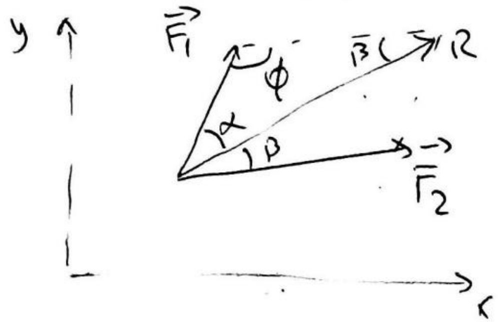
YILDIZ TECHNICAL UNIVERSITY
CIVIL ENGINEERING DEPARTMENT
DIVISION OF MECHANICS

THIRD CHAPTER
RIGID BODIES AND EQUIVALENT SYSTEMS OF FORCES

RIGID BODIES AND FORCE RESULTANT (~~IN PLANE CASE~~ COPLANAR FORCES)

3.1 Resultant of two forces (concurrent forces)



$$\vec{R} = \vec{F_1} + \vec{F_2}$$

$$\vec{R} = \left(F_{1x}\vec{i} + F_{1y}\vec{j}\right) + \left(F_{2x}\vec{i} + F_{2y}\vec{j}\right)$$

$$= \left(F_{1x} + F_{2x}\right)\vec{i} + \left(F_{1y} + F_{2y}\right)\vec{j}$$

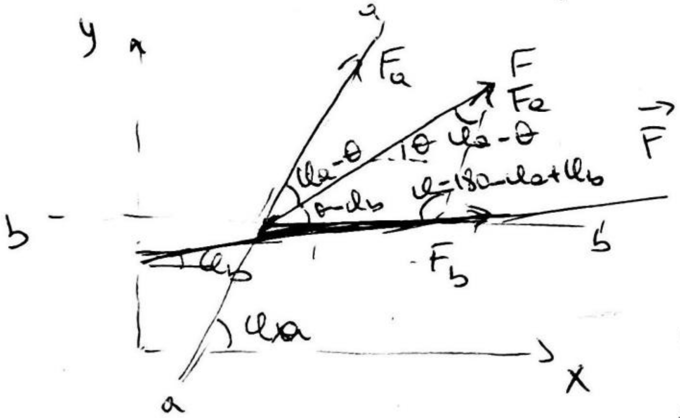
Theory of sinus

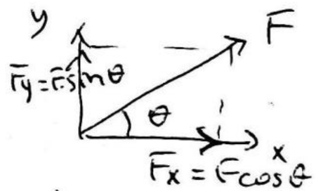
$$\frac{R}{\sin\phi} = \frac{F_1}{\sin\beta} = \frac{F_2}{\sin\alpha}$$

Cosine law

$$R = \sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos\phi}$$

3.2. Resolving [redacted] force into its components





$$\vec{F} = \vec{F_a} + \vec{F_b}$$

$$\vec{F} = F_x\,i + F_y\,j \rightarrow$$ (Rectangular Components in cartesian coordinates)

$$F_x i + F_y j = F_{ax} i + F_{ay} j + F_{bx} i + F_{by} j$$

$$F\cos\theta = F_x = F_{ax} + F_{bx} = F_a\cos\alpha_a + F_b\cos\alpha_b$$

$$F\sin\theta \quad F_y = F_{ay} + F_{by} = F_a\sin\alpha_a + F_b\sin\alpha_b$$

Solve for F_a and F_b

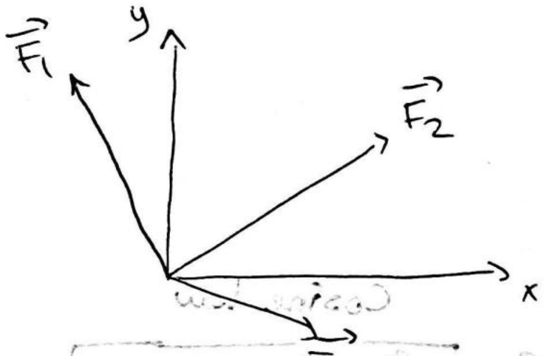
α_a and α_b are known

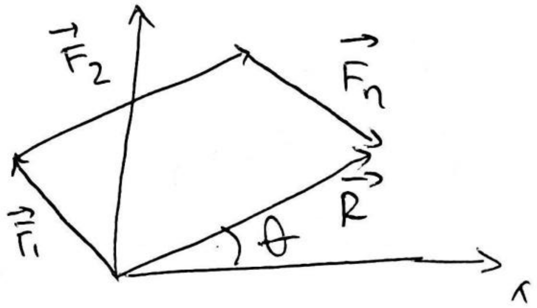
$$\frac{F}{\sin\alpha} = \frac{F_a}{\theta - \alpha_b} = \frac{F_b}{\alpha_a - \theta}$$

θ is the angle between F and x axis

$$\vec{F} = \left(F_{ax} + F_{bx}\right) i + \left(F_{ay} + F_{by}\right) j$$

3.3 Resultant of System of Forces





$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

Using Rectangular components

$$\vec{R} = R_x \vec{i} + R_y \vec{j}$$

$$R_x = \sum F_{ix} = F_{1x} + F_{2x} + F_{nx}$$

$$R_y = \sum F_{iy} = F_{1y} + F_{2y} + F_{ny}$$

Magnitude $R = \sqrt{R_x^2 + R_y^2}$

Direction (Angle betwen R and x axis) $\tan \theta_x = \dfrac{R_y}{R_x}$

$\theta_x = \arctan \dfrac{R_y}{R_x}$ $\qquad \theta_y = \dfrac{\pi}{2} - \theta_x$

~~3.3 Resultan~~ 3.4 Equilibrium of Particle

Condition for Equilibrium is simple:

The resultant of all the forces acting on the particle must be zero:

$$\vec{R} = 0$$

$$\vec{R} = R_x \vec{i} + R_y \vec{j} + R_z \vec{k} = 0$$

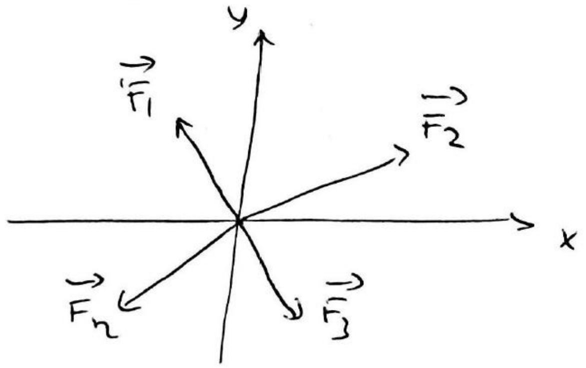
$$R_x = F_{1x} + F_{2x} + F_{3x} + F_{nx} = 0$$

$$= \Sigma F_{ix} = 0$$

$$R_y = F_{2y} + F_{2y} + F_{3y} + F_{ny} = 0$$

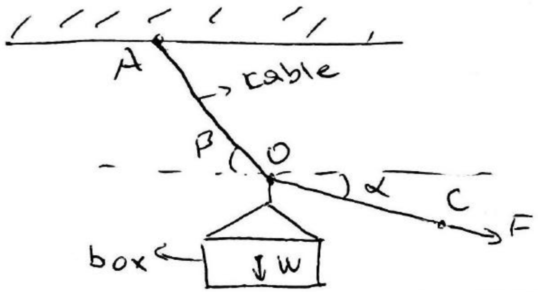
$$= \Sigma F_{iy} = 0$$

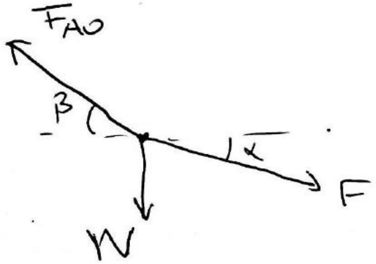
$\vdots$



3.5 Free Body Diagram (FBD)

FBD is a sketch of the outlined shape of the body, whic represents it as being "free" from its surroundings. (e.g. supperst)





if the system is in equilibrium Then,

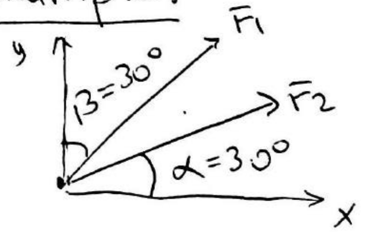
$$R_x = \Sigma F_x = 0$$

$$-F_{AO} \cos\beta + F \cos\alpha = 0$$

$$R_y = \Sigma F_y = 0$$

$$F_{AO} \sin\beta - F \sin\alpha - W = 0$$

Example:



