YILDIZ TECHNICAL UNIVERSITY
CIVIL ENGINEERING DEPARTMENT
DIVISION OF MECHANICS

STATICS

CHAPTER NINE
MOMENTS OF INERTIA

9. Moments of Inertia of Areas

9.1. Definitions

In the torsion and bending problems, there are some integral quantities which are relatd to cross-section of bars (Fig. 9.1).

A cross-section, A, is assumed to be in the xy plane. dA shows an infinitesimal element of the cross-sectional area. Before investigating moments of areas, the following expressions called static moments of cross-sectional areas are defined.

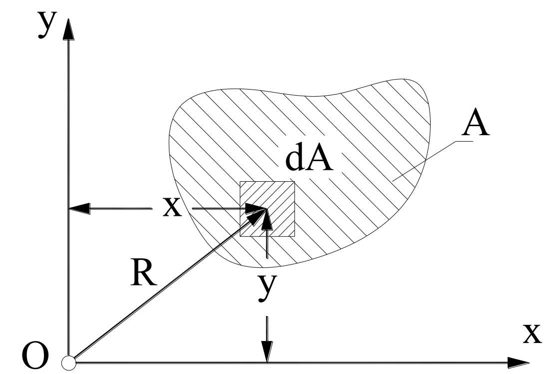


Figure 9.1

$$S_x = \iint_A y\,dA \qquad S_y = \iint_A x\,dA \qquad (9.1)$$

If the x, y axes are to be at the center of gravity of cross-sectional area, these expressions (first moments of areas) must be zero. The quantities defined as follows are called as moment of inertia or the second moment of the cross-section of the beam.

$$I_x = \iint_A y^2 dA \qquad I_y = \iint_A x^2 dA \qquad \text{ve} \qquad I_{xy} = \iint_A x\,y\,dA \quad (9.2)$$

The first and second of these expressions are the second moments of areas with respect to x and y axes respectively. The third of them is known as the product of inertia of the area A with respect to the x and y axes.

Their units are in *cm⁴*; I_x , and I_y are always positive unlike I_{xy} can be positive, negative, or zero. Thus,

$$I_x > 0\,, \quad I_y > 0 \quad ve \quad I_{xy} \geq 0 \quad veya \quad I_{xy} \leq 0 \qquad (9.3)$$

An integral of great importance in problems concerning the torsion of cylindrical shafts and in rpblems dealing with the rotation of slabs is called as the polar moment of inertia of the area, A, w.r.t. the pole O.

$$J_0 = \iint_A R^2 dA \quad (9.4) \qquad \text{Indeed,} \quad x^2 + y^2 = R^2$$

The polar moment of inertia of a given area can be computed from the rectangular moments of inertia. $J_0 = I_x + I_y$

There are some quantities derived from the second moments of areas:

$$i_x = \sqrt{\frac{I_x}{A}} \qquad i_y = \sqrt{\frac{I_y}{A}} \qquad \text{veya} \qquad I_x = i_x^2\,A \qquad \text{ve} \qquad I_y = i_y^2\,A \qquad (9.6)$$

i_x ve i_y are known as radius of gyration of the areas w.r.t. the x and y axes and the dimesion is in cm.

9.2. Parallel-Axis Theorem

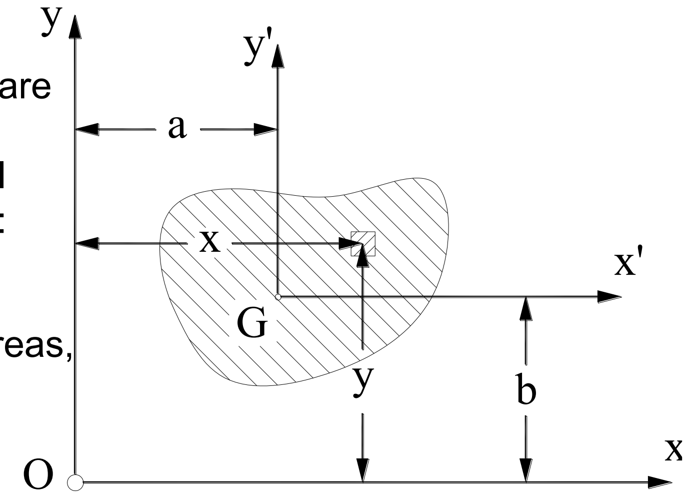


Figure 9.2

Lets consider the system of axes x,y and the system of axes (x',y') which are parallel to each other. The origin of the second system of rectangular coordinates x' and y' is taken at the center of gravity of the cross-sectional area of the beam. The coordinate transformation equations are as follows:

$$x = a + x' \quad \text{and} \quad y = b + y'$$

Using these transformations and the definition of the second moment of areas,

$$I_x = \iint_A y^2 dA = \iint_A (b+y')^2 dA = \iint_A y'^2 dA + 2b\iint_A y' dA + b^2 \iint_A dA$$

$$I_y = \iint_A x^2 dA = \iint_A (a+x')^2 dA = \iint_A x'^2 dA + 2a\iint_A x' dA + a^2 \iint_A dA$$

$$I_{xy} = \iint_A x\,y\,dA = \iint_A x'y' dA + a\iint_A y' dA + b\iint_A x' dA + a\,b\iint_A dA$$

Because x' and y' are centroidal axes, the first moment of area about these axes must be zero. Therefore, the transformation expressions can be written as follows:

$$I_x = I_{x'} + b^2 A \qquad I_y = I_{y'} + a^2 A \qquad I_{xy} = I_{x'y'} + a\,b\,A \qquad (9.7)$$

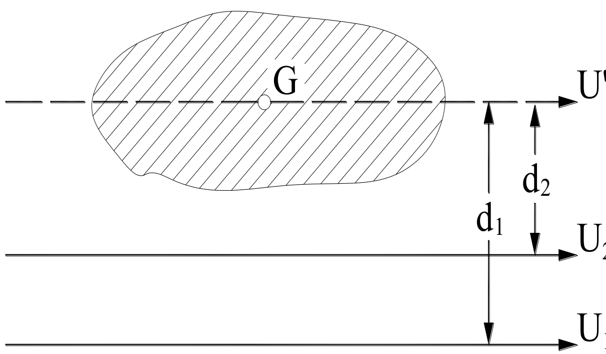


Figure 9.3

This theorem is known as the parallel-axıs theorem or Steiner theorem. If the two axes do not pass through the center of area, then it is possible to find the transformation formula by using Eq. (9.7) as shown in Fig. 9.3. By using Eq. (9.7),

$I_{u_1} = I_{u'} + d_1^2 A$ and $I_{u_2} = I_{u'} + d_2^2 A$ If $I_{u'}$ İs eliminated, then $I_{u_1} = I_{u_2} + (d_1^2 - d_2^2) A$ (9.8)

This is the transformation formula between the two axes. The first and second expressions of Eq. (9.7) indicate that the second moment of area is minimum according to the centroidal axes.

Determine the moment of inertia for the rectangular area shown in Fig. 10–5 with respect to (a) the centroidal *x* axis, (b) the axis *xb* passing
through the base of the rectangle, and (c) the pole or *z* axis perpendicular to the *xy* plane and passing through the centroid *C* .

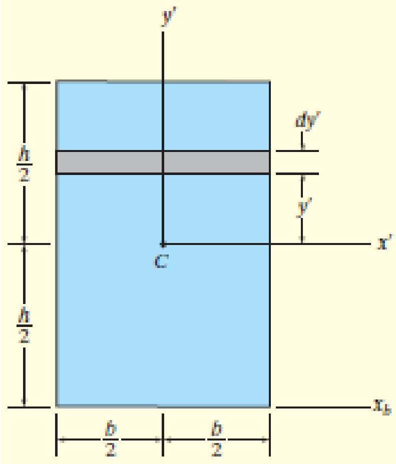


Fig. 10–5

Part (a).

$$\bar{I}_{x'} = \int_A y'^2\, dA = \int_{-h/2}^{h/2} y'^2 (b\, dy') = b \int_{-h/2}^{h/2} y'^2\, dy'$$

$$\bar{I}_{x'} = \frac{1}{12} bh^3$$

Part (b).

$$I_{x_b} = \bar{I}_{x'} + Ad_y^2$$

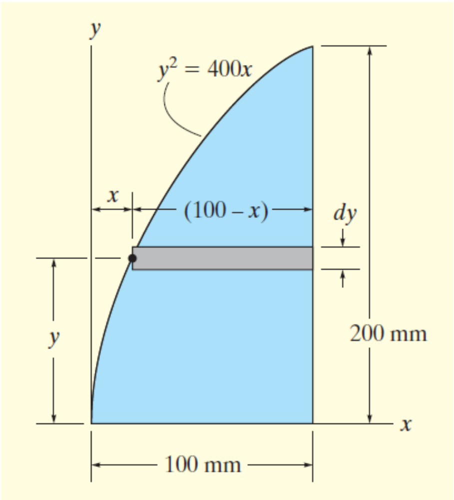
$$= \frac{1}{12} bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3} bh^3$$

Part (c).

$$\bar{I}_{y'} = \frac{1}{12} hb^3$$

$$\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12} bh(h^2 + b^2)$$

Determine the moment of inertia for the shaded area shown in Fig. 10–6 *a* about the *x* axis.

