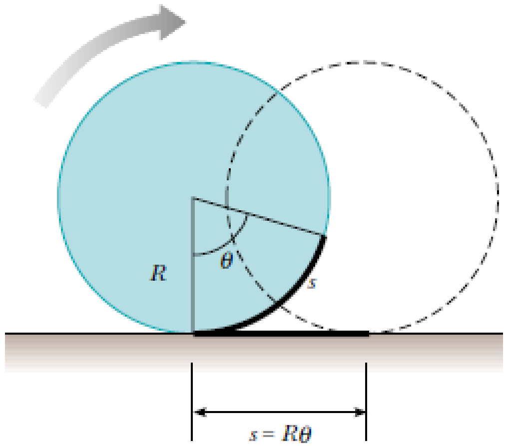
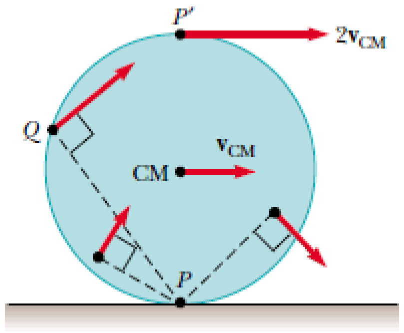
Rolling Motion of a Rigid Object





Rolling motion without slipping is the condition for ***pure rolling motion.***

$$s = R\theta$$

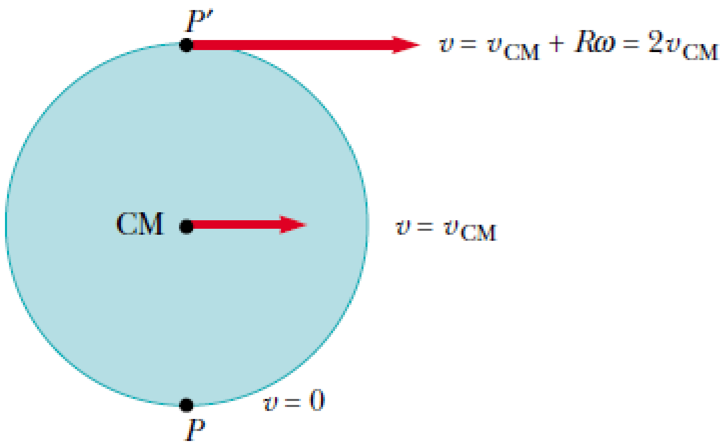
$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

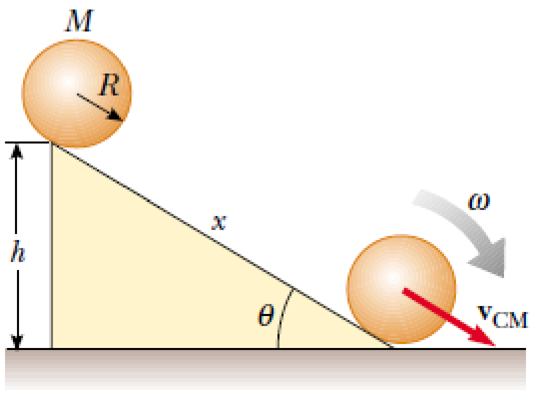
$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha$$

$$K = \tfrac{1}{2} I_P \omega^2$$

$$K = \tfrac{1}{2} I_{CM}\omega^2 + \tfrac{1}{2} MR^2\omega^2$$

$$\boxed{K = \tfrac{1}{2} I_{CM}\omega^2 + \tfrac{1}{2} M v_{CM}^2}$$





$$v_{CM} = R\omega$$

$$K = \frac{1}{2} I_{CM} \left(\frac{v_{CM}}{R} \right)^2 + \frac{1}{2} M v_{CM}{}^2$$