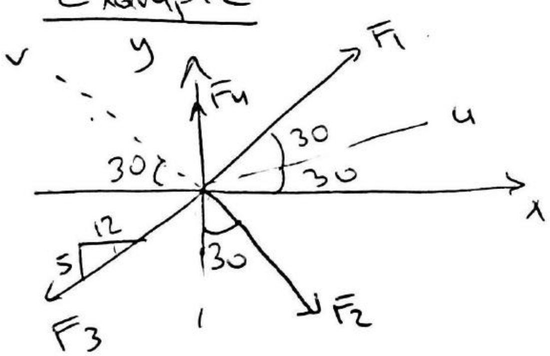
$F_1 = 100$ kN
$F_2 = 50$ kN
Find the resultant of R of the force system given in Fig.
Find the direction of R.

$$\vec{R} = R_x \vec{i} + R_y \vec{j}$$

$$R_x = \Sigma F_x = F_1 . \sin 30 + F_2 . \cos 30 = 93{,}30 \text{ kN}$$
$$R_y = \Sigma F_y = F_1 . \cos 30 + F_2 . \sin 30 = 111.6 \text{ kN}$$
$$R = \sqrt{93{,}30^2 + 111{,}6^2} = 145{,}46 \text{ kN}$$

direction $\theta = \arctan \dfrac{R_y}{R_x} \Rightarrow \theta = 50{,}1^\circ$

Example



$F_1 = F_2 = F_4 = 5$ kN
$F_3 = 26$ kN

a) Finde the resultant of the given Force system.
b) Resolve the resultant of R into its components in u an v director

$$R_x = 5.\cos 60 + 5.\sin 60 - F_3 . \frac{12}{13} = -19 \text{ kN}$$
$$R_y = 5.\sin 60 + 5 - 5.\cos 30 - 26.\frac{5}{13} = -5 \text{ kN}$$
$$R = \sqrt{18^2 + 5^2} = 18.65 \text{ kN}$$

direction $\tan \beta = \dfrac{-5}{-19} \rightarrow \beta = 14{,}74^\circ$

b) $R_x = F_u \cdot \cos 30 - F_v \cdot \cos 30 = -19$

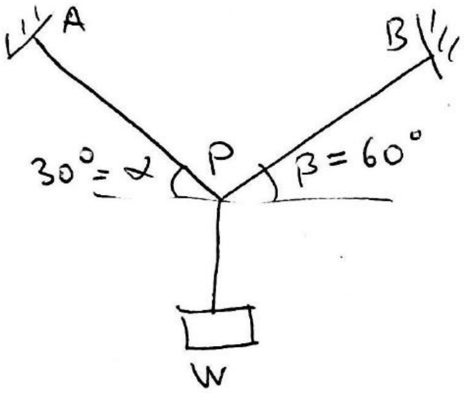
$R_y = F_u \cdot \sin 30 + F_v \cdot \sin 30 = -5$

Solving for F_u and F_v

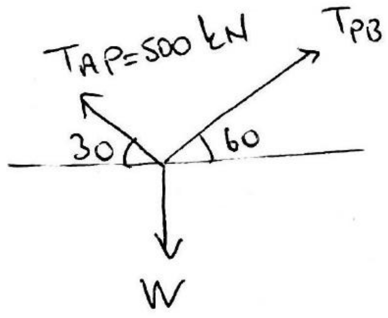
$F_u = -15{,}97$ kN

$F_v = 5{,}97$ kN

Example



The maximum load that can be carried by the AP rope is 500 kN.
Find the maximum weight W which can be carried by the ropes?



$\vec{R} = 0$

$R_x = \Sigma F_x = 0$

$-T_{AP} \cdot \cos 30 + T_{PB} \cdot \cos 60 = 0$

$T_{PB} = 866.03$ kN

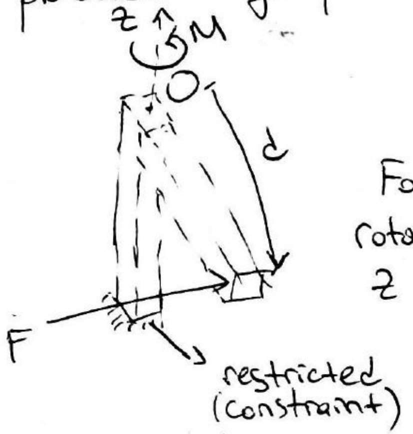
$R_y = \Sigma F_y = 0$

$T_{AP} \cdot \sin 30 + T_{PB} \cdot \sin 60 - W = 0$

$W = 1000$ kN

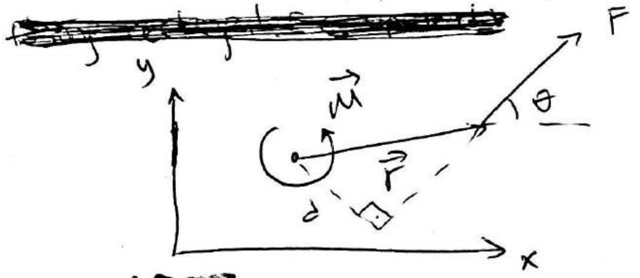
Moment of a Force

Moment of a Force about a point O can be expressed by vector product of position vector $\vec{r}$ and force F.



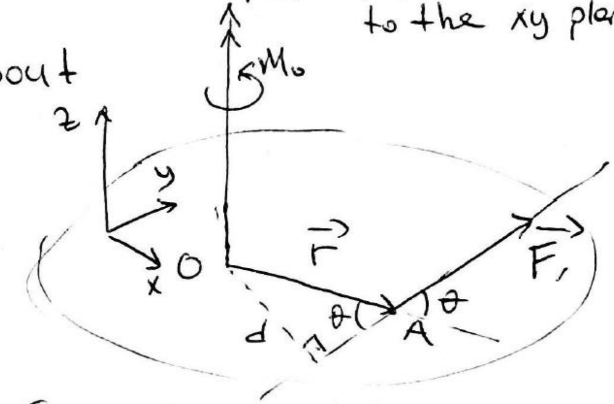
$M = F.d$

Force F tends to rotate the bar about z axis.



