Problems for Ch 9 and 10

A particle with mass $m_1 = 2\ kg$ and velocity $\vec{v}_1 = 10\hat{\imath}\ (m/s)$ makes a head-on (central) collision with another particle with mass $m_2 = 4\ kg$ and velocity $\vec{v}_2 = -2\hat{\imath}\ (m/s)$.

(i) If the collision is perfectly inelastic;

a) Find the final velocities of the particles.

$$m_1\vec{V}_1 + m_2\vec{V}_2 = (m_1 + m_2)\vec{V}_s$$

$$2 \cdot 10\hat{\imath} + 4 \cdot (-2\hat{\imath}) = (2+4)\vec{V}_s$$

$$12\hat{\imath} = 6\vec{v}_s$$

$$\boxed{\vec{v}_s = 2\hat{\imath}\ m/s}$$

b) If the collision between the particles has lasted $10^{-3} s$, find the average force exerted by m_2 on m_1.

$$\vec{F} = \frac{\Delta \vec{P}_1}{\Delta t} = \frac{m_1(\vec{V}_s - \vec{V}_1)}{\Delta t}$$

$$\vec{F} = \frac{2(2\hat{\imath} - 10\hat{\imath})}{10^{-3}} = \frac{-16\hat{\imath}}{10^{-3}}$$

$$\boxed{\vec{F} = 16 \times 10^3 (-\hat{\imath})\ N}$$

(ii) If the collision is elastic, find the final velocities of the particles after the collision ($\vec{v}_1'$ ve $\vec{v}_2'$).

$$m_1\vec{V}_1 + m_2\vec{V}_2 = m_1\vec{V}_1' + m_2\vec{V}_2'$$

$$12\hat{\imath} = 2\vec{V}_1' + 4\vec{v}_2'$$

$$\boxed{6\hat{\imath} = \vec{v}_1' + 2\vec{V}_2'}\ (1)$$

$$\frac{1}{2}m_1 V_1^2 + \frac{1}{2}m_2 V_2^2 = \frac{1}{2}m_1 V_1'^2 + \frac{1}{2}m_2 \vec{V}_2'^2$$

$$\vec{V}_1 + \vec{V}_1' = \vec{V}_2 + \vec{V}_2'$$

$$\boxed{10\hat{\imath} + \vec{V}_1' = -2\hat{\imath} + \vec{V}_2'}\ (2)$$

$$\vec{V}_2' = 12\hat{\imath} + \vec{V}_1' \Rightarrow (1)$$

$$(1) \Rightarrow 6\hat{\imath} = \vec{V}_1' + 24\hat{\imath} + 2\vec{V}_1'$$

$$-18\hat{\imath} = 3\vec{V}_1' \Rightarrow \boxed{\vec{V}_1' = -6\hat{\imath}\ m/s}$$

$$\vec{V}_2' = 12\hat{\imath} - 6\hat{\imath}$$

$$\boxed{\vec{V}_2' = 6\hat{\imath}\ m/s}$$

2- A disk of mass M and radius R starts from rest and rotates with constant angular acceleration about a fixed axis.

$(I_{disk} = \frac{1}{2}MR^2)$

a) Find the time "t", when the tangential acceleration (a_t) of a point on the disk at a distance of r from the rotation axis equal to radial acceleration (a_r). Express your answer in terms of the angular acceleration (α).

$$a_t = r\alpha$$

$$a_r = r\omega^2 \quad \omega = \cancel{\omega_0}_{0} + \alpha t$$

$$a_t = a_r$$

$$\cancel{r}\alpha = \cancel{r}(\alpha t)^2 \Rightarrow \boxed{t = \frac{1}{\sqrt{\alpha}}}$$

b) Find the angular displacement (θ) at that time.

$$\Delta\theta = \cancel{\omega_0}_{0} + \frac{1}{2}\alpha t^2$$

$$\Delta\theta = \frac{1}{2}\alpha \cdot \frac{1}{\alpha}$$

$$\boxed{\Delta\theta = 0.5 \text{ rad}}$$

c) Find the work done on the disk in that time interval in terms of M, R and α.