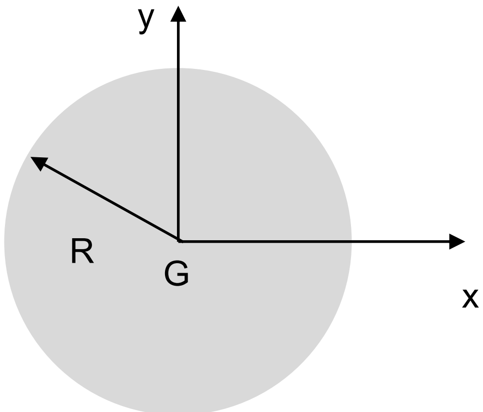


$$dA = (100 - x)\, dy.$$

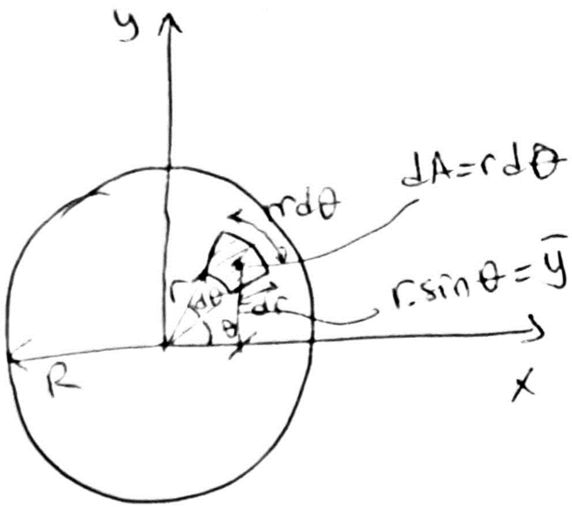
$$I_x = \int_A y^2\, dA = \int_0^{200\text{ mm}} y^2(100 - x)\, dy$$

$$= \int_0^{200\text{ mm}} y^2\left(100 - \frac{y^2}{400}\right) dy = \int_0^{200\text{ mm}} \left(100y^2 - \frac{y^4}{400}\right) dy$$

$$= 107(10^6)\text{ mm}^4$$



Determine the moment of inertia of the circular section about *x* axis



$$I_x = \int_A \bar{y}^2\, dA$$

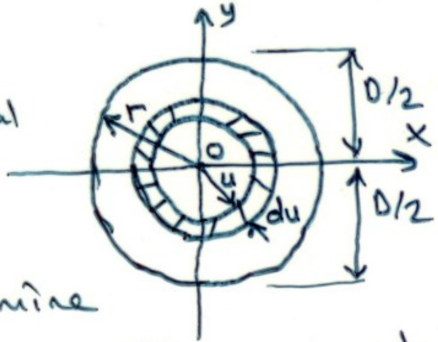
$$= \int_0^R \int_0^{2\pi} (r\sin\theta)^2\, r\, d\theta\, dr$$

$$= \frac{R^4}{4} \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= \frac{R^4}{4} \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right] \Bigg|_0^{2\pi}$$

$$= \frac{\pi R^4}{4}$$

to any parallel axis.



Sample Problem: (a) Determine the centroidal polar moment of inertia of a circular area by direct integration.

(b) Using the result of part (a), determine the moment of inertia of a circular area with respect to a diameter.

Solution:

(a) Polar moment of inertia: An annular differential element of area is chosen to be dA. Since all portions of the differential area are at the same distance from the origin, we write

$$dI_o = u^2 dA \quad ; \quad dA = 2\pi u\, du$$

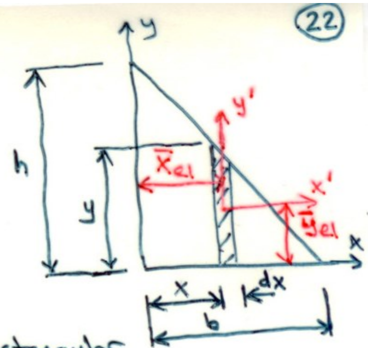
$$I_o = \int dI_o = \int_0^r u^2 (2\pi u\, du) = 2\pi \int_0^r u^3 du = \underline{\frac{\pi r^4}{2}} = \underline{\frac{\pi D^4}{32}} \text{ (D is diameter)}$$

(b) Moment of inertia with respect to a diameter: Because of the symmetry of the circular area, we have $I_x = I_y$. We then write

$$I_o = I_x + I_y = 2I_x = 2I_y \rightarrow \frac{\pi r^4}{2} = \frac{\pi D^4}{32} = 2I_x$$

$$I_{diameter} = I_x = I_y = \underline{\frac{\pi r^4}{4}} = \underline{\frac{\pi D^4}{64}}$$

Sample Problem: Determine the product of inertia of the right triangle shown (a) with respect to the x and y axes and (b) with respect to centroidal axes parallel to the x and y axes.



Solution:

(a) Product of inertia I_{xy}: A vertical rectangular strip is chosen as the differential element of area. Using the parallel-axis theorem, we write

$$dI_{xy} = dI_{x'y'} + \bar{x}_{el} \cdot \bar{y}_{el} \cdot dA$$

Since the element is symmetrical with respect to the x' and y' axes, we note that $dI_{x'y'} = 0$. From the geometry of the triangle, we obtain

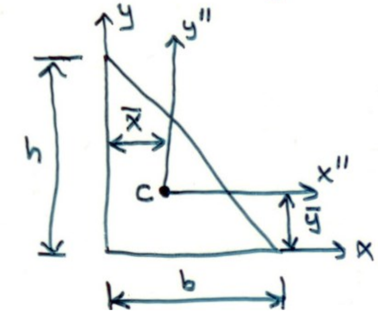
$$y = h\left(1 - \frac{x}{b}\right) \;; \quad dA = y\,dx = h\left(1 - \frac{x}{b}\right)dx$$

$$\bar{x}_{el} = x \quad ; \quad \bar{y}_{el} = \frac{1}{2}y = \frac{1}{2}h\left(1 - \frac{x}{b}\right)$$

Integrating dI_{xy} from $x = 0$ to $x = b$, we obtain

$$I_{xy} = \int dI_{xy} = \int \bar{x}_{el} \cdot \bar{y}_{el} \cdot dA = \int_0^b x \cdot \frac{1}{2} h^2 \left(1 - \frac{x}{b}\right)^2 dx$$

$$I_{xy} = h^2 \int_0^b \left(\frac{x}{2} - \frac{x^2}{b} + \frac{x^3}{2b^2}\right)dx = h^2\left[\frac{x^2}{4} - \frac{x^3}{3b} + \frac{x^4}{8b^2}\right]_0^b = \frac{1}{24}b^2h^2$$



(b) Product of inertia $I_{x''y''}$:

The coordinates of the centroid of the triangle relative to the x and y axes are

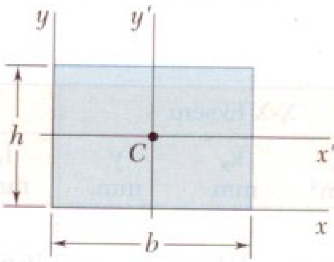
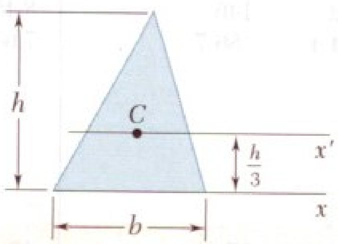
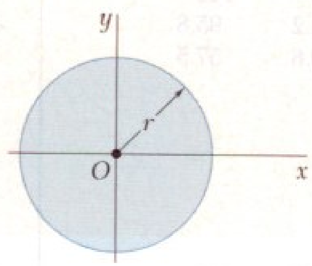
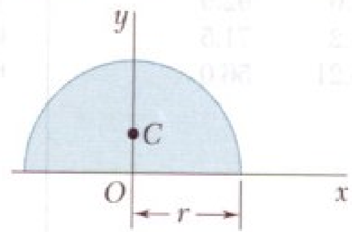
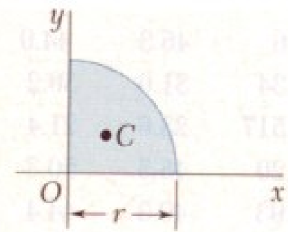
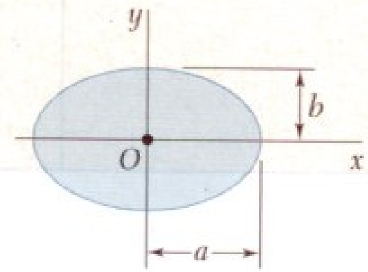
$$\bar{x} = \frac{b}{3} \quad ; \quad \bar{y} = \frac{h}{3}$$

Using the expression for I_{xy} obtained in part (a), we apply the parallel-axis theorem and write

$$I_{xy} = I_{x''y''} + \bar{x}\,\bar{y}\,A \rightarrow I_{x''y''} = I_{xy} - \bar{x}\,\bar{y}\,A$$

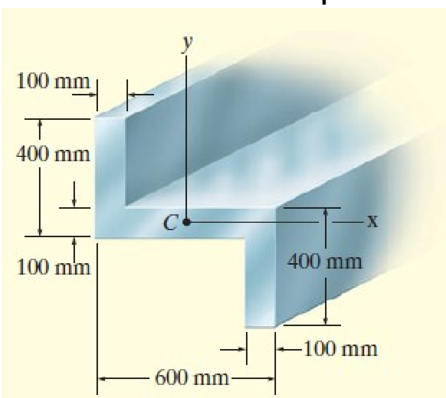
$$I_{x''y''} = \frac{b^2h^2}{24} - \frac{b}{3}\,\frac{h}{3}\,\frac{bh}{2} = -\frac{b^2h^2}{72}$$

Table 9.1 Moment of Inertia of some Cross-Sections

Dikdörtgen		$\bar{I}_{x'} = \frac{1}{12} bh^3$ $\bar{I}_{y'} = \frac{1}{12} b^3h$ $I_x = \frac{1}{3} bh^3$ $I_y = \frac{1}{3} b^3h$ $J_C = \frac{1}{12} bh(b^2 + h^2)$
Üçgen		$\bar{I}_{x'} = \frac{1}{36} bh^3$ $I_x = \frac{1}{12} bh^3$
Daire		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Yarım Daire		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Çeyrek Daire		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Elips		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Moments of Inertia for Composite Areas

- A composite area consists of a series of connected "simpler" parts or shapes, such as rectangles, triangles, and circles.
- moment of inertia for the composite area about this axis equals the *algebraic sum* of the moments of inertia of all its parts.