$$K = \frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}{}^2$$

$$K_f + U_f = K_i + U_i$$

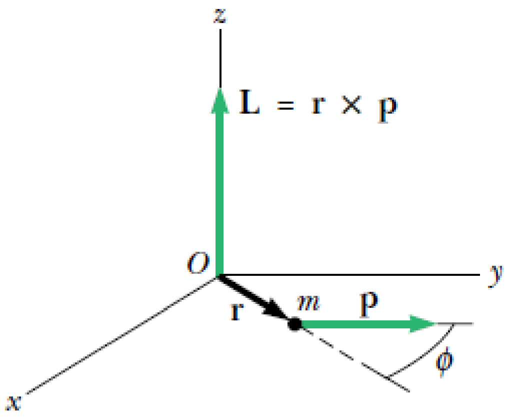
$$\frac{1}{2} \left(\frac{I_{CM}}{R^2} + M \right) v_{CM}{}^2 + 0 = 0 + Mgh$$

$$v_{CM} = \left(\frac{2gh}{1 + (I_{CM}/MR^2)} \right)^{1/2}$$

Angular Momentum

The instantaneous **angular momentum L** of a particle relative to the origin O is defined by the cross product of the particle's instantaneous position vector **r** and its instantaneous linear momentum **p**:

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}$$



$$L = mvr \sin \phi$$

$$\sum \boldsymbol{\tau} = \mathbf{r} \times \sum \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

Torque can be written as

$$\sum \boldsymbol{\tau} = \frac{d(\mathbf{r} \times \mathbf{p})}{dt}$$

Because

$$\sum \boldsymbol{\tau} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p}$$

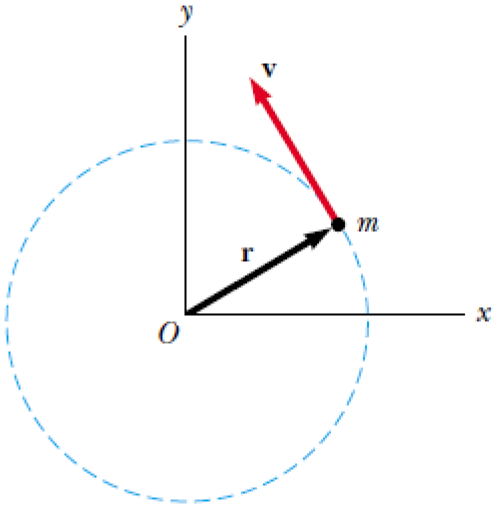
And

$\frac{d\mathbf{r}}{dt} \times \mathbf{p},$ is zero

Then

$$\boxed{\sum \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}}$$

Ex/ A particle moves in the *xy* plane in a circular path of Radius *r*, as shown in Figure. Find the magnitude and direction of its angular momentum relative to *O* when its linear velocity is v.



$$L = mvr \sin 90° = \boxed{mvr}$$

$$\boxed{\mathbf{L} = \mathbf{r} \times \mathbf{p}}$$

$$= m\mathbf{r} \times \mathbf{v}$$

$$\mathbf{L} = (mvr)\hat{\mathbf{k}}.$$

Angular Momentum of a System of Particles

$$\sum \mathbf{F}_{\text{ext}} = \frac{d\mathbf{p}_{\text{tot}}}{dt}$$

$$\mathbf{L}_{\text{tot}} = \mathbf{L}_1 + \mathbf{L}_2 + \cdots + \mathbf{L}_n = \sum_i \mathbf{L}_i$$

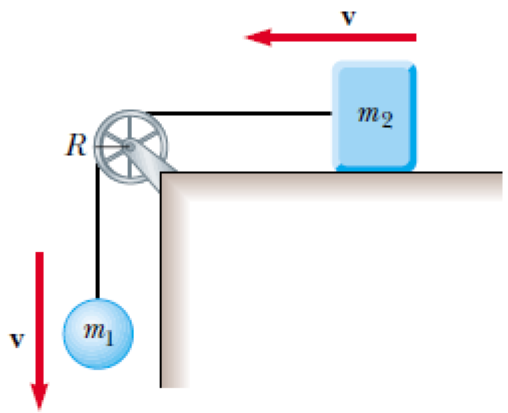
$$\frac{d\mathbf{L}_{\text{tot}}}{dt} = \sum_i \frac{d\mathbf{L}_i}{dt} = \sum_i \boldsymbol{\tau}_i$$

$$\sum \boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}_{\text{tot}}}{dt}$$

the net external torque acting on a system about some axis passing through an origin in an inertial frame equals the time rate of change of the total angular momentum of the system about that origin.

