a cylinder of the same length ($L > D_2$)

$$S = \frac{2\pi L}{\cosh^{-1}\left(\dfrac{D_1^2 + D_2^2 - 4z^2}{2D_1D_2}\right)}$$



(8) Large plane wall

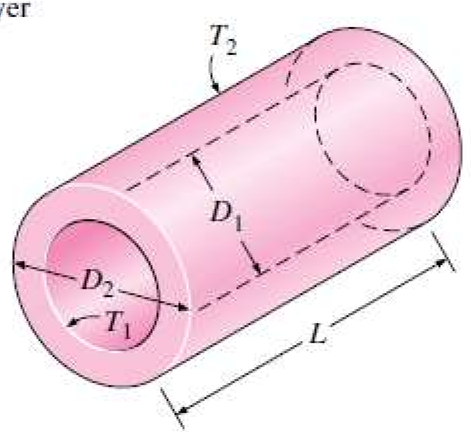
$$S = \frac{A}{L}$$



(Continued)

(9) A long cylindrical layer

$$S = \frac{2\pi L}{\ln (D_2/D_1)}$$



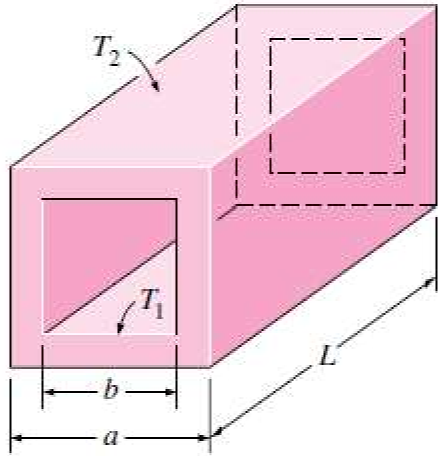
(10) A square flow passage

(*a*) For $a/b > 1.4$,

$$S = \frac{2\pi L}{0.93 \ln (0.948\, a/b)}$$

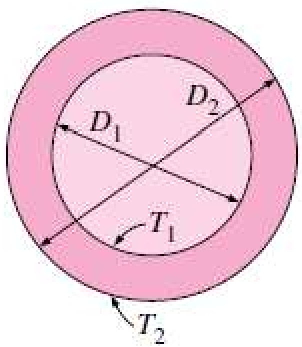
(*b*) For $a/b < 1.41$,

$$S = \frac{2\pi L}{0.785 \ln (a/b)}$$



(11) A spherical layer

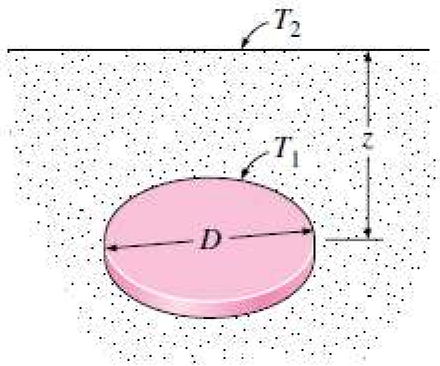
$$S = \frac{2\pi D_1 D_2}{D_2 - D_1}$$



(12) Disk buried parallel to the surface in a semi-infinite medium ($z >> D$)

$$S = 4D$$

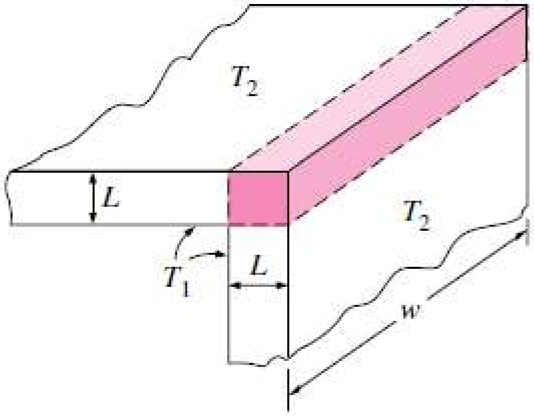
($S = 2D$ when $z = 0$)



(Continued)

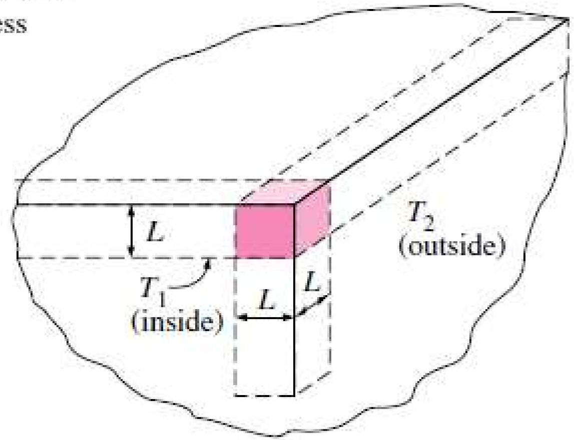
(13) The edge of two adjoining walls of equal thickness

$$S = 0.54\,w$$



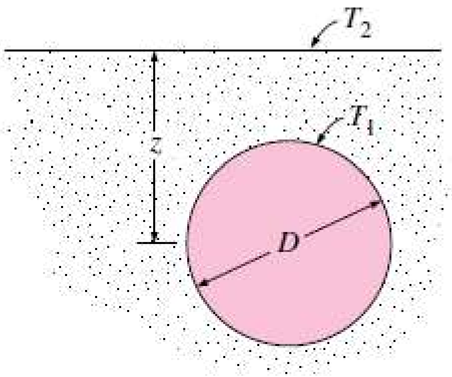
(14) Corner of three walls of equal thickness

$$S = 0.15L$$



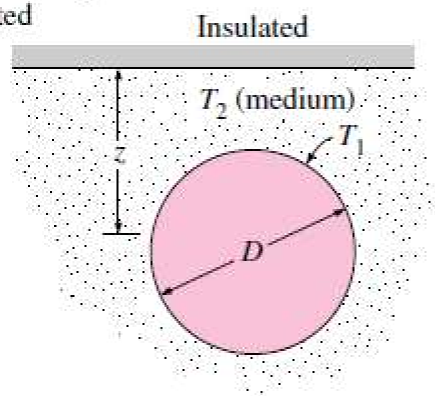
(15) Isothermal sphere buried in a semi-infinite medium

$$S = \frac{2\pi D}{1 - 0.25D/z}$$



(16) Isothermal sphere buried in a semi-infinite medium at T_2 whose surface is insulated

$$S = \frac{2\pi D}{1 + 0.25D/z}$$



Concluding Points:

- ❖ Steady and One-Dimensional Modeling of Heat Transfer through a Wall
- ❖ Conduction and Convection Resistances
- ❖ Analogy between Thermal and Electrical Resistances
- ❖ Radiation and Combined Heat Transfer Coefficients
- ❖ Overall Heat Transfer Coefficient
- ❖ Heat Transfer through a Plane and Multilayer Plane Walls
- ❖ Thermal Contact Resistance
- ❖ Generalized Thermal Resistance Networks
- ❖ Heat Conduction in Multilayered Cylinders and Spheres
- ❖ Critical Radius of Insulation for Cylindrical and Spherical Bodies
- ❖ Heat Transfer from Finned Surfaces
- ❖ Fin Efficiency, Fin Effectiveness and Overall Effectiveness
- ❖ Important Considerations in the Design and Selection of Fins
- ❖ Heat Transfer in Common Configurations and Conduction Shape Factors