$$W = \Delta K_D.$$

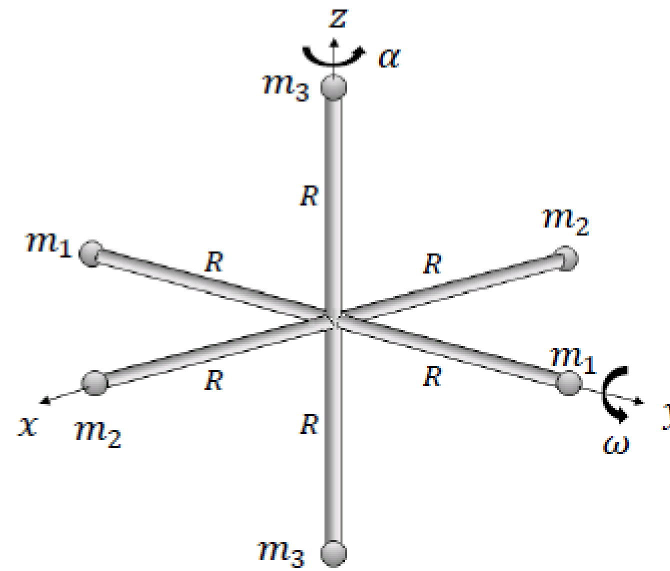
$$W = \frac{1}{2}I\omega_s^2 - \frac{1}{2}I\cancel{\omega_i^2}_{0} \; ; \; \omega_i = 0$$

$$\omega_s = \alpha t = \frac{\alpha}{\sqrt{\alpha}} \Rightarrow \boxed{\omega_s = \sqrt{\alpha} \; \frac{\text{rad}}{\text{s}}}$$

$$W = \frac{1}{2} \cdot \frac{1}{2}MR^2 \cdot (\sqrt{\alpha})^2$$

$$\boxed{W = \frac{MR^2\alpha}{4}}$$

3) Three identical rods, each of length $2R$ and mass $3M$ are placed perpendicular to each other as shown in figure. Particles with masses $m_1 = \frac{M}{2}$, $m_2 = M$ and $m_3 = 2M$ are attached to the ends of the rods. **(For a rod of length l and mass m, the moment of inertia is $I_{CM} = \frac{1}{12} ml^2$).**



a) If the system rotates about the y-axis with constant angular velocity (ω), find the angular momentum of this system ($\vec{L}$) in terms of M, R and ω.

for the rod of $3M$ and $2R$

$$I^{q}_{KM} = \frac{1}{12} 3M (2R)^2 \Rightarrow \boxed{I^{q}_{KM} = MR^2}$$

$$L = I_y \cdot W$$

$$I_y = 2 \cdot I^{q}_{KM} + 2 \cdot m_2 R^2 + 2 \cdot m_3 R^2$$

$$I_y = 2[MR^2 + MR^2 + 2MR^2] \Rightarrow \boxed{I_y = 8MR^2}$$

$$\boxed{\vec{L} = 8MR^2 \omega \hat{j}}$$

b) If the system rotates about the z-axis with constant angular acceleration (α), find the torque ($\vec{\tau}$) in terms of M, R and α.

$$\tau = I_z \alpha$$

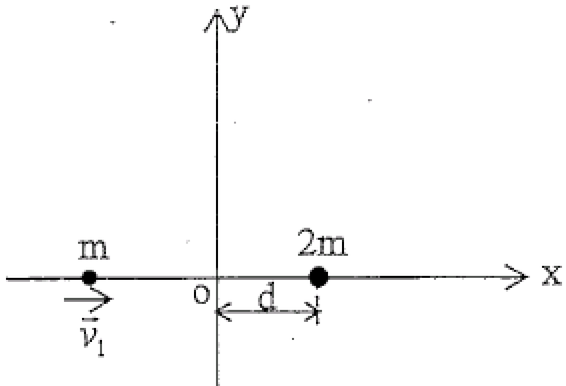
$$I_z = 2 I^{q}_{KM} + 2 \cdot m_1 R^2 + 2 m_2 R^2$$

$$I_z = 2\left(MR^2 + \frac{M}{2} R^2 + MR^2\right)$$

$$\boxed{I_z = 5MR^2}$$

$$\boxed{\vec{\tau} = 5MR^2 \alpha \hat{k}}$$

4) A particle of mass ***m*** moving with a velocity of $\vec{v}_1 = v_1\hat{\imath}$ collides with a target particle of mass **2*m*** that is initially at rest at $x = d$ as shown in Figure-1 just before the collision. After the collision (Figure-2), the mass ***m*** moves with the velocity of $\vec{v}_1' = \frac{v_1}{2}\hat{\jmath}$ at an angle of $\theta = 90^0$ with respect to the original line of motion.