(a)

Determine the moments of inertia for the cross-sectional area of the member shown in Fig. 10–9 *a* about the *x* and *y* centroidal axes.

The cross section can be subdivided into the three rectangular areas *A*, *B* , and *D*

Rectangles A and D

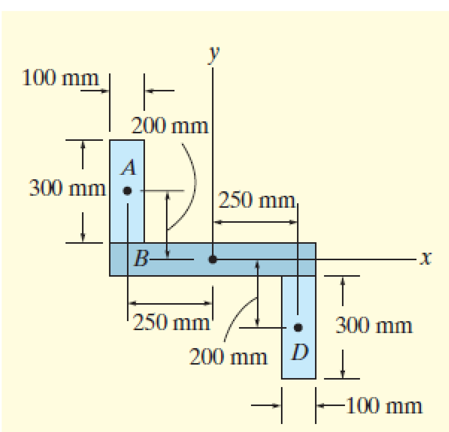
$$I_x = \bar{I}_{x'} + Ad_y^2 = \frac{1}{12}(100)(300)^3 + (100)(300)(200)^2$$
$$= 1.425(10^9)\ \text{mm}^4$$

$$I_y = \bar{I}_{y'} + Ad_x^2 = \frac{1}{12}(300)(100)^3 + (100)(300)(250)^2$$
$$= 1.90(10^9)\ \text{mm}^4$$

Rectangle B

$$I_x = \frac{1}{12}(600)(100)^3 = 0.05(10^9)\ \text{mm}^4$$

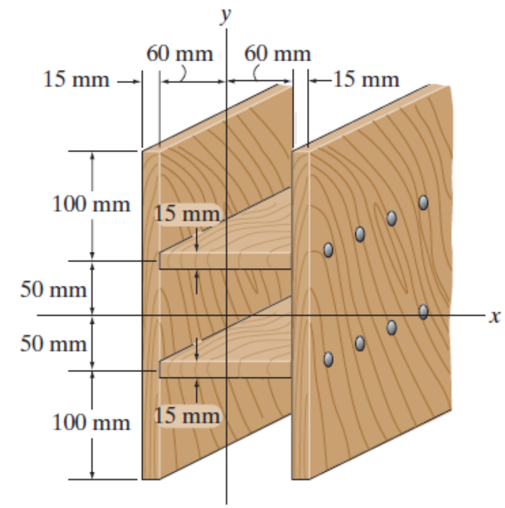
$$I_y = \frac{1}{12}(100)(600)^3 = 1.80(10^9)\ \text{mm}^4$$



Summation. The moments of inertia for the entire cross section are thus

$$I_x = 2[1.425(10^9)] + 0.05(10^9)$$
$$= 2.90(10^9)\ \text{mm}^4 \qquad \textit{Ans.}$$
$$I_y = 2[1.90(10^9)] + 1.80(10^9)$$
$$= 5.60(10^9)\ \text{mm}^4 \qquad \textit{Ans.}$$

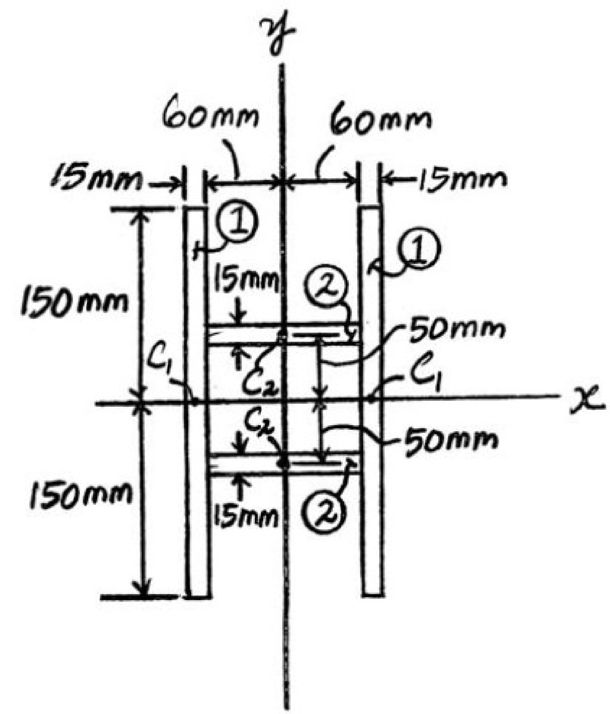
Determine moment of inertia of the beam's cross section



$$I_x = \bar{I}_{x'} + A(d_y)^2$$

$$= \left[2\left(\frac{1}{12}(15)(300^3)\right) + 2(15)(300)(0)^2\right] + \left[2\left(\frac{1}{12}(120)(15^3)\right) + 2(120)(15)(50)^2\right]$$

$$= 67.5(10^6) + 9.0675(10^6) = 76.6(10^6)\,\text{mm}^4$$

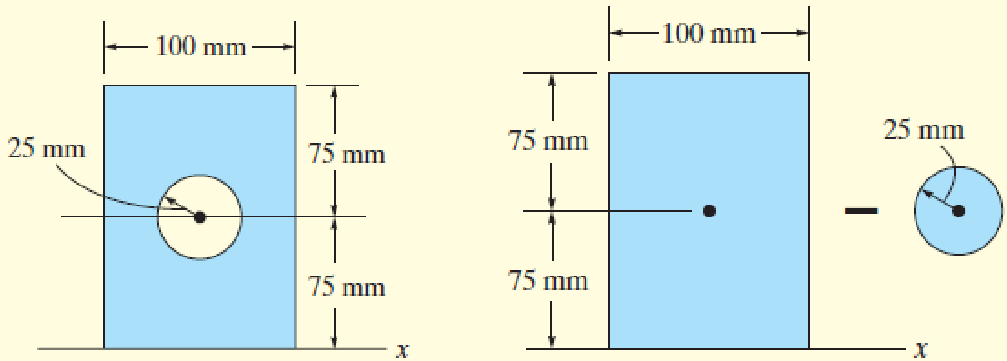


$$I_y = \bar{I}_{y'} + A(d_x)^2$$

$$= \left[2\left(\frac{1}{12}(300)(15^3)\right) + 2(300)(15)(67.5)^2\right] + \left[2\left(\frac{1}{12}(15)(120^3)\right) + 2(120)(15)(0)^2\right]$$

$$= 41.175(10^6) + 4.32(10^6) = 45.5(10^6)\,\text{mm}^4$$

Determine the moment of inertia of the area shown in Fig. 10–8*a* about the x axis.



$\bar{I}_x = \frac{1}{4}\pi r^4;$

Circle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4(10^6)\ \text{mm}^4$$

$\bar{I}_x = \frac{1}{12}bh^3,$

Rectangle

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6)\ \text{mm}^4$$

Summation. The moment of inertia for the area is therefore

$$I_x = -11.4(10^6) + 112.5(10^6)$$

$$= 101(10^6)\ \text{mm}^4$$

9.3. Principal Axes and Principal Moments of Inertia

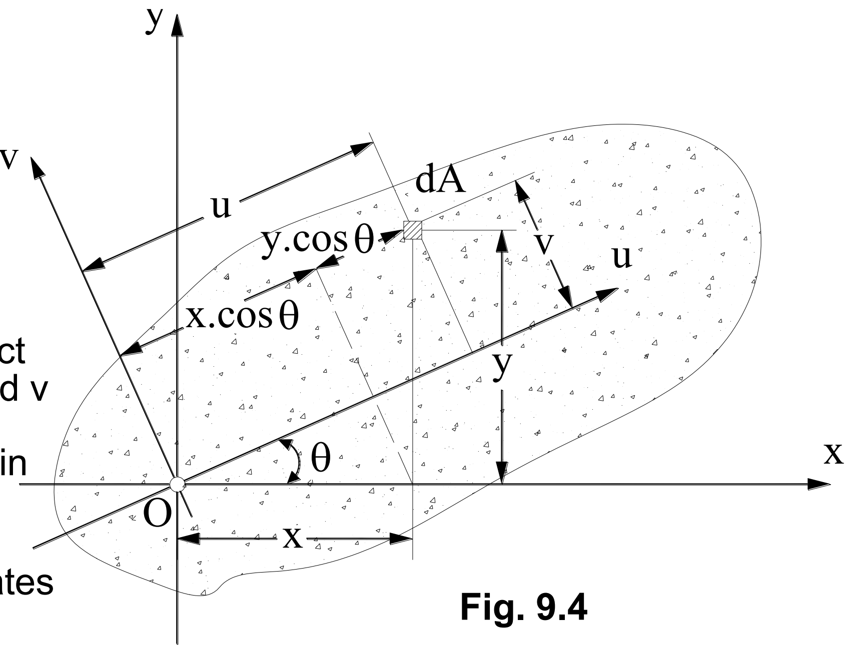


Fig. 9.4

Consider the area A and the coordinate axes x and y, Fig.9.4. assuming that the moments of inertia:

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$

and product of inertia $\quad I_{xy} = \int xy dA$

are known. We propose to determine the moments and product of inertia I_u, I_v, and I_{uv} of A with respect to new axes u and v

which are obtained by rotating the original axes about the origin through an angle θ.