The resultant torque acting on a system about an axis through the center of mass equals the time rate of change of angular momentum of the system regardless of the motion of the center of mass.

Ex/ A sphere of mass m_1 and a block of mass m_2 are connected by a light cord that passes over a pulley, as shown in Figure. The radius of the pulley is R, and the mass of the rim is M. The spokes of the pulley have negligible mass. The block slides on a frictionless, horizontal surface. Find an expression for the linear acceleration of the two objects, using the concepts of angular momentum and torque.



Angular Momentum of a Rotating Rigid Object

$$L = m_1 vR + m_2 vR + MvR = (m_1 + m_2 + M)vR$$

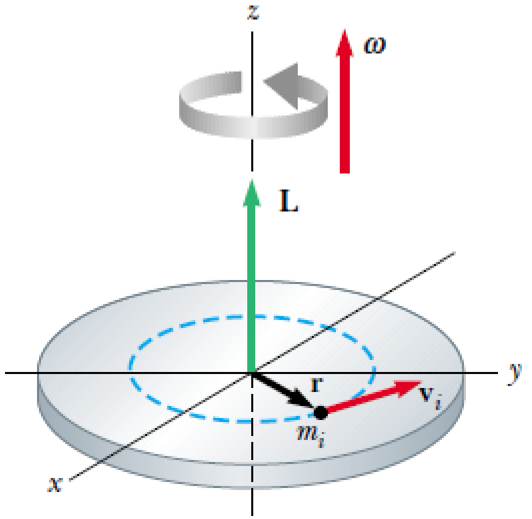
$$\sum \tau_{\text{ext}} = \frac{dL}{dt}$$

$$m_1 gR = \frac{d}{dt}[(m_1 + m_2 + M)vR]$$

$$m_1 gR = (m_1 + m_2 + M)R\frac{dv}{dt}$$

$$dv/dt = a,$$

$$a = \boxed{\frac{m_1 g}{m_1 + m_2 + M}}$$



$$v_i = r_i\omega,$$

$$L_i = m_i v_i r_i = m_i r_i^2 \omega$$

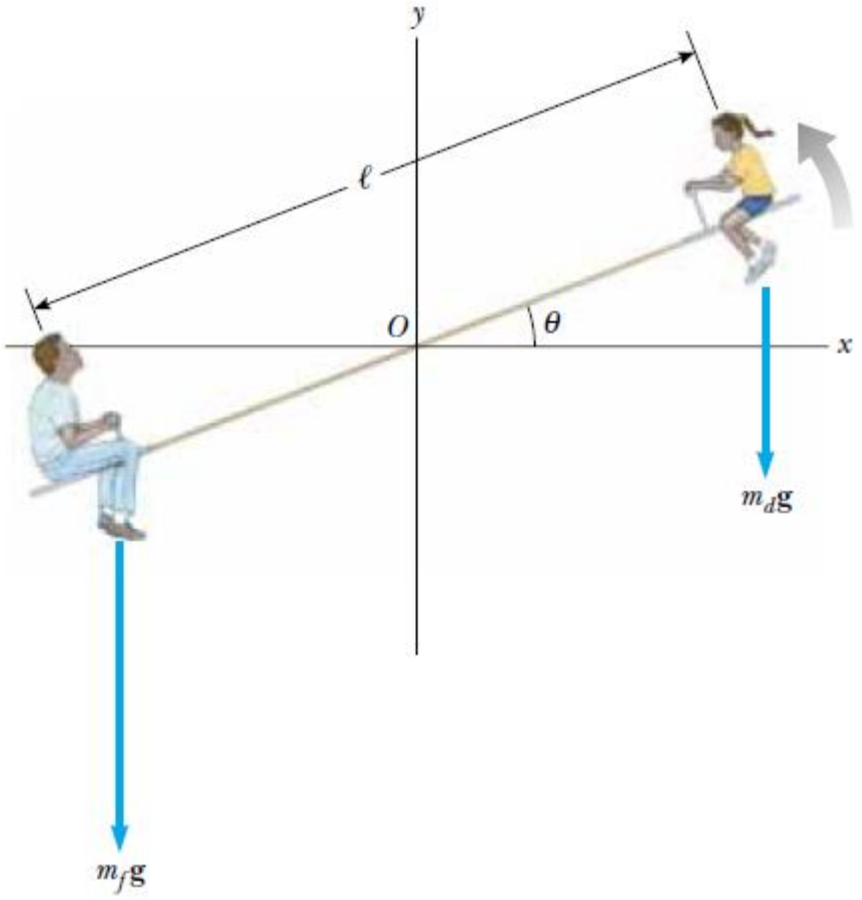
$$L_z = \sum_i L_i = \sum_i m_i r_i^2 \omega = \left(\sum_i m_i r_i^2\right)\omega$$