Using Vector Notation

$\vec{M}_o \to$ perpendicular to the xy plane



$\vec{M}_o = \vec{r} \times \vec{F}$

$M_o = rF\sin\theta$

$d = r.\sin\theta$

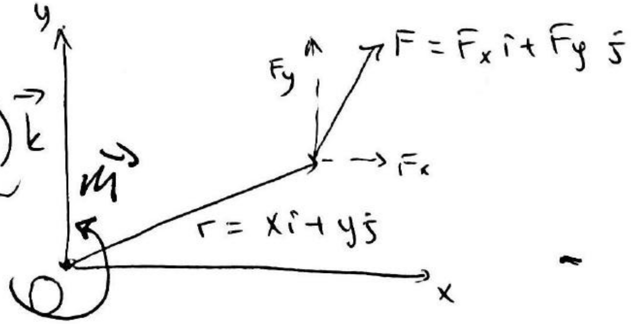
$M_o = F.d$

Sign of moment M_o can be determined by right hand rule

$\vec{F} = F_x\vec{i} + F_y\vec{j}$

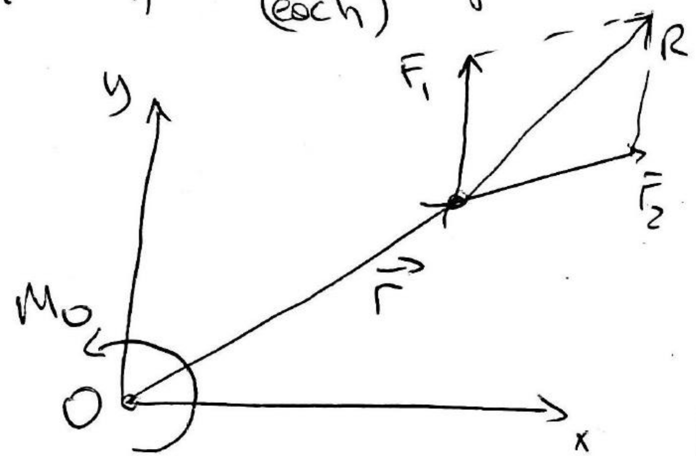
$\vec{r} = x\vec{i} + y\vec{j}$

$$\vec{M}_o = \vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ x & y & \\ F_x & F_y & \end{vmatrix} = \underbrace{(xF_y - yF_x)}_{\text{magnitute of } M_o}\vec{k}$$



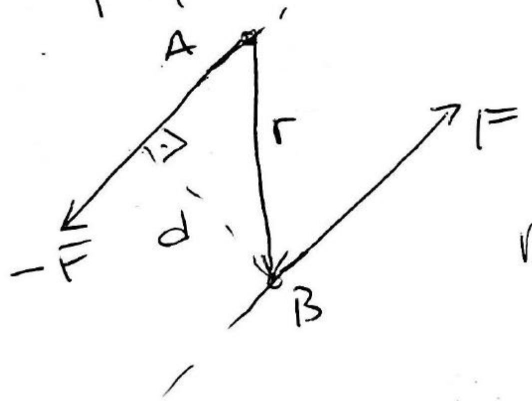
Varignon Theorem: Moment of the Resultant of a system of forces about a point equals to the sum of moments of individual forces (each)

$$\vec{M}_o = \vec{r} \times \vec{R}$$
$$= \vec{r} \times (\vec{F}_1 + \vec{F}_2)$$
$$M_o = \underbrace{\vec{r} \times \vec{F}_1}_{\vec{M}_1} + \underbrace{\vec{r} \times \vec{F}_2}_{\vec{M}_2} = \vec{r} \times \vec{R}$$

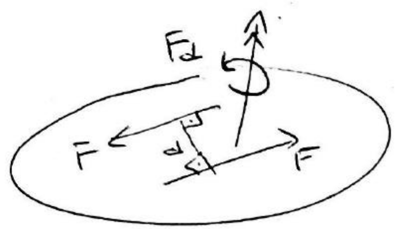


Moment of a Couple

<u>Couple</u> is defined as two parallel forces that have the same magnitude, have opposite direction and separated by a perpendicular distance d. Moment represents by free vectors

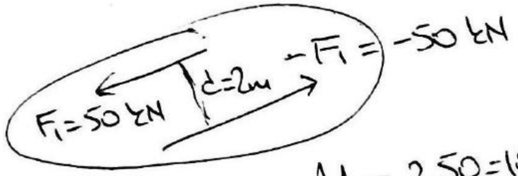


$$\vec{M}_A = r \times F$$
$$\vec{M}_B = (-r) \times (-F) = \vec{r} \times \vec{F} = \vec{M}_B$$
$$M_A = M_B = M = Fd$$

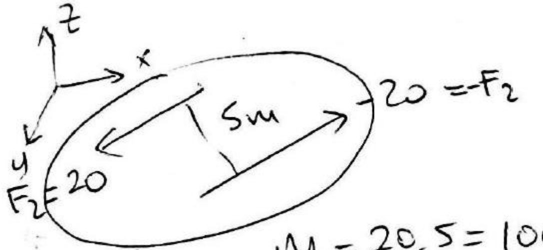


Equivalent Couples

Two Couples are said to be equivalent if they produce same moment.

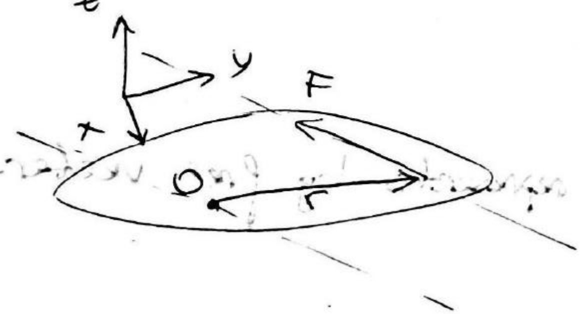


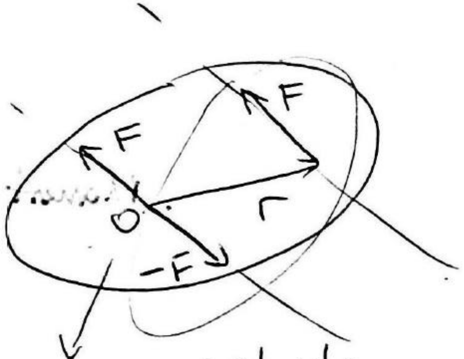
$M = 2.50 = 100$ kN



$M = 20.5 = 100$ kN

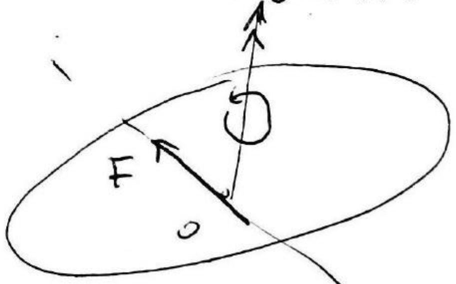
Moving a Force to a point out of its line of action





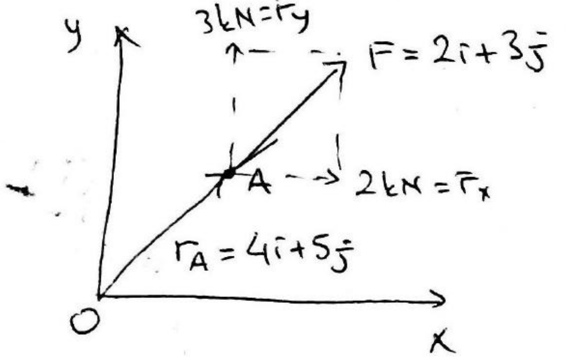
Forces with same magnitude but opposite sense are added

$\vec{M}_o = \vec{r} \times \vec{F}$



Force Couple system

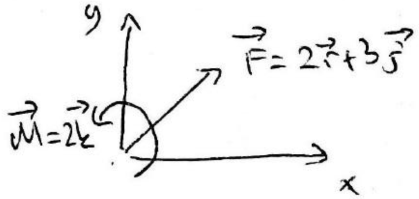
Example:



A(4,5)'in noktasına etkiyen $F = 2i + 3j$'lik kuvveti O noktasına taşıyın.