We first note that the following relations between the coordinates u, v and x, y of an element of area dA:

$$u = x\cos\theta + y\sin\theta \qquad (9.9a) \qquad\qquad v = y\cos\theta - x\sin\theta \qquad (9.9b)$$

Substituting for v in the expression for I_u, we write

$$I_u = \int v^2 dA = \int (y\cos\theta - x\sin\theta)^2 dA = \int y^2\cos^2\theta.dA - 2\int xy\cos\theta\sin\theta.dA + \int x^2\sin^2\theta.dA \qquad (9.10)$$

$$I_u = I_x\cos^2\theta - 2I_{xy}\sin\theta\cos\theta + I_y\sin^2\theta \qquad (9.11a)$$

Similarly,

$$I_v = I_x\sin^2\theta + 2I_{xy}\sin\theta\cos\theta + I_y\cos^2\theta \qquad (9.11b)$$

$$I_{uv} = I_x\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta) - I_y\sin\theta\cos\theta \qquad (9.11c)$$

Adding (9.11a) and (9.11b) we observe that

$$I_u + I_v = I_x + I_y \qquad (9.12)$$

This result could have been anticipated, since both members of (9.12) are equal to the polar moment of inertia, I_0. Therefore, the sum of the moments of inertia is independent of the coordinate rotation, namely it is invariant.

Recalling the trigonometric relations

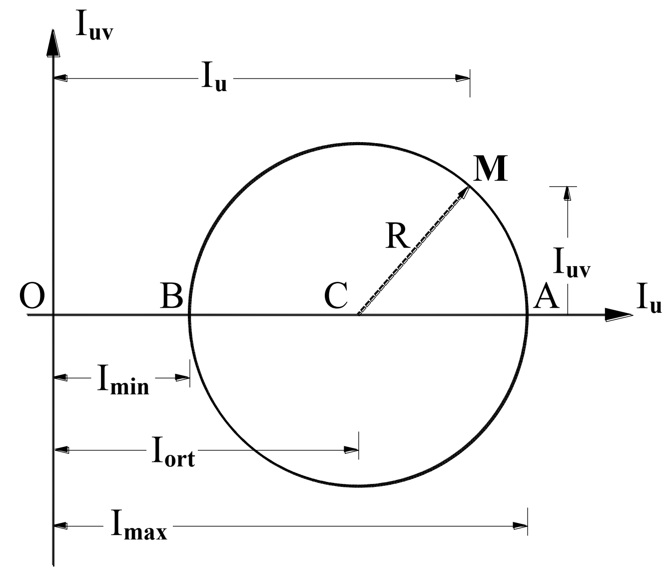
$$\sin 2\theta = 2\sin\theta\cos\theta \qquad cos2\theta = cos^2\theta - sin^2\theta$$

We can write (9.11a), (9.11b), and (9.11c) as follows:

$$I_u = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta \qquad (9.13a)$$

$$I_v = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta \qquad (9.13b)$$

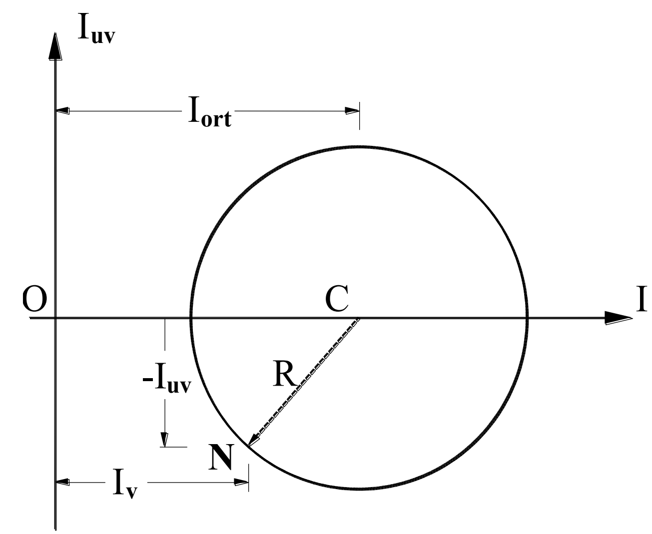
$$I_{uv} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta \qquad (9.13c)$$



Equations (9.13a) and (9.13c) are the parametric equations of a circle.

This means that if we choose a set of rectangular axes and plot a point M of abscissa I_u and ordinate I_{uv} for any given value of the parameter θ, all of the points thus obtained will lie on a circle.

To establish this property, we eliminate θ from Eqs. (9.13a) and (9.13c).



We write

$$\left(I_u - \frac{I_x + I_y}{2}\right)^2 + I_{uv}^2 = \left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2 \qquad (9.14)$$

Setting $\quad I_{ort} = \frac{I_x + I_y}{2} \quad ve \quad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \qquad (9.15)$

We write the identity (9.14) in the form $\quad (I_u - I_{ort})^2 + I_{uv}^2 = R^2 \qquad (9.16)$

This is the equation of a circle of radius R centered at the point C whose x and y coordinates are I_{ave} and 0, respectively, Fig. 9.5. the two points A and B where the above circle intersects the horizontal axis, Fig. 9.5, are of special interest: Point A corresponds to the maximum value of the moment of inertia I_u, while point B corresponds to its minimum value. In addition, both points correspond to a zero value of the product of inertia I_{uv}.

Thus, the values θ_m of the parameter of which correspons to the points A and B can be obtained by setting $I_{uv} = 0$ İn Eq. (9.13c). We obtain

$$tg2\theta_m = -\frac{2I_{xy}}{I_x - I_y} \qquad (9.17)$$

This equation defines two values $2\theta_m$ which are 180° apart and thus two values θ_m which are 90° apart. One of these corresponds to point A in Fig. 9.5 and to an axis through 0 in Fig. 9.4.

With respect to which the moment of inertia of the given area is maximum; the other value corresponds to point B and to an axis through 0 with respect to which the moment of inertia of the area is minimum. The two axes thus defined, which are perpendicular to each other, are called the principal axes of the area about 0, and the corresponding values I_{max} and I_{min} of the moment of inertia are called the principal moments of inertia of the area about 0.

Since the two values θ_m defined by Eq. (9.17) were obtained by setting $I_{uv}= 0$ in Eq. (9.13c), it is clear that the product of inertia of the given area with respect to its principal axes is zero. We observe from Fig. 9.5 that

$$I_{\max} = I_{ort} + R \quad ve \quad I_{\min} = I_{ort} - R \qquad (9.18)$$

Using the values for I_{ave} and R from formulas (9.15), we write

$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \qquad (9.19)$$

If an area possesses an axis of symmetry through a point O, this axis must be a principal axis of the area about O. On the toher hand, a principal axis does not need to be an axis of symmetry whether or not an area possesses any axes of symmetry, it will have two principal axes of inertia about any point O. The properties we have established hold for any point O located inside or outside the given area. If the point O is chosen to coincide with the centroid of the area, any axis through O is a centroidal axis; the two principla axis of the area about its centroid are referred to as the principal axes of the area.

9.4. Mohr's Circle for Moments and Products of Inertia

If the moments and product of inertia of an area A are known with respect to two rectangular x and y axes which pass through a point O,Mohr's circle first introduced by the German engineer Otto Mohr can be used to graphically determine (a) the principal axes and principal moments of inertia of the area about O and, (b) the moments and product of inertia of the area w.r.t. any other pair of rectangular axes u and v through O.

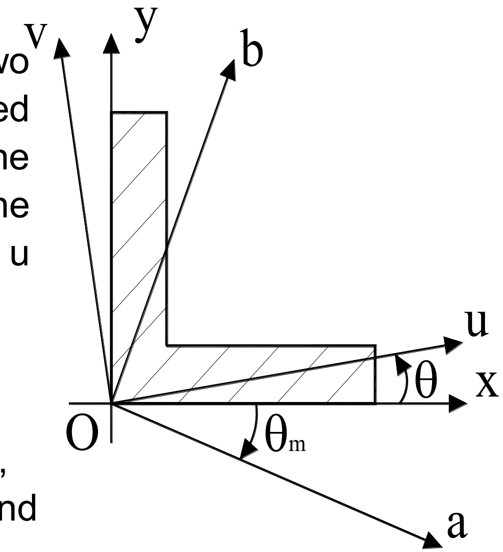


Figure 9.6a

Consider a given area A and two rectangular coordinate axes x and y, Fig. 9.6a.

Assuming that the moments of inertia I_x, I_y and the product of inertia I_{xy} are known, we will represent them on a diagram by plotting a point X of coordinates Ix and Iy and a point Y of coordinates :I_y and $-I_{xy}$, Fig. 9.6b.

Joining X and Y with a straight line, we denote by C the point of intersection of line XY with the horizontal axis and draw the circle of center C and diameter XY. Noting that the abscissa of C and the radius of the circle are respectively equal to the quantities I_{ave} and R defined by the formula (9.15). The angle θm, which defines in Fig. 9.6a the principal axis Oa corresponding to point A in Fig. 9.6b, is equal to half of the angle XCA of Mohr's circle.

Similarly, the point U of coordinates I_u and I_{uv} and the point V of coordinates I_v and $-I_{uv}$ are thus located on Mohr's circle, and the angle UCA in Fig. 9.6b must be equal to twice the angle uOa in Fig. 9.6a. The following can be written from Figs. 9.6a and 9.6b.

$$I_u = I_{ort} + R\cos(2\theta_m + 2\theta) \qquad I_{uv} = R\sin(2\theta_m + 2\theta)$$

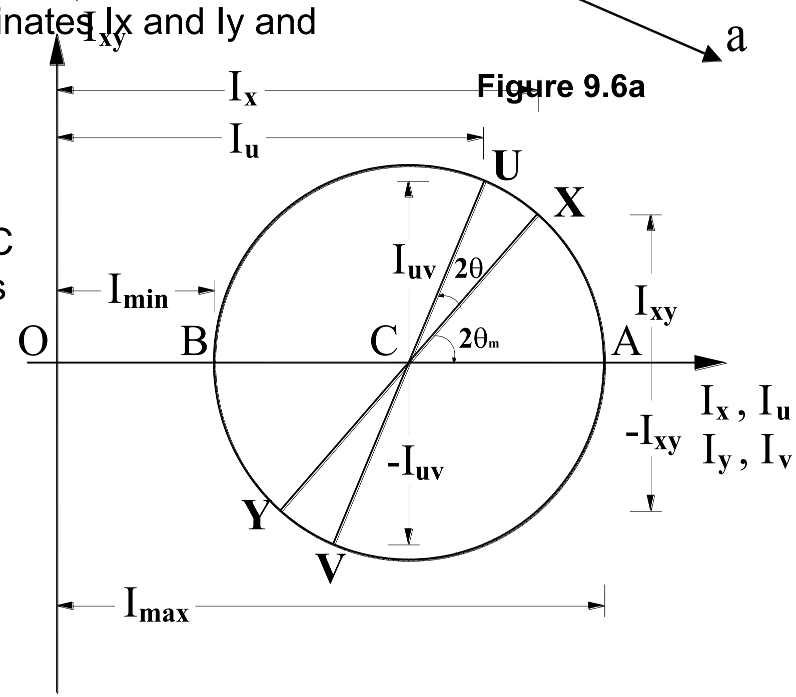


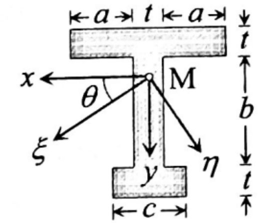
Figure 9.6b

The rotation which brings the diameter XY into diameter UV in Fig. 9.6b has the same sense as the rotation which brings the x and y axes into the u and v axes in Fig. 9.6a. If it is counter clock-wise, it is positive. If not, it is negative.