$$\boxed{L_z = I\omega}$$

$$\frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha$$

$$\boxed{\sum \tau_{\text{ext}} = I\alpha}$$

Ex/ A father of mass m_f and his daughter of mass m_d sit on opposite ends of a seesaw at equal distances from the pivot at the center. The seesaw is modeled as a rigid rod of mass M and length ℓ and is pivoted without friction. At a given moment, the combination rotates in a vertical plane with an angular speed ω.



(A) Find an expression for the magnitude of the system's angular momentum.

(B) Find an expression for the magnitude of the angular acceleration of the system when the seesaw makes an angle θ with the horizontal.

(A) $I = mr^2$ for each person

$$I = \frac{1}{12}M\ell^2 + m_f\left(\frac{\ell}{2}\right)^2 + m_d\left(\frac{\ell}{2}\right)^2 = \frac{\ell^2}{4}\left(\frac{M}{3} + m_f + m_d\right)$$

$$L = I\omega = \boxed{\frac{\ell^2}{4}\left(\frac{M}{3} + m_f + m_d\right)\omega}$$

(B) $\Sigma\tau_{\text{ext}} = I\alpha$

$$\tau_f = m_f g\frac{\ell}{2}\cos\theta \qquad \tau_d = -m_d g\frac{\ell}{2}\cos\theta$$

$$\sum\tau_{\text{ext}} = \tau_f + \tau_d = \tfrac{1}{2}(m_f - m_d)g\ell\cos\theta$$

$$\alpha = \frac{\sum\tau_{\text{ext}}}{I} = \boxed{\frac{2(m_f - m_d)g\cos\theta}{\ell\left(\frac{M}{3} + m_f + m_d\right)}}$$

Conservation of Angular Momentum

The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero, that is, if the system is isolated.

$$\sum \boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}_{\text{tot}}}{dt} = 0$$