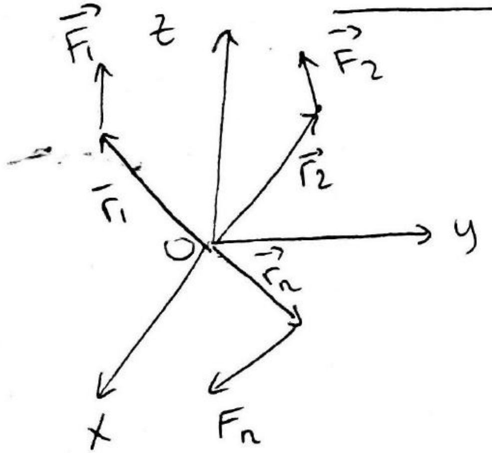
$$M_O = r \times F = \begin{vmatrix} i & j \\ 4 & 5 \\ 2 & 3 \end{vmatrix} = (4\times3 - 5\times2)\,k = \underline{2k} \text{ kNm}$$

$$M = F_y \cdot x_A - F_x\, y_A = 3\cdot4 - 2\cdot5 = 2 \text{ kNm} \circlearrowleft$$



~~Kuvvetler Sisteminin~~

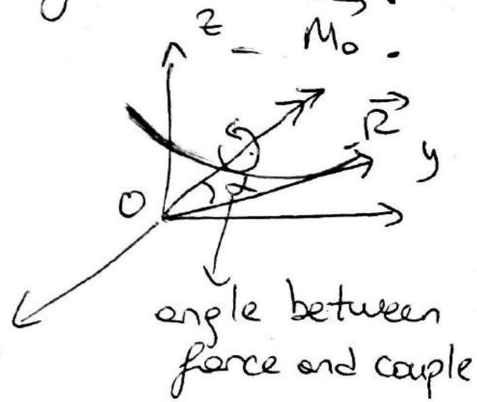
Reduction of Force System to a Point



Reduction results in force-couple system:

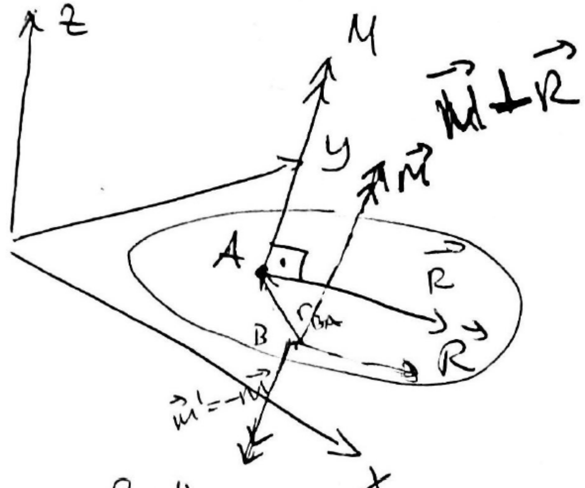
$$\vec{R} = \Sigma \vec{F_i} = \vec{F_1} + \vec{F_2} + \vec{F_3}$$

$$M = \Sigma\, r_i \times F_i = \vec{r_1} \times \vec{F_1} + \vec{r_2} \times \vec{F_2} + \vec{r_n} \times \vec{F_n}$$

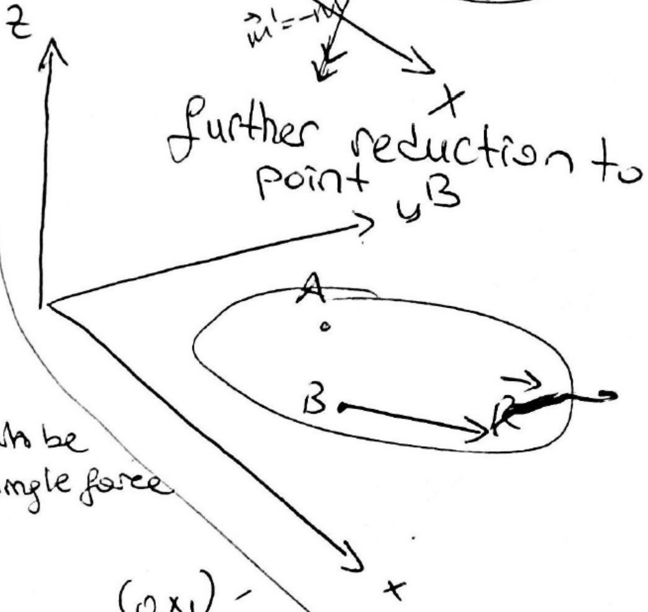


Further Reduction of force system to a Single force

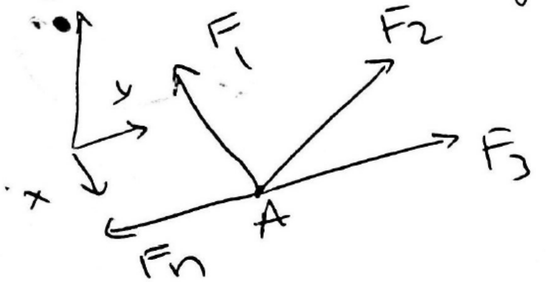
if force vector $\vec{R}$ is perpendicular to $\vec{M}$ then its possible to reduce force system to a single force



further reduction to point B



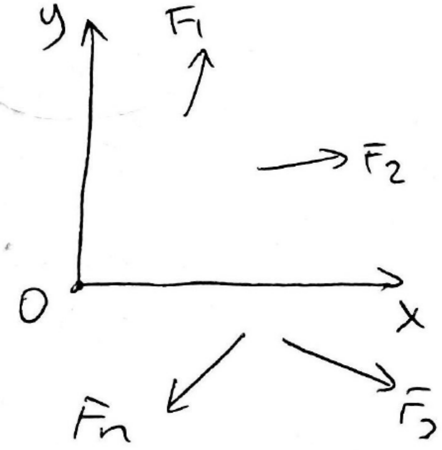
a) Concurrent forces

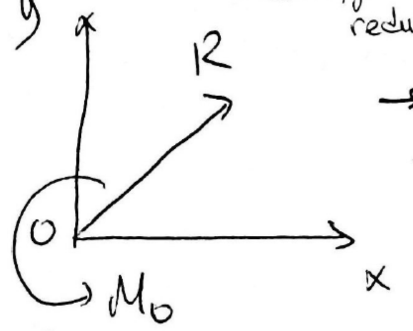


system can be reduced to $\vec{R}$

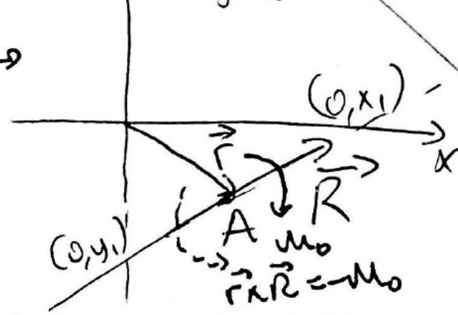
$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_n$

b) Coplanar Forces: If Forces are moved to arbitrary point O the results is force couple system





since $\vec{M} \perp \vec{R}$, force system can be reduced to a single force



The moment of R with respect to point O is;

$$\vec{M}_o = M_o\vec{k} = \begin{vmatrix} \vec{i} & \vec{j} \\ x & y \\ R_x & R_y \end{vmatrix} = (xR_y - yR_x)\vec{k}$$