PROBLEM 10-22 Şekil (10-P22) deki düzlemsel alanın ağırlık merkezi M üzerine yerleştirilmiş (x,y) ve (ξ,η) gibi iki tane dik eksen takımı tanımlanmıştır. Bu dik eksen takımlarının x ve ξ doğrultuları arasındaki açı $\theta=30^\circ$ olup, boyutlar $a=30\,\text{mm}$, $b=80\,\text{mm}$, $c=40\,\text{mm}$ ve $t=20\,\text{mm}$ dir. Buna göre,

a). I_x, I_y ve I_{xy} eylemsizlik momentlerini bulunuz,

b). I_ξ, I_η ve $I_{\xi\eta}$ eylemsizlik momentlerini hesaplayınız.



Şekil (10-P22)

ÇÖZÜM: Eylemsizlik momenti hesabı için önce ağırlık merkezi M nin konumu belirlenmelidir. Şekil (P22.a) daki alan, y eksenine göre simetrik olduğundan sadece y_M koordinat değerini bulmak yeterlidir. Bunun için alanın simetri eksenini ve tabanını kapsayacak biçimde yerleştirilecek bir $(\bar{x},\bar{y})$ eksen takımını kullanalım. Şimdi alanı üç tane dikdörtgenin birleşimi olarak düşünürsek, bunların Şekil (P22.a) daki $(\bar{x},\bar{y})$ takımına göre ağırlık merkezleri $\overline{\mathrm{M}}_1(0,110)$, $\overline{\mathrm{M}}_2(0,60)$, $\overline{\mathrm{M}}_3(0,10)$ dir. Şu halde,

$$A_1=20\times80=1600\,\text{mm}^2,\quad A_2=20\times80=1600\,\text{mm}^2,\quad A_3=20\times40=800\,\text{mm}^2$$

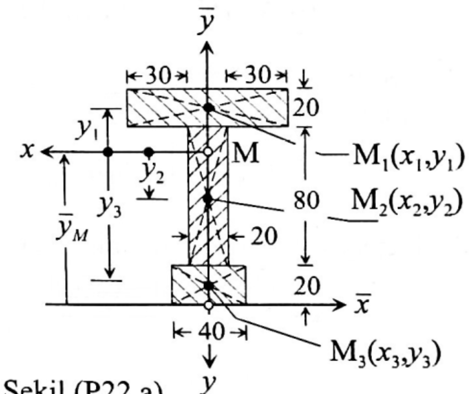
$$\bar{y}_M=\frac{\sum_{i=1}^{3}A_i\bar{y}_i}{\sum_{i=1}^{3}A_i}=\frac{1600\times110+1600\times60+800\times10}{1600+1600+800}=70\,\text{mm}$$

elde edilir. Buna göre (x,y) eksen takımında,

$$y_1=\bar{y}_M-\bar{y}_1=70-110=-40\,\text{mm}$$
$$y_2=\bar{y}_M-\bar{y}_2=70-60=10\,\text{mm}$$
$$y_3=\bar{y}_M-\bar{y}_3=70-10=60\,\text{mm}$$

bulunur. Alan y eksenine göre simetrik olduğundan:

$$x_1=x_2=x_3=0$$



Şekil (P22.a)

a). (x,y) eksen takımında alanların ağırlık merkezleri $\mathrm{M}_1(0,-40)$, $\mathrm{M}_2(0,10)$ ve $\mathrm{M}_3(0,60)$ dır (Bakınız Şekil P22b). Alanların eylemsizlik momentlerini kendi ağırlık merkezlerinde hesapladıktan sonra bunları tüm alanın ağırlık merkezi M ye taşıyalım ve daha sonra toplam yaparak sonuca ulaşalım. Şöyle ki:

$$I_x=\sum_{i=1}^{3}\left(\tfrac{1}{12}b_ih_i^3+A_iy_i^2\right)$$
$$=\left[\tfrac{1}{12}\left(80\times20^3\right)+1600(-40)^2\right]+\left[\tfrac{1}{12}\left(20\times80^3\right)+1600\times10^2\right]+\left[\tfrac{1}{12}\left(40\times20^3\right)+800\times60^2\right]\cong653\times10^4\,\text{mm}^4$$

$$I_y=\sum_{i=1}^{3}\tfrac{1}{12}h_ib_i^3=\tfrac{1}{12}\left(20\times80^3\right)+\tfrac{1}{12}\left(80\times20^3\right)+\tfrac{1}{12}\left(20\times40^3\right)\cong101\times10^4\,\text{mm}^4$$

$$I_{xy}=0$$

Alan y eksenine göre simetrik olduğundan, son eşitlik doğrudan yazılmıştır.

b). ξ ekseni ile x ekseni arasındaki açı 30° dir. Buna göre I_ξ eylemsizlik momenti, dönüşüm bağıntısından,

$$I_\xi=I_x\cos^2 30^\circ+I_y\sin^2 30^\circ-2I_{xy}\sin 30^\circ\cos 30^\circ=\left(653\times10^4\right)0.75+\left(101\times10^4\right)0.25-0\cong 515\times10^4\,\mathrm{mm}^4$$

bulunur. Ama istersek aynı sonuç, çift açılar cinsinden yazılmış dönüşüm bağıntısı kullanılarak,

$$I_\xi=\tfrac{1}{2}\left(I_x+I_y\right)+\tfrac{1}{2}\left(I_x-I_y\right)\cos(2\theta)-I_{xy}\sin(2\theta)\cong 515\times10^4\,\mathrm{mm}^4$$

biçiminde de elde edilir. Burada $2\theta=60^\circ$ dir. η ekseni ile x ekseni arasındaki açı ise 120° dir. Buna göre,

$$\begin{aligned}I_\eta&=I_x\cos^2 120^\circ+I_y\sin^2 120^\circ-2I_{xy}\sin 120^\circ\cos 120^\circ\\&=\left(653\times10^4\right)(-0.5)^2+\left(101\times10^4\right)0.866^2+0\cong 239\times10^4\,\mathrm{mm}^4\end{aligned}$$

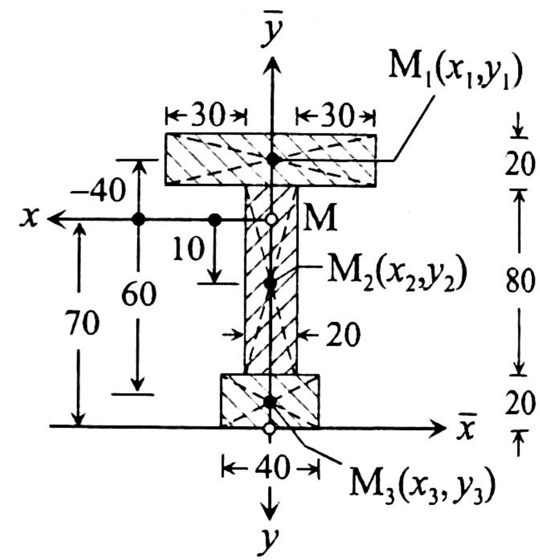
olur. Çarpım eylemsizlik momenti ise 30° için, dönüşüm bağıntısından,

$$\begin{aligned}I_{\xi\eta}&=\left(I_x-I_y\right)\sin 30^\circ\cos 30^\circ+I_{xy}\left(\cos^2 30^\circ-\sin^2 30^\circ\right)\\&=\left(653\times10^4-101\times10^4\right)(0.5\times0.866)+0\cong 239\times10^4\,\mathrm{mm}^4\end{aligned}$$

elde edilir. Ama istersek aynı sonuca, dönüşüm bağıntısının çift açılar cinsinden yazılmış,

$$I_{\xi\eta}=\tfrac{1}{2}\left(I_x-I_y\right)\sin(2\theta)+I_{xy}\cos(2\theta)\cong 239\times10^4\,\mathrm{mm}^4$$

denklemiyle de varmak mümkündür. Burada $2\theta=60^\circ$ dir.

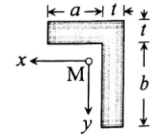


Şekil (P22.b)

PROBLEM 10-23 Şekil (10-P23) de verilmiş olan düzlemsel alanın boyutları $a = 50\,\text{mm}$, $b = 80\,\text{mm}$ ve $t = 20\,\text{mm}$ olup (x, y) eksen takımı ağırlık merkezi M dir. Buna göre,

a). I_x, I_y ve I_{xy} eylemsizlik momentlerini bulunuz,

b). Asal eylemsizlik momentleri ile asal eksen doğrultusunu hesaplayınız.