Figure–1: Before the collision

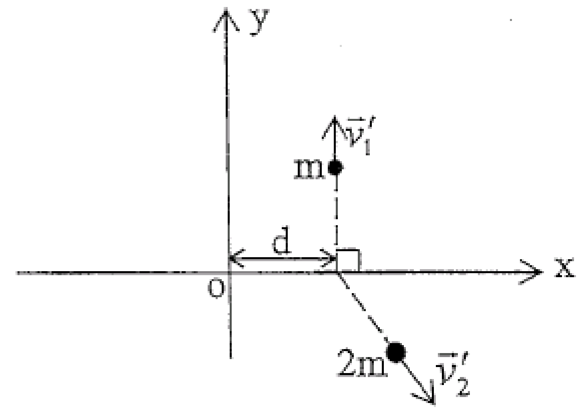


Figure –2: After the collision

a) Find the velocity $\vec{v}_2'$ of the mass **2*m*** after the collision.

$$\sum \vec{P}_i = \sum \vec{P}_s$$

$$m_1\vec{V}_1 + m_2\underbrace{\vec{V}_2}_{0} = m_1\vec{V}_1' + m_2\vec{V}_2'$$

$$\cancel{m} v_1\hat{\imath} = \cancel{m}\frac{v_1}{2}\hat{\jmath} + 2\cancel{m}\vec{V}_2'$$

$$v_1\hat{\imath} - \frac{v_1}{2}\hat{\jmath} = 2\vec{V}_2'$$

$$\boxed{\vec{V}_2' = \frac{V_1}{2}\left(\hat{\imath} - \frac{\hat{\jmath}}{2}\right) = \frac{V_1}{4}(2\hat{\imath} - \hat{\jmath})}$$

b) Calculate the velocity of the center of mass of the particles after the collision.

$$\vec{v}_{KM}' = \frac{m_1\vec{V}_1' + m_2\vec{V}_2'}{m_1 + m_2}$$

$$\vec{v}_{KM}' = \frac{\cancel{m}\cancel{\frac{V_1}{2}\hat{\jmath}} + 2\cancel{m}\frac{V_1}{4}(2\hat{\imath} - \cancel{\hat{\jmath}})}{3\cancel{m}}$$

$$\boxed{\vec{v}_{KM}' = \frac{1}{3}V_1\hat{\imath}}$$

$$(\vec{v}_{KM})_i = (\vec{v}_{KM}')_s$$

c) Define the type of the collision and **explain the reason.**

Check conservation of kinetic energy

$$K_i = \frac{1}{2} m |\vec{V}_1|^2 = \boxed{\frac{1}{2} m_1 V_1^2}$$

$$K_s = \frac{1}{2} m |\vec{V}_1'|^2 + \frac{1}{2} 2m |\vec{V}_2'|^2$$

$$|V_1'| = \frac{V_1^2}{4} \quad , |\vec{V}_2'|^2 = \frac{5V_1^2}{16}$$

$$K_s = \frac{1}{2} m \frac{V_1^2}{4} + \frac{1}{\cancel{2}} \cancel{2} m \, 5 \frac{V_1^2}{16}$$

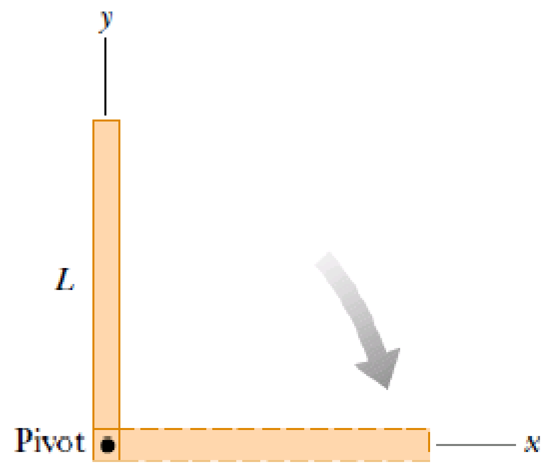
$$\boxed{K_s = \frac{7}{16} m V_1^2}$$

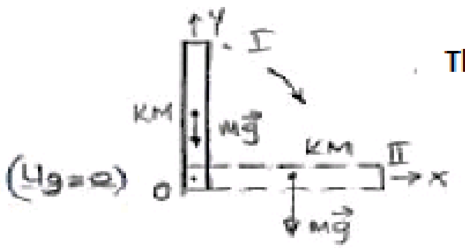
$$\Delta K = K_s - K_i \neq 0$$

Therefore it is inelastic collision

A long, uniform rod of length L and mass M is pivoted about a horizontal, frictionless pin passing through one end. The rod is released from rest in a vertical position, as shown Figure. At the instant the rod is horizontal, find

a) Its angular speed,
b) The magnitude of its angular acceleration,
c) The x and y components of the acceleration of its center of mass, and
d) The components of the reaction force at the pivot.