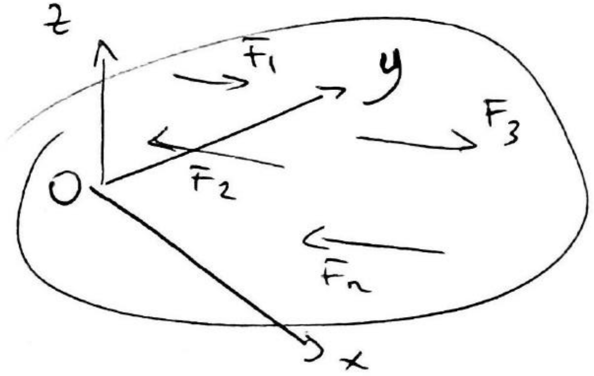
$M_o = xR_y - yR_x \rightarrow$ equation of line of action

for $y=0$ $\quad x_1 = \dfrac{M_o}{R_y}$

for $x=0$ $\quad y_1 = \dfrac{-M_o}{R_x}$

$\Big\}$ points that intersect the x and y axis

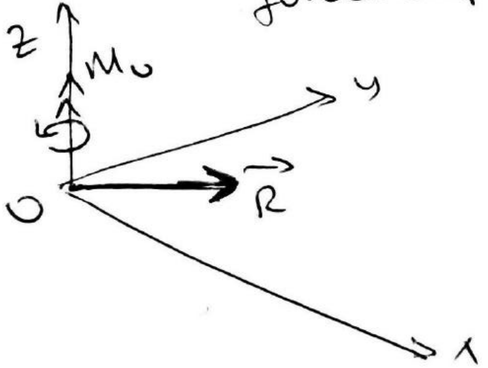
c) Parallel Forces: Forces lie in the same plane (xy plane)



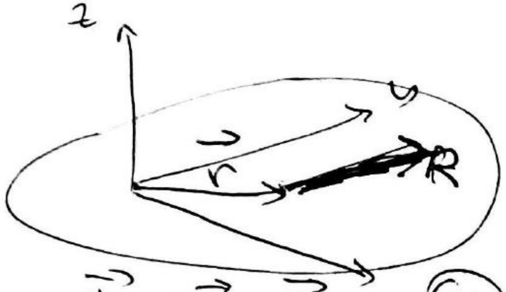
Resultant $\vec{R} = R_x\vec{i} + R_y\vec{j}$

Moment at O $\quad \vec{M} = \sum \vec{r}_i \times \vec{F}_i = M_o\vec{k}$

Then it is obvious that $R \perp M$, Hence force-couple system can be reduced to a single force

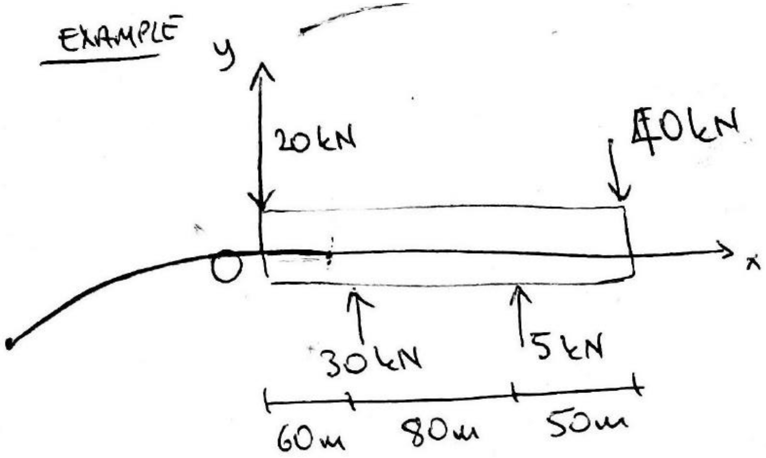


further reduction $\longrightarrow$



$\vec{M}_o = \vec{r} \times \vec{F}$

EXAMPLE



Find the resultant Force of ~~given~~ Parellel forces applied on the rigid body?

$R = R_x\vec{i} + R_y\vec{j}$

$R_x = 0$

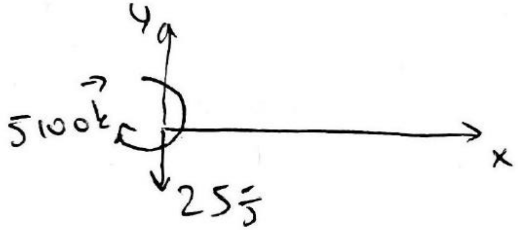
$R_y = -20 + 30 + 5 - 40 = -25$ kN

$\vec{R} = -25\vec{j}$

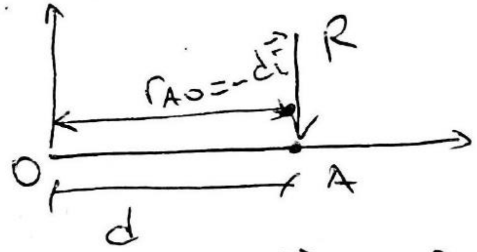
Moment at O

$+\circlearrowleft M_o = 60 \cdot 30 + 140 \cdot 5 - 190 \cdot 40 = -5100$ kN m

$M_o = -5100\vec{k}$



if we are to find a single resultent force we have to move the force-couple system to point A



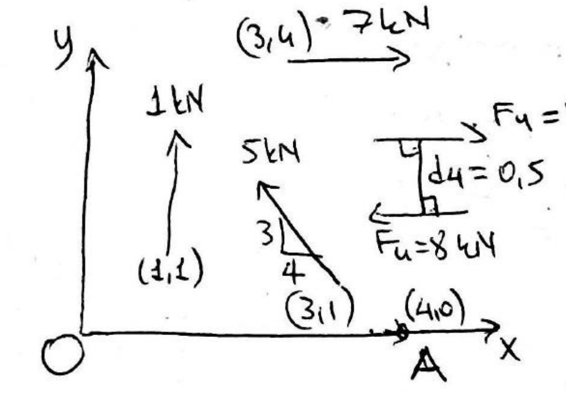
$\vec{M}_o = \vec{r} \times \vec{R}$

$M_o\vec{k} = (xR_y - yR_x)\vec{k}$

$-5100 = x(-25)$

$x = 204$ m

EXAMPLE:



For the given force system

a) Find the resultant at point O (reduce the forces system to point O)

b) Find the equation for line of action

c) Reduce the force system to point A

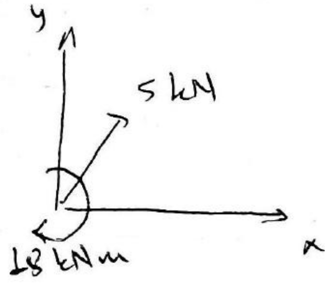
a) $R_x = 7 + 8 - 8 - 5 \cdot \frac{4}{5} = 3$ kN

$R_y = 1 + 5 \cdot \frac{3}{5} = 4$ kN

$+\circlearrowleft M_O = 1 \cdot 1 + 3 \cdot \frac{3}{5} + 1 \cdot \frac{4}{5} - 8 \cdot 0,5 - 7 \cdot 4 = -18$ kNm

$R = \sqrt{R_x^2 + R_y^2} = 5$ kN

$\tan\phi = \frac{R_y}{R_x} = \frac{4}{3}$ $\quad \phi = 53.13°$

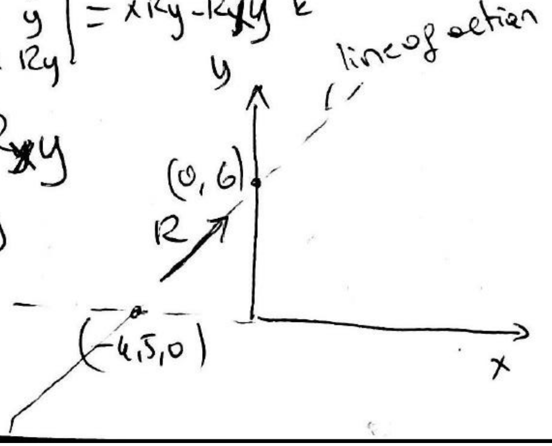


b) Equation ~~of~~ of line of action

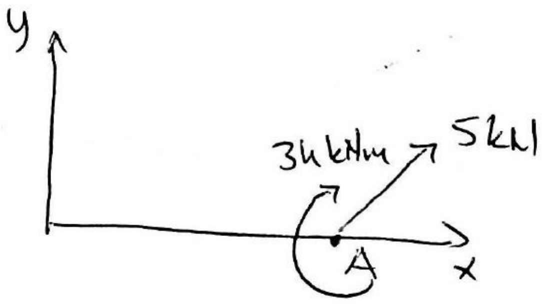
$\vec{M_O} = \vec{r} \times \vec{R} = \begin{vmatrix} i & j \\ x & y \\ R_x & R_y \end{vmatrix} = xR_y - R_x y \ \vec{k}$