Şekil (10-P23)

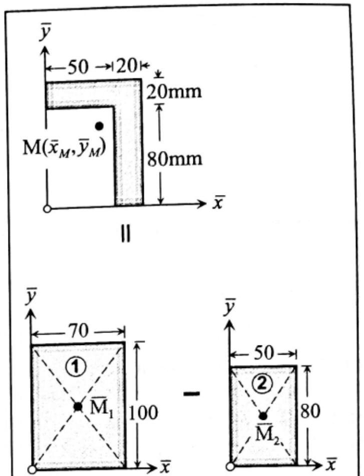
ÇÖZÜM: Eylemsizlik momenti hesabına geçebilmek için öncelikle alanın ağırlık merkezi M nin konumu belirlenmelidir. Alanda herhangi bir simetri ekseni bulunmadığı için Şekil (P23.a) da çizilmiş olan $(\bar{x}, \bar{y})$ eksen takımı kullanılarak ağırlık merkezi $\mathrm{M}(\bar{x}_M, \bar{y}_M)$ nin konumu bu koordinat takımında elde edilir. Ayrıca bu alan Şekil (P23.a) daki iki dikdörtgen alanın farkı diye de düşünülebilir. O zaman bu alanlar,

$$\left.\begin{aligned} A_1 &= 70 \times 100 = 7000\,\text{mm}^2 \\ A_2 &= 50 \times 80 = 4000\,\text{mm}^2 \end{aligned}\right\} \Rightarrow \quad A = A_1 - A_2 = 3000\,\text{mm}^2$$

olur. $(\bar{x}, \bar{y})$ takımında alanların ağırlık merkezleri $\mathrm{M}_1(35{,}50)$, $\mathrm{M}_2(25{,}40)$ dir. Tüm alanın ağırlık merkezi ise,

$$\bar{x}_M = \frac{\sum_{i=1}^{2} A_i \bar{x}_i}{A} = \frac{7000 \times 35 + (-4000)25}{3000} \cong 48\,\text{mm}$$

$$\bar{y}_M = \frac{\sum_{i=1}^{2} A_i \bar{y}_i}{A} = \frac{7000 \times 50 + (-4000)40}{3000} \cong 63\,\text{mm}$$



Şekil (P23.a)

a). Kesitin eylemsizlik momentlerini bulmak için alanın kollarını düşünerek hesap yapalım. O zaman Şekil (P23.b) deki ① ve ② kollarının alanları ile ağırlık merkezleri:

$$A_1 = 70 \times 20 = 1400\,\text{mm}^2, \quad \bar{\mathrm{M}}_1(35{,}90)$$

$$A_2 = 80 \times 20 = 1600\,\text{mm}^2, \quad \bar{\mathrm{M}}_2(60{,}40)$$

(x, y) takımında alanların ağırlık merkezlerinin koordinatları,

$$x_1 = \bar{x}_M - \bar{x}_1 = 48 - 35 = 13\,\text{mm}$$

$$x_2 = \bar{x}_M - \bar{x}_2 = 48 - 60 = -12\,\text{mm}$$

$$y_1 = \bar{y}_M - \bar{y}_1 = 63 - 90 = -27\,\text{mm}$$

$$y_2 = \bar{y}_M - \bar{y}_2 = 63 - 40 = 23\,\text{mm}$$

Alanların eylemsizlik momentlerini kendi ağırlık merkezlerinde hesapladıktan sonra hepsini önce tüm alanın ağırlık merkezi M ye taşıyalım ve daha sonra bunları toplayalım. Şimdi Şekil (P23.c) den yararlanırsak,

$$I_x = \sum_{i=1}^{2}\left(\tfrac{1}{12} b_i h_i^3 + A_i y_i^2\right) = \left[\tfrac{1}{12}\left(70 \times 20^3\right) + 1400(-27)^2\right] + \left[\tfrac{1}{12}\left(20 \times 80^3\right) + 1600 \times 23^2\right] = 276.7 \times 10^4\,\text{mm}^4$$

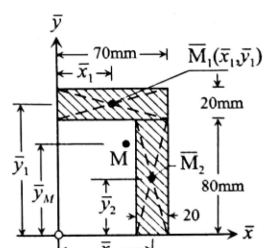
$$\begin{aligned} I_y &= \sum_{i=1}^{2}\left(\tfrac{1}{12} h_i b_i^3 + A_i x_i^2\right) \\ &= \left[\tfrac{1}{12}\left(20 \times 70^3\right) + 1400 \times 13^2\right] + \left[\tfrac{1}{12}\left(80 \times 20^3\right) + 1600(-12)^2\right] \\ &\cong 109.2 \times 10^4\,\text{mm}^4 \end{aligned}$$

olur. Hiçbir simetri ekseni olmayan alanda çarpım eylemsizlik momenti,

$$I_{xy} = \sum_{i=1}^{2} A_i x_i y_i = 1400(-27)13 + 1600 \times 23(-12) = -93.3 \times 10^4\,\text{mm}^4$$

b). Asal eylemsizlik momentlerinin doğrultusu,

$$\tan(2\phi_o) = -\frac{2I_{xy}}{I_x - I_y} = 1.114 \quad \Rightarrow \quad 2\phi_o \cong 48^\circ$$



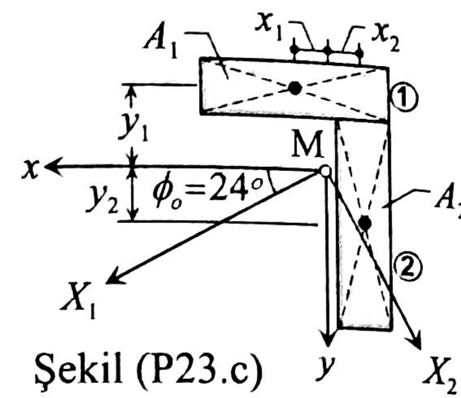
Şekil (P23.b)

dir. (X_1, X_2) asal eksenleri Şekil (P23.c) de görülmektedir. Buna göre; asal eylemsizlik momentleri:

$$I_{1,2}=\tfrac{1}{2}\left(I_x+I_y\right)\pm\sqrt{\left[\tfrac{1}{2}\left(I_x-I_y\right)\right]^2+I_{xy}^2}$$

$$=\left[\tfrac{1}{2}(276.7+109.2)\pm\sqrt{\left[\tfrac{1}{2}(276.7-109.2)\right]^2+(-93.3)^2}\right]10^4$$

➥ $I_1 \cong 318.3\times10^4\,\text{mm}^4\,,\quad I_2 \cong 67.6\times10^4\,\text{mm}^4$



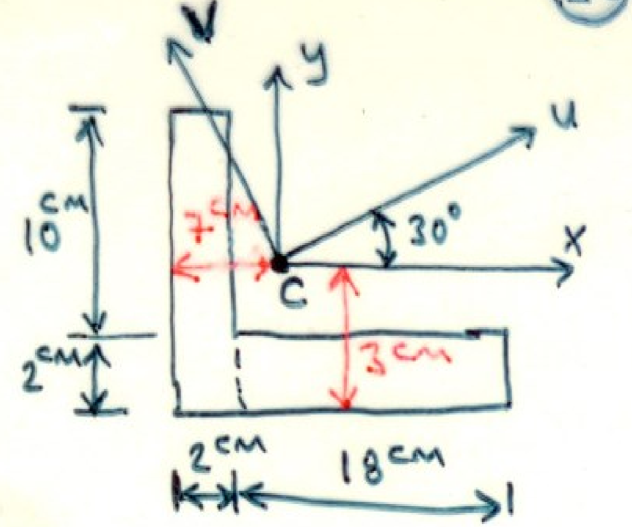
Şekil (P23.c)

Sample problem: For the section shown at the right hand side, determine

(a) center of gravity,

(b) the moments and product of inertia with respect to the centroidal x and y axes,

(c) the moments and product of inertia with respect to the centroidal u and v axes which form an angle of 30° with the x and y axes. Use the Mohr's circle for solving the part (c).



Solution:

(a) Center of gravity:

$$y_c = \frac{12.2.6 + 18.2.1}{12.2 + 18.2} = \frac{180}{60} = 3^{cm}$$

$$x_c = \frac{12.2.1 + 18.2.11}{12.2 + 18.2} = \frac{420}{60} = 7^{cm}$$

(b) The moments and product of inertia with respect to the centroidal x and y axes:

$$I_x = \frac{2.12^3}{12} + (2.12)(6-3)^2 + \frac{18.2^3}{12} + (18.2)(1-3)^2 = 288+216+12+144 = \underline{660\ cm^4}$$

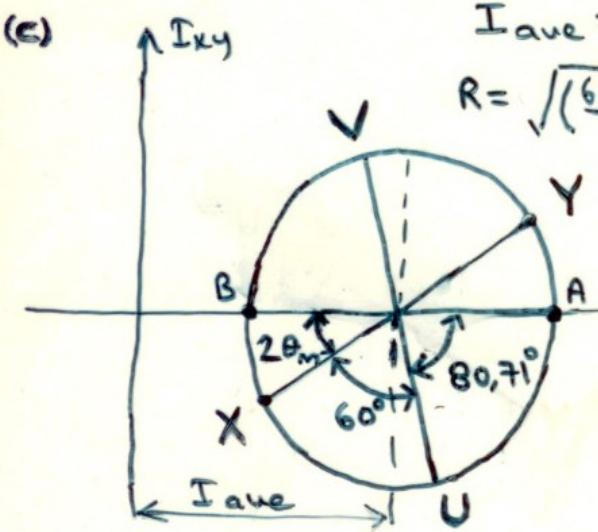
$$I_y = \frac{12.2^3}{12} + (12.2)(1-7)^2 + \frac{2.18^3}{12} + (2.18)(11-7)^2 = 8+864+972+576 = \underline{2420\ cm^4}$$

$$I_{xy} = (2.12)(1-7)(6-3) + (18.2)(11-7)(1-3) = -432-288 = \underline{-720\ cm^4}$$