The moment of inertia of the rod through the center of mass : $I_{KM} = \frac{1}{12}ML^2$

$$I_o = I_{km} + Md^2$$

The moment of inertia about "O"

$$= \frac{1}{2}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$

a)

$$E_I = E_{II}$$

$$\underbrace{K_I}_{0} + U_I = K_{II} + \underbrace{U_{II}}_{0}$$

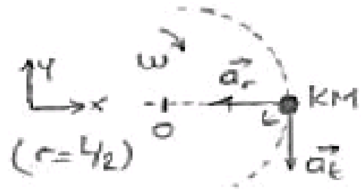
$$mg\left(\frac{L}{2}\right) = \frac{1}{2}I_o\omega^2 \Rightarrow \omega = \sqrt{\frac{3g}{L}}$$

b)

$$\Sigma\tau_o = I\cdot\alpha$$

$$(Mg)\left(\frac{L}{2}\right) = \frac{1}{3}ML^2\cdot\alpha \Longrightarrow \alpha = \frac{3g}{2L}$$

c)



$$a_t = \alpha \cdot r = \left(\frac{3g}{2L}\right)\left(\frac{L}{2}\right) = \frac{3}{4}g$$

$$a_r = \frac{v^2}{r} = \omega^2 r = \left(\frac{3g}{L}\right)\left(\frac{L}{2}\right) = \frac{3}{2}g$$

$$\vec{a} = \left[-\frac{3}{2}g\,\vec{i} - \frac{3}{4}g\,\vec{j}\right]\ m/s^2$$

d) According to the Newton's Second Law: $\boldsymbol{F} = \boldsymbol{ma}$

$$\Rightarrow \begin{cases} \Sigma F_x = m \cdot a_x \\ R_x = m \cdot \left(-\frac{3}{2}g\right) = -\frac{3}{2}mg \end{cases}$$

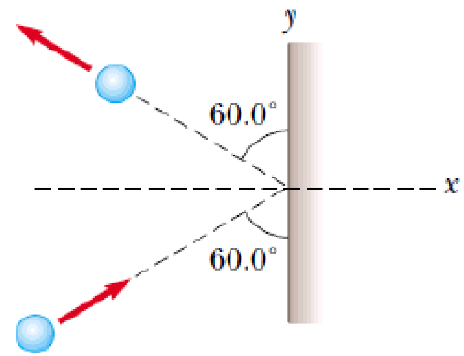
$$\Sigma F_y = m \cdot a_y$$

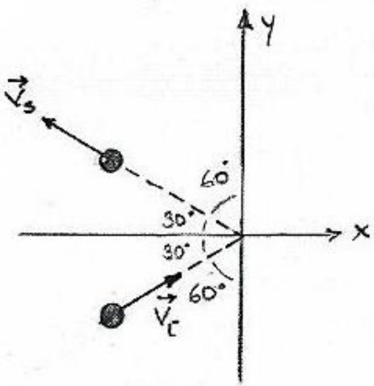
$$R_y - mg = m \cdot \left(-\frac{3}{4}g\right)$$

$$R_y = \frac{mg}{4}$$

6)

A 3.00-kg steel ball strikes a wall with a speed of 10.0 m/s at an angle of 60.0° with the surface. It bounces off with the same speed and angle as in Figure. If the ball is in contact with the wall for 0.200 s, what is the average force exerted by the wall on the ball ?