$M_O = xR_y - R_x y$

$-18 = x4 - 3y$



c) $M_A = -4 \cdot 4 - 18 = -34$ kNm



Resultant of a two vector

Equilibrium of coplanar forces

Equilibrium of ~~three~~ forces

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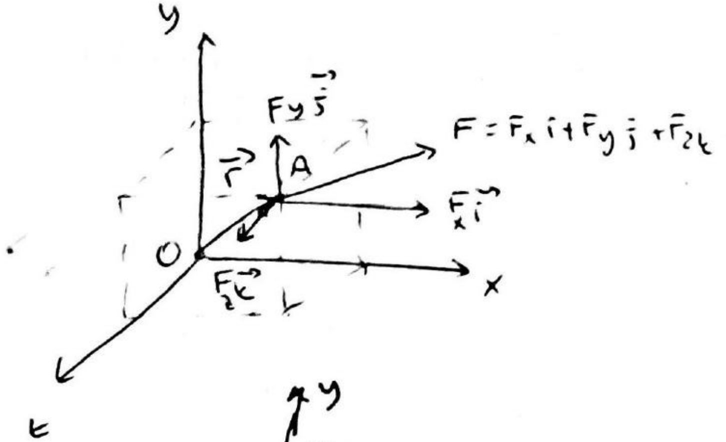
Rectangular components of the Moment of a Force



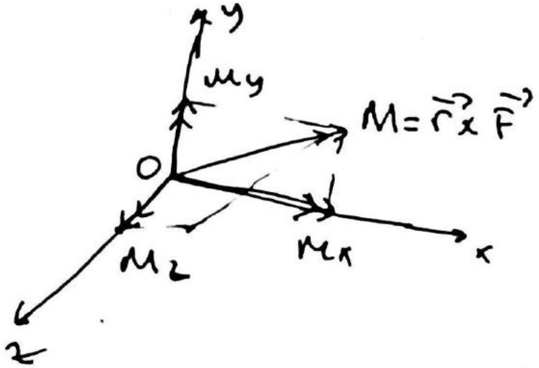
Consider the force applied at point A and position vector $\vec{r}$

We can write $\vec{r}$ and $\vec{F}$ using their rectangular components

$$\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

$$\vec{F} = F_x\,\vec{i} + F_y\,\vec{j} + F_z\,\vec{k}$$

the moment of the force about point O



$$\vec{M}_o = \vec{r} \times \vec{F} = M_x\,\vec{i} + M_y\,\vec{j} + M_z\,\vec{k}$$

$$= \begin{vmatrix} i & j & k \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = (y F_z - z F_y)\, i + (z F_x - x F_z)\, j + (x F_y - y F_x)\, k$$

In the case of problem involving two dimensions

$$M_o \vec{k} = \vec{M}_o = (x F_y - y F_x)\,\vec{k}$$

Hence;

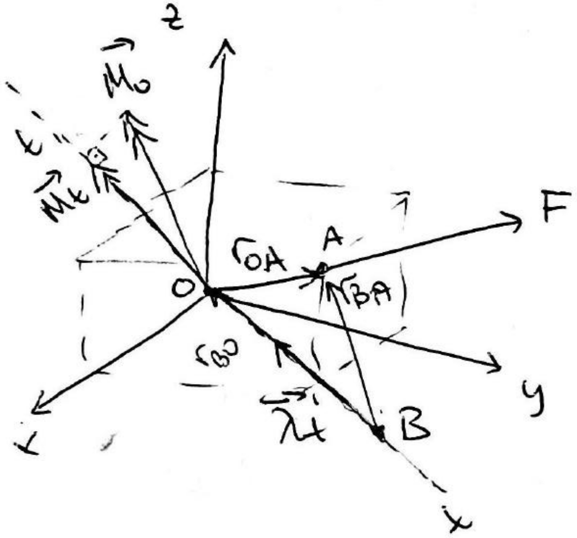
$$M_x = y F_z - z F_y$$

$$M_y = z F_x - x F_z$$

$$M_z = x F_y - y F_x$$

Moment of a Force About a Given Axis

Moment of a Force about an axis always remains constant



let moment of Force $\vec{F}$ about poin O in which the axis t-t passes through be $\vec{M_o}$

Moment of $\vec{F}$ about t-t axis is calculated as follow

- take the moment of $\vec{F}$ about a point lies on the t-t axis. Say point O;

  $\vec{M_o} = \vec{r}_{OA} \times \vec{F}$

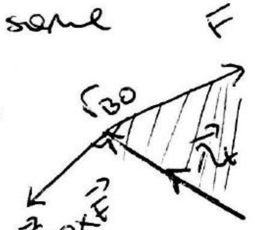
- Calculate the unit vector of t-t axis $\vec{\lambda}_t$
- take the dot product to calculate the magnitude

  $M_t = \vec{M_o} \cdot \vec{\lambda}_t$ (projection of $\vec{M_o}$ on t-t axis)

- Write M_t in Vectorial form

  $\vec{M_t} = M_t \vec{\lambda}_t$

if the moment $\vec{M_B}$ is calculated first result will be same

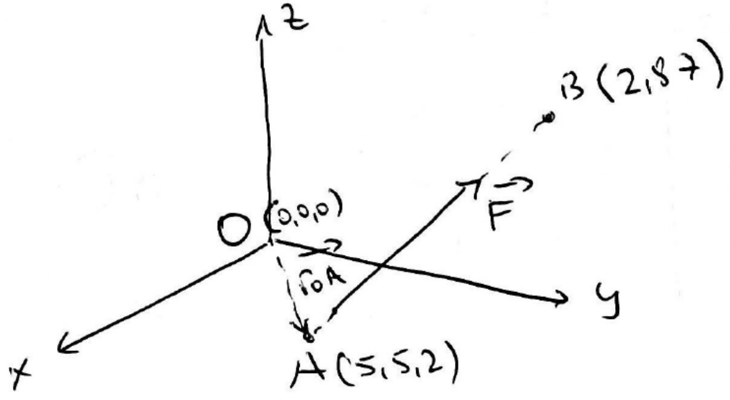


$$\vec{M_B} = \vec{r}_{BA} \times \vec{F}$$
$$= (\vec{r}_{BO} + \vec{r}_{OA}) \times \vec{F}$$
$$= \vec{r}_{BO} \times \vec{F} + \vec{M_o}$$

$$M_t = \vec{M_B} \cdot \vec{\lambda}_t$$
$$= \underbrace{(\vec{r}_{BO} \times \vec{F})\, \vec{\lambda}_t}_{=0} + \vec{M_o} \vec{\lambda}_t$$

$$M_t = \vec{M_o} \vec{\lambda}_t$$

EXAMPLE



The force F is acting on the direction AB and its magnitude is F = 10 kN. Move $\vec{F}$ to point O.