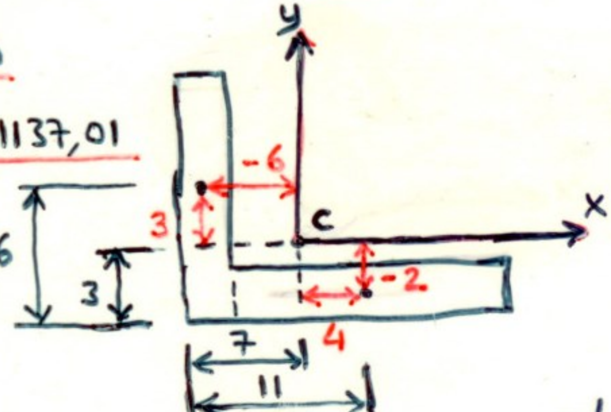
(c)

$$I_{ave} = \frac{660+2420}{2} = \underline{1540}$$

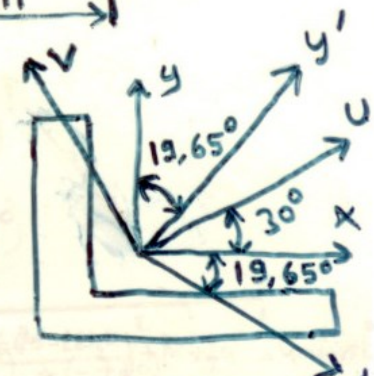
$$R = \sqrt{\left(\frac{660-2420}{2}\right)^2 + (-720)^2} = \underline{1137,01}$$



$X(I_x\ ;\ I_{xy})$
$Y(I_y\ ;\ -I_{xy})$



$$tg\,2\theta_M = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2(-720)}{660-2240} = -0,81\overline{81} \rightarrow \underline{2\theta_M = 39,29°}$$
$$\underline{\theta_M = 19,645°}$$



x', y' principal axes. $\theta_M = 19,65°$

$$I_u = I_{ave} + R.\cos 80,71° = \underline{1723,55\ cm^4}$$

$$I_{uv} = -R.\sin 80,71° = \underline{-1122,097\ cm^4}$$

$$I_u + I_v = I_x + I_y \rightarrow 1723,55 + I_v = 660 + 2420 \rightarrow \underline{I_v = 1356,45\ cm^4}$$

MOMENTS OF INERTIA OF MASSES

9.5. Moment of Inertia of a Mass

Consider a small mass Δm mounted on a rod of negligible mass which can rotate freely about an axis AA', Fig. 9.7a. If a couple is applied to the system, the rod and mass, assumed to be initially at rest, will start rotating about AA'. The details of this motion will be studied later in dyanmics. At present, we wish only to inidicate that the time required for the system to reach a given speed of rotation is proportional to the mass Δm and to the square of the distance r.

The product $r^2 \Delta m$ provides, therefore, a measure of the resistance the system offers when we try to set it in motion. For this reason, the product $r^2 \Delta m$ is called the moment of inertia of the mass Δm with respect to the axis AA'.

Consider now a body of mass m which is to be rotated about an axis AA', Fig. 9.7b. Dividing the body into elements of mass $\Delta m_1, \Delta m_2, ..., \Delta m_n$, we find that the body's resistance to being rotated is measured by the sum

$r_1^2 \Delta m_1 + r_2^2 \Delta m_2 + ... + r_n^2 \Delta m_n$.

This sum defines, therefore, the moment of inertia of the body w.r.t. the axis AA'. Increasing the number of elements, we find that the moment of inertia is equal, in the limit, to the integral

$$I = \int r^2 \, dm \qquad (9.21)$$

The radius of gyration k of the body w.r.t. the axis AA' is defined by the relation:

$$I = k^2 m \quad \text{or} \quad k = \sqrt{I / m} \qquad (9.22)$$

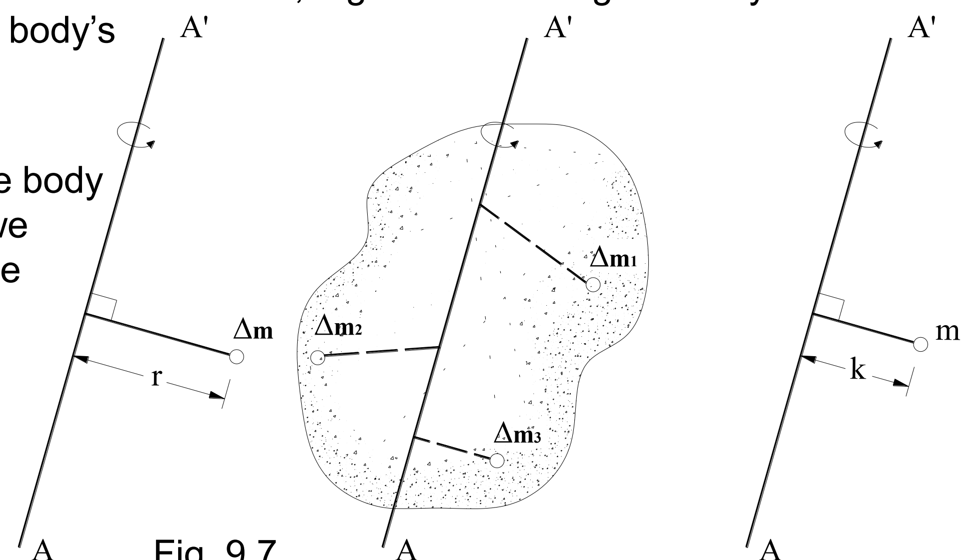
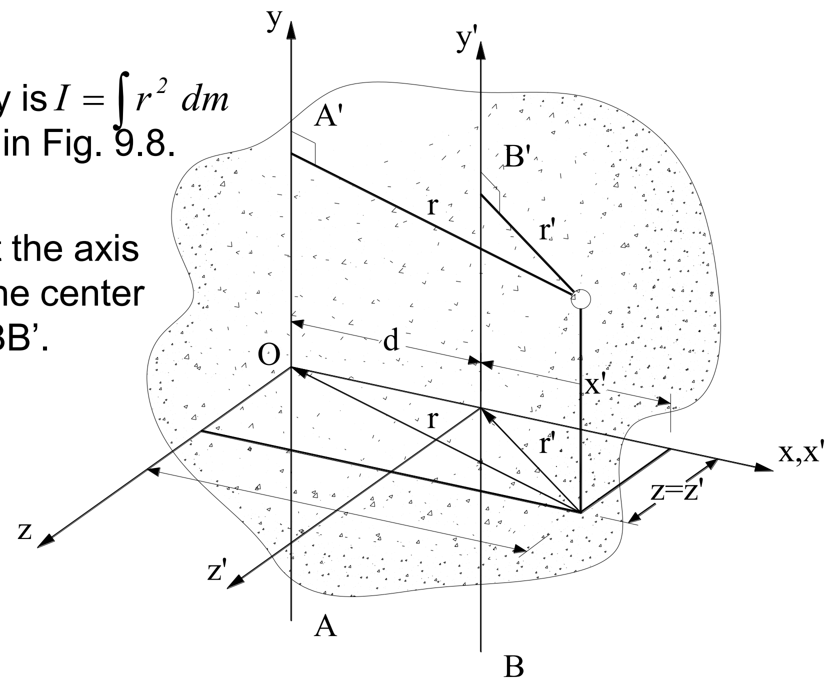


Fig. 9.7

The radius of gyration k represents, therefore, the distance at which the entire mass of the body should be concentrated if its moment of inertia w.r.t. AA' is to remain unchanged, Fig. 9.7c. Whether it is kept in its original shape, Fig. 9.7b, or whether it is concentrated as shown in Fig. 9.7c, the mass m will react in the same way to a rotation, or gyration, about AA'. The radius of gyration is expressed in centimeters and the mass is in kilograms, and the moment of inertia of a mass is *kgcm²*.

9.6. Parallel-axis theorem



Consider a body of mass, m. The moment of inertia of the body is $I = \int r^2 \, dm$ about AA' axis. *r dm* is the distance of the mass to the axis AA' in Fig. 9.8.

Similarly, the moment of inertia of the body is $\bar{I} = \int r'^2 \, dm$ w.r.t the axis BB' whose axes are parallel to AA' axis and whose origin is at the center of gravity of the body. *r'* is the distance of the mass to the axis BB'.

Lets choose the two systems of axes as shown in Fig. 9.8.

The followings can be written:

$$r^2 = x^2 + z^2 \qquad r'^2 = x'^2 + z'^2$$

Considering the distance between AA' and BB' is equal to d,

$x = x' + d$ and $z = z'$ can be written. Then, $r^2 = (x'+d)^2 + z'^2 = r'^2 + 2x'd + d^2$

can be found. Setting r^2 in $I = \int r^2 \, dm$, the moment of inertia w.r.t AA' can be obtained as follows:

$$I = \int r^2 dm = \int r'^2 \, dm + 2d \int x' \, dm + d^2 \int dm$$

The first integral in this expression represents the moment of inertia w.r.t. B' axis, the second integral represents the first moment of the body w.r.t. the y'z'plane, and, since G is in the plane, the integral is zero, the last integral is equal to the total mass m of the body. Therefore,

$$I = \bar{I} + m d^2 \qquad (9.23)$$

Expressing the moments of inertia in terms of the corresponding radii of gyration, we can also write

$$k^2 = \bar{k}'^2 + d^2 \qquad (9.24)$$

where k and $\bar{k}'$ represent the radii of gyration of he body about AA' and BB', respectively.

9.7 Moments of Inertia of Thin Plates

Consider a thin plate of uniform thickness t, which is made of a homogeneous material of density ρ (density=mass per unit volume).