$m = 3\,kg$

$\Delta t = 0{,}2\,s$

$V_i = 10\,m/s$

$V_s = 10\,m/s$

$\vec{V}_i = V_x\vec{i} + V_y\vec{j}$

$= V_i(\cos 30\vec{i} + \sin 30\vec{j})$

$\vec{V}_s = V_i(-\cos 30\vec{i} + \sin 30\vec{j})$

$\vec{P}_i = m\cdot\vec{V}_i \quad , \quad \vec{P}_s = m\cdot\vec{V}_s$

In the case of constant force

$\Delta\vec{P} = \vec{F}\cdot\Delta t \quad ,$

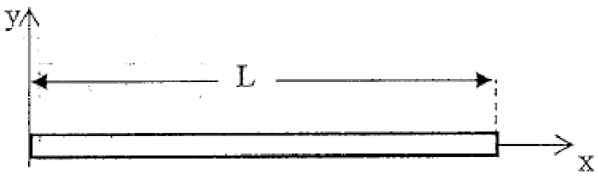
$\vec{F} = \vec{\bar{F}}$

$$\Delta\vec{P} = \vec{\bar{F}}\cdot\Delta t \longrightarrow \vec{\bar{F}} = \frac{\Delta\vec{P}}{\Delta t} = \frac{\vec{P}_s - \vec{P}_i}{\Delta t} = \frac{mV_i\left[(-\cos 30\vec{i} + \sin 30\vec{j}) - (\cos 30\vec{i} + \sin 30\vec{j})\right]}{\Delta t}$$

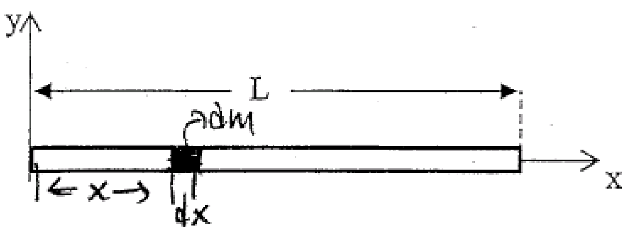
$$\vec{\bar{F}} = \frac{-2mV_i\cos 30\vec{i}}{\Delta t} = \frac{-2\cdot 3\cdot 10\cdot\cos 30}{0{,}2}\vec{i}$$

$$= \underline{\underline{-260\vec{i}\ N}}$$

7) Consider a non-uniform rod of mass M and length L placed along the x-axis with one end at the origin, as shown in the figure. Linear mass density of the rod varies with x according to the expression $\lambda = Ax^2$ (A is a positive constant).



a) Find the total mass M of the rod in terms of **A and L.**



$$M = \int dm = \int \lambda dx$$

$$M = \int_0^L Ax^2 dx = A\frac{x^3}{3}\Big|_0^L$$

$$\boxed{M = A\frac{L^3}{3}}$$

b) Find the position of the center of mass of the rod.

$$X_{KM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int x\lambda dx$$

$$X_{KM} = \frac{1}{M}\int_0^L x(Ax^2)dx$$

$$X_{KM} = \frac{1}{M}\cdot A\frac{x^4}{4}\Big|_0^L$$

$$X_{KM} = \frac{3}{\cancel{L^3}\cancel{A}}\cdot\cancel{A}\frac{L^4}{4}$$

$$\boxed{X_{KM} = \frac{3}{4}L}$$

c) Calculate the moment of inertia of the rod about y-axis in terms of **M and L.**

$$I_y = \int r^2 dm \quad , \; r = x \, , \; dm = \lambda dx$$

$$I_y = \int_0^L x^2 \lambda dx = \int_0^L x^2 (Ax^2) dx$$

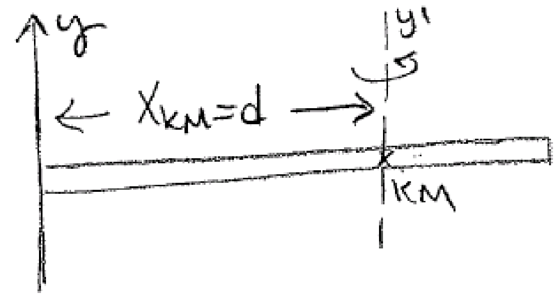
$$I_y = A \int_0^L x^4 dx = A \frac{x^5}{5} \Big|_0^L$$

$$I_y = A \frac{L^5}{5} \; ;$$

$$M = A \frac{L^3}{3} \Rightarrow AL^3 = 3M$$

$$\boxed{I_y = \frac{3}{5} M L^2}$$

d) Calculate the moment of inertia about an axis perpendicular to the rod and passing through its center of mass in terms of **M and L *by using parallel axis theorem*.**



Parallel axis theorem

$$I_y = I_{KM} + Md^2$$

$$I_{KM} = I_y - M\left(\frac{3}{4}L\right)^2$$

$$I_{KM} = \frac{3}{5} ML^2 - \frac{9}{16} ML^2$$

$$\boxed{I_{KM} = \frac{3}{80} ML^2}$$