$$d_x = (2-5) = -3\,m$$
$$d_y = (8-5) = 3\,m$$
$$d_z = (7-2) = 5\,m$$
$$d = \sqrt{d_x^2 + d_y^2 + d_z^2} = 6{,}56\,m$$

$$\lambda_x = \frac{d_x}{d} = -0{,}457$$
$$\lambda_y = \frac{d_y}{d} = 0{,}457$$
$$\lambda_z = \frac{d_z}{d} = 0{,}762$$
$$\vec{\lambda} = \lambda_x \vec{i} + \lambda_y \vec{j} + \lambda_z \vec{k}$$

$$\vec{r}_{OA} = 5\vec{i} + 5\vec{j} + 2\vec{k}$$

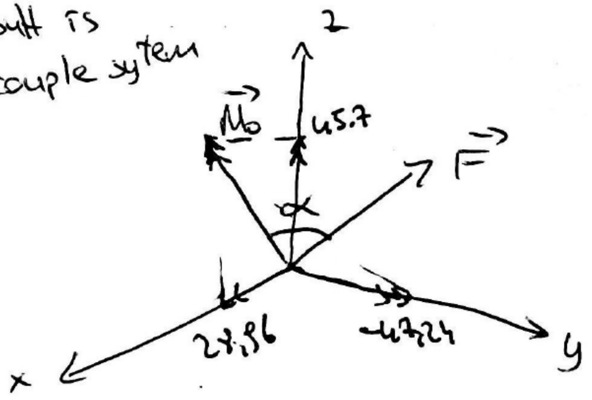
$$\vec{F} = F\vec{\lambda}$$
$$= -4{,}57\vec{i} + 4{,}57\vec{j} + 7{,}62\vec{k}$$

$$\vec{M}_O = \vec{r}_{OA} \times \vec{F} = \begin{vmatrix} i & j & k \\ 5 & 5 & 2 \\ -4{,}57 & 4{,}57 & 7{,}62 \end{vmatrix}$$

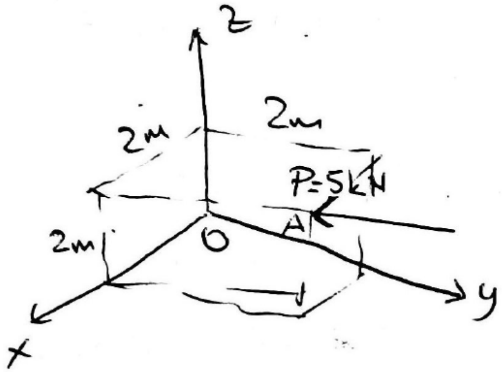
$$\vec{M}_o = 28{,}96\vec{i} - 47{,}24\vec{j} + 45.7\vec{k}$$

$$\vec{F} \cdot \vec{M}_o = F M_o \cos\alpha$$
$$\cos\alpha = \frac{\vec{F} \cdot \vec{M}_o}{M_o F}$$
$$\alpha = \frac{\pi}{2}$$

The result is force-couple sytem



EXAMPLE

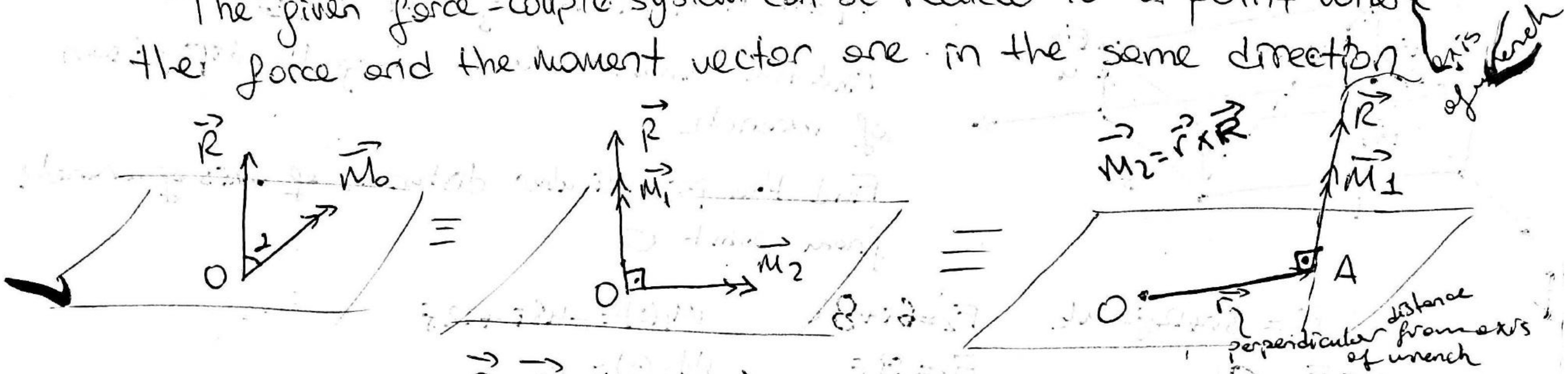


P kuvvetini O noktasına taşıyın.

$$\vec{M_O} = \vec{r_{OA}} \times \vec{F} = \begin{vmatrix} i & j & k \\ 2 & 2 & 2 \\ 0 & -5 & 0 \end{vmatrix} = 10\vec{i} - 10\vec{k}$$

Reducing the force-couple system to a wrench:

The given force-couple system can be reduced to a point where the force and the moment vector are in the same direction



The magnitude of $M_1 = \frac{\vec{R} \cdot \vec{M_0}}{R}$ (M_1 is the projection of M_0) pitch of wrench $= \frac{M_1}{R} = \frac{\vec{R} \cdot \vec{M_0}}{R^2}$

$r = \frac{\vec{R} \times \vec{M_0}}{R^2}$

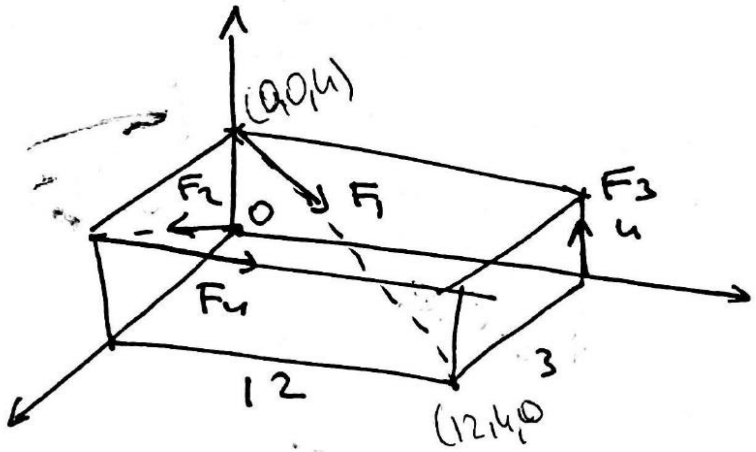
In order to define axis of wrench

we move the force-couple system from O to point A and the following equation can be written.

$$\vec{M_2} + \vec{M_1} = \vec{M_0}$$

$$\vec{r} \times \vec{R} + \vec{M_1} = \vec{M_0}$$ ($\vec{M_0}$, $\vec{M_1}$ and $\vec{R}$ are known) or $r = \frac{\vec{R} \times M_0}{R^2}$

the component of $\vec{r}$ (e.g. $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$) can be found by the above equation.