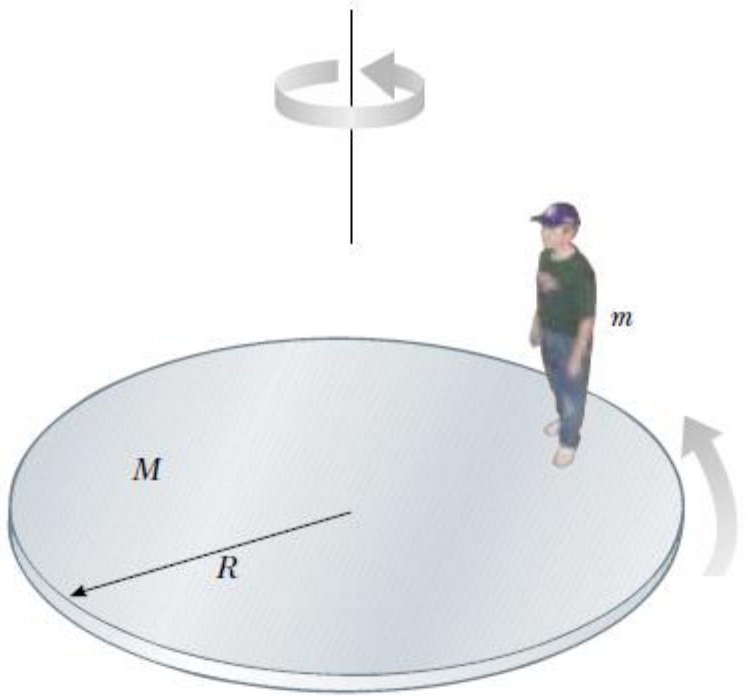
$$\mathbf{L}_{\text{tot}} = \text{constant} \qquad \text{or} \qquad \mathbf{L}_i = \mathbf{L}_f$$

For an isolated rotating system;

$$I_i\omega_i = I_f\omega_f = \text{constant}$$

$$\left.\begin{aligned} E_i &= E_f \\ \mathbf{p}_i &= \mathbf{p}_f \\ \mathbf{L}_i &= \mathbf{L}_f \end{aligned}\right\} \quad \text{For an isolated system}$$

Ex/ A horizontal platform in the shape of a circular disk rotates freely in a horizontal plane about a frictionless vertical axle. The platform has a mass M=100 kg and a Radius R=2 m. A student whose mass is m=60 kg walks slowly from the rim of the disk toward its center. If the angular speed of the system is 2 rad/s when the student is at the rim, what is the angular speed when he reaches a point r=0.50 m from the center?



$$I_i = I_{pi} + I_{si} = \tfrac{1}{2}MR^2 + mR^2$$

$$I_f = I_{pf} + I_{sf} = \tfrac{1}{2}MR^2 + mr^2$$

$$I_i\omega_i = I_f\omega_f$$

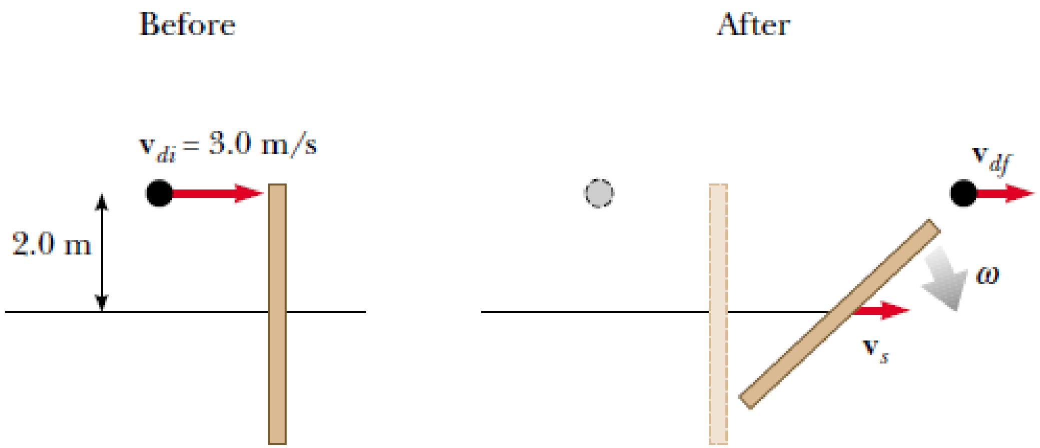
$$(\tfrac{1}{2}MR^2 + mR^2)\omega_i = (\tfrac{1}{2}MR^2 + mr^2)\omega_f$$

$$\omega_f = \left(\frac{\frac{1}{2}MR^2 + mR^2}{\frac{1}{2}MR^2 + mr^2}\right)\omega_i$$

$$= \left(\frac{\frac{1}