The mass moment of inertia of the plate w.r.t. an axis AA'düzleminde bulunan AA' contained in the plate of the plate, Fig. 9.9a, is

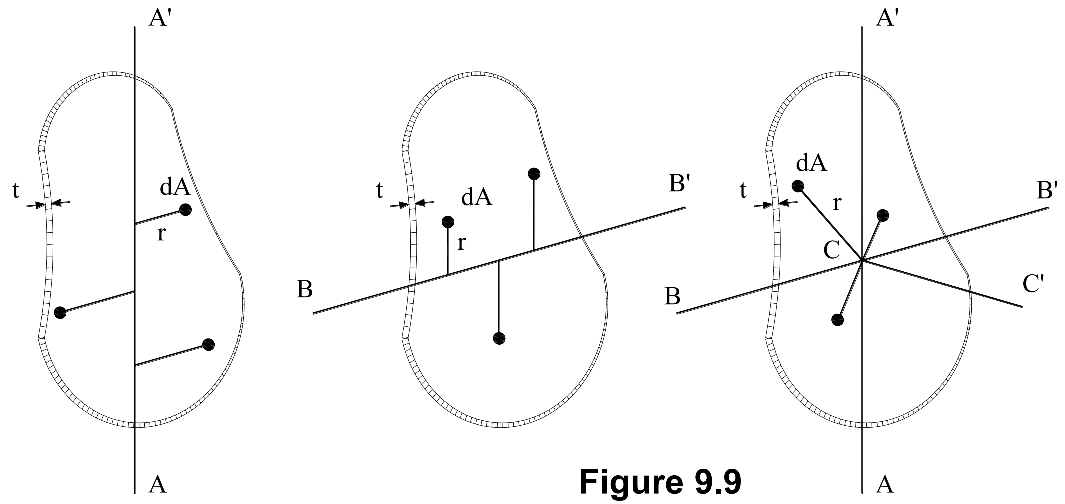


Figure 9.9

$I_{AA'kütle} = \int r^2 dm$ since $dm = \rho t dA$ we write

$$I_{AA'kütle} = \rho t \int r^2 dA$$

But r represent the distance of the element of area dA to the axis AA'; the integral is therefore equal to the moment of inertia of the area of the plate w.r.t. AA'. We have

$$I_{AA'\,kütle} = \rho t I_{AA'alan} \qquad (9.25)$$

Similarly, for an axis BB' which is contained in the plane of the plate and is perpendicular to AA', Fig. 9.9b, we have

$$I_{BB'\,kütle} = \rho t I_{BB'alan} \qquad (9.26)$$

Considering now the axis CC' which is perpendicular to the plate and passes through the point of intersection C of AA' and BB', Fig. 9.9c, we write

$$I_{CC'\,kütle} = \rho t\, I_{C\,alan} \qquad (9.27)$$

where $I_{C\,alan}$ is the polar moment of inertia of the area of the plate w.r.t. point C.

Recalling the relation $I_C = I_{AA'} + I_{BB'}$ which exists between polar and rectangular moments of inertia of an area, we write the following relation between the mass moments of inertia of a thin plate:

$$I_{CC'} = I_{AA'} + I_{BB'} \qquad (9.28)$$

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Rectangular Plate: in the case of a rectangular plate of sides a and b, Fig. 9.13, we obtain in the following mass moments of inertia w.r.t. axes through the center of gravity of the plate:

$$I_{AA'\,kütle} = \rho t\, I_{AA'alan} = \rho t\,(a^3 b/12)$$

$$I_{BB'\,kütle} = \rho t\, I_{BB'alan} = \rho t\,(ab^3/12)$$

Observing that the product $\rho a b t$ is equal to the mass m of the plate, we write the mass moments of inertia of a thin rectangular plate as follows:

$$I_{AA'} = ma^2/12 \qquad I_{BB'} = mb^2/12 \qquad (9.29)$$

$$I_{CC'} = I_{AA'} + I_{BB'} = \frac{m}{12}(a^2 + b^2) \qquad (9.30)$$

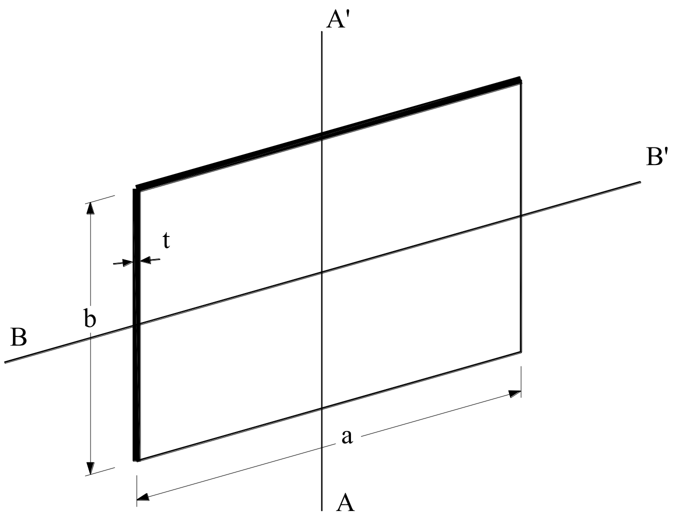

Figure 9.10

Circular Plate: In the case of a circular plate, or disk, of radius r, Fig. 9.10, we write

$$I_{AA'\,kütle} = \rho t I_{AA'alan} = \rho t(\pi r^4/4)$$

Observing that the product $\rho \pi r^2 t$ is equal to the mass m of the plate and that $I_{AA'} = I_{BB'}$ we write the mass moments of inertia of a circular plate as follows:

$$I_{AA'} = I_{BB'} = mr^2/4 \qquad (9.31)$$

$$I_{CC'} = I_{AA'} + I_{BB'} = mr^2/2 \qquad (9.32)$$

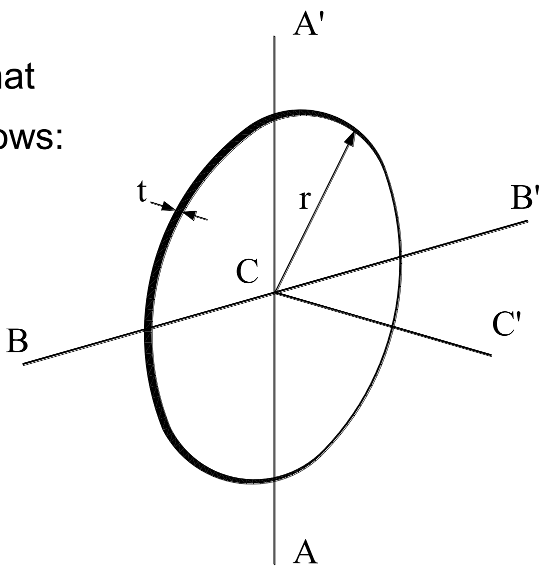

Figure 9.11

9.8. Determination of the Moment of Inertia of a Three-Dimensional Body by Integration

The moment of inertia of a three-dimensional body is obtained by evaluating the integral $I = \int r^2 dm$.

If the body is made of a homogenous material of density ρ, the element of mass, dm, is equal to $dm = \rho\, dV$ and we can write $I = \rho \int r^2 dV$.

This integral depends only upon the shape of the body. Thus, in order to compute the moment of inertia of a three-dimensional body, it will generally be necessary to perform a triple, or at least a double integration.

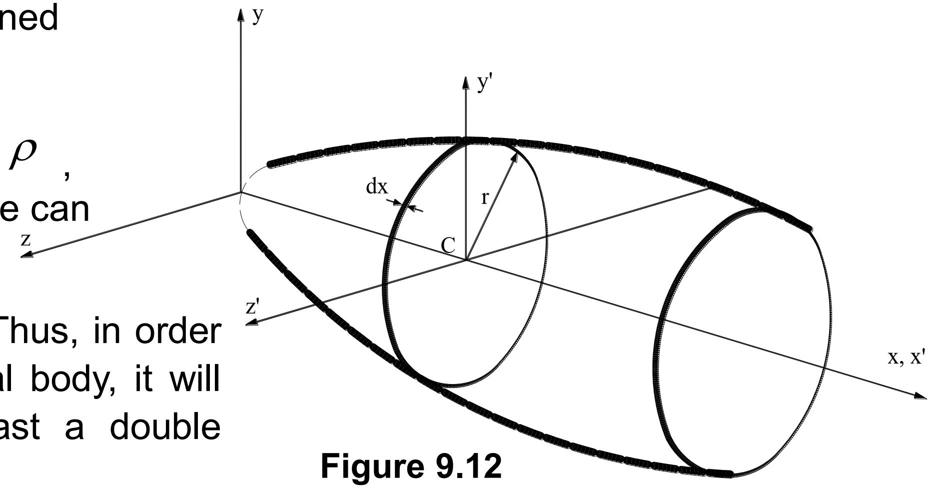

Figure 9.12

However, if the body possesses two planes of symmetry it is usually possible to determine the body's moment of inertia with a single integration by choosing as the element of mass dm a thin slab which is perpendicular to the planes of symmetry.

In the case of bodies of revolution, for example, the element of mass would be a thin disk, Fig. 9.15. Using formula (9.32), the moment of inertia of the disk with respect to the axis of revolution can be expressed as indicated in Fig. 9.12. Its moment of inertia w.r.t. each of the other two coordiante axes is obtained by using formula (9.31) and the parallel-axis theorem. Integration of the expression obtained yields the desired moment of inertia of the body.

9.9. Moments of Inertia of Composite Bodies

The moments of inertia of a few common shapes are shown in Fig. 9,12. For a body consisting of several of these simple shapes,the moment of inertia of the body w.r.t. a given axis can be obtained by first computing the moments of inertia of its component parts about the desired axis and then adding them together. As was the case for areas, the radius of gyration of a composite body cannot be obtained by adding the radii of gyration of its component parts.