For the given system of forces

$F_1 = 13\ N \quad F_2 = 10\ N \quad F_3 = 8\ N \quad F_4 = 9\ N$

Find the moment values along the direction of wrench

Find the perpendicular distance of axis of wrench from point O

$\vec{F_1} = 3i + 12j - 4k \qquad F_2 = 6i + 8k \qquad M_1(0) = -48i + 12j$

$F_3 = 8k \qquad F_4 = 9j \qquad M_2(0) = 0$

$M_3(0) = 96i$

$M(0) = \sum M_i(0) = 12i + 12j + 27k \qquad M_4(0) = -36i + 27k$

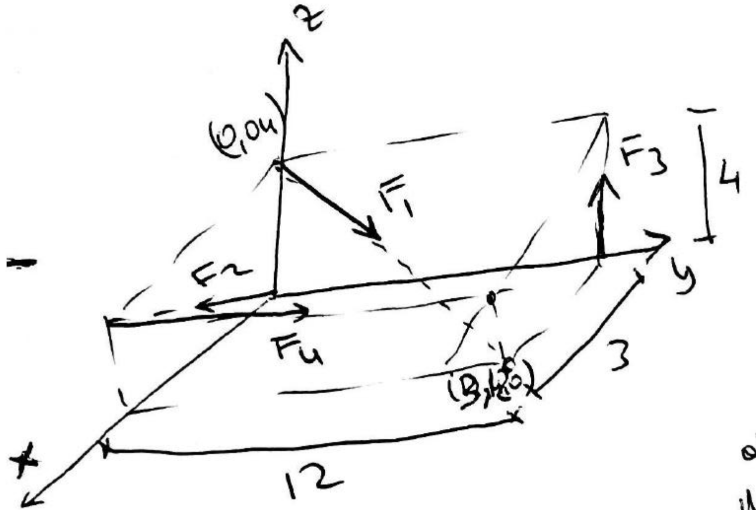
$R(0) = \sum F_i = 6i + 21j + 12k$

$M_1 = \dfrac{R(0) \cdot M(0)}{|R(0)|} = \dfrac{688}{31{,}88} = 21{,}45\ N.m$

$\vec{r} = \dfrac{R(0) \times M(0)}{R(0)^2} = \dfrac{1}{(6i + 21j + 12k)(6i + 21j + 12k)} \cdot \begin{vmatrix} i & j & k \\ 6 & 21 & 12 \\ 60 & 12 & 27 \end{vmatrix}$

$= 0{,}681\, i + 0{,}888\, j - 1{,}813\, k$

$|\vec{r}| = 2{,}22$



For the given system of forces

$F_1 = 13\,N \quad F_2 = 10\,N \quad F_3 = 8\,N \quad F_4 = 9\,N$

Find the moment along the direction of wrench

Find the perpendicular distance of axis of wrench from point O

$F_1 = 3i + 12j - 4k$

$F_2 = 6i + 8k$

$F_3 = 8k$

$F_4 = 9j$

about point O

$$M_1 = \begin{vmatrix} i & j & k \\ 0 & 0 & 4 \\ 3 & 12 & -4 \end{vmatrix} = -48i + 12j$$

$$M_2 = 0$$

$$M_3 = \begin{vmatrix} i & j & k \\ 0 & 12 & 0 \\ 0 & 0 & 8 \end{vmatrix} = 96i$$

$$\lambda_1 = \frac{3i + 12j - 4k}{\sqrt{3^2+12^2+4^2}} = \frac{3}{13}i + \frac{12}{13}j - \frac{4}{13}k \qquad M_4 = \begin{vmatrix} i & j & k \\ 3 & 0 & 4 \\ 0 & 9 & 0 \end{vmatrix} = -36i + 27k$$

$$\vec{M} = M_1 + M_2 + M_3 + M_4$$
$$= 12i + 12j + 27k$$

$$\vec{R} = F_1 + F_2 + F_3 + F_4$$
$$= 9i + 21j + 12k$$

$$\vec{\lambda}_R = 0{,}0135i + 0{,}032j + 0{,}018k$$

$$M_1 = \frac{\vec{R} \cdot \vec{M}}{|R|} = \frac{684}{25{,}81} = 26{,}5\ N.m \rightarrow \vec{M}_1 = M_1 \vec{\lambda}_R = 0{,}358i + 0{,}836j + 0{,}477k$$

$$\vec{r} \times \vec{R} + \vec{M}_1 = \vec{M}_O$$

$$(xi + yj + zk) \times (0{,}358i + 0{,}836j + 0{,}477k) = 12i + 12j + 27k$$

$$(0{,}477y - 0{,}836z)i - (0{,}477x - 0{,}358z)j + (0{,}836x - 0{,}358y)k = 12i + 12j + 27k$$

$$\left.\begin{aligned} &0{,}477y - 0{,}836z = 12 \\ &-0{,}477x + 0{,}358z = 12 \\ &0{,}836x - 0{,}358y = 27 \end{aligned}\right.$$

x =

y =

z =

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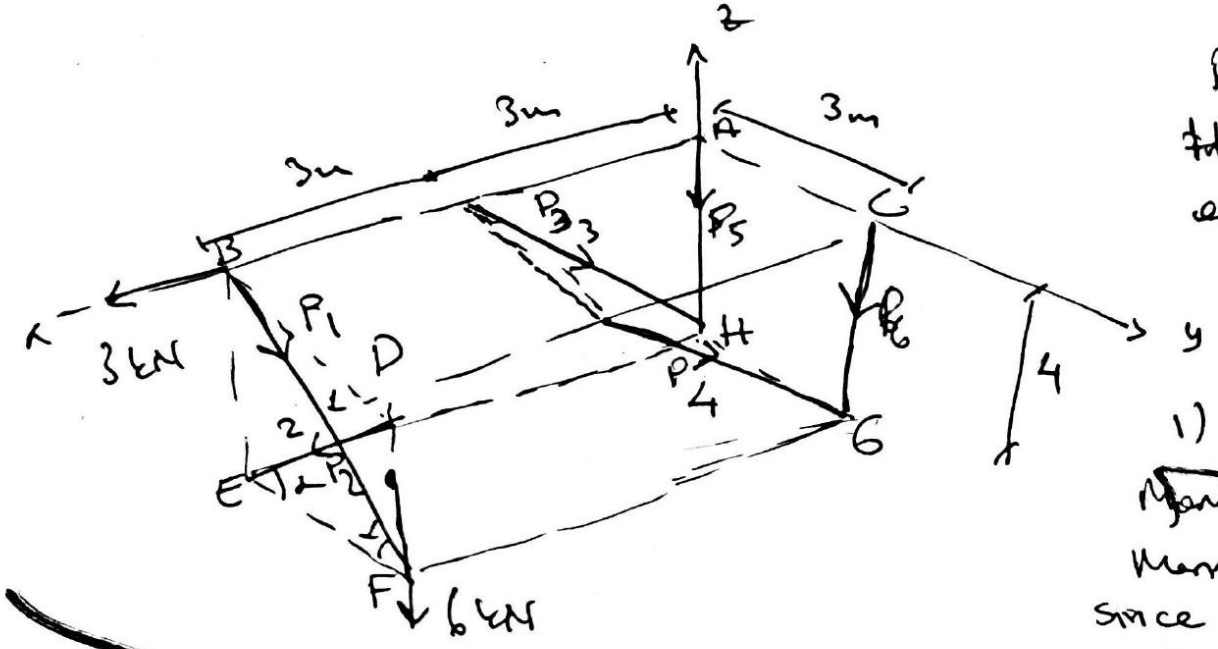

Resolve the given forces into their components along the given axes.

Using Varignon Theorem

1) Moment about BE axis

Moment of P_1, P_2, P_3 and $P_5 \cdot P_6$ is zero

Moment of the given forces is zero since they are parallel or intersect BE axis

then $\boxed{P_4 = 0}$

~~6. Moment about AB axis~~

2) Resultant along the BD axis

$$P_1 \cos\alpha - P_2 \cos\alpha = 0 \qquad \boxed{P_1 = P_2}$$

$\sin\alpha = \frac{4}{5}$

$\cos\alpha = \frac{3}{5}$

6. Resultant along AB axis

$$-P_3 \cos\alpha - P_4 \cos\alpha = 3$$

$$\boxed{P_3 = -5} \text{ kN}$$

3) Moment about HG axis

$$P_1 \sin\alpha\, 6 + P_2 \sin\alpha\, 6 = 6.6 + 3.4$$

$$\boxed{P_1 = P_2 = 5 \text{ kN}}$$

4) Moment about AB ax

$$P_2 \sin\alpha\, 3 + P_4 \sin\alpha\, 3 + P_6 \cdot 3 = 6.3$$

$$\boxed{P_6 = 2 \text{ kN}}$$

5) Moment about DC axis

$$P_1 \sin\alpha\, 3 + P_3 \sin\alpha\, 3 + P_5 \cdot 3 = 0 \qquad \boxed{P_5 = 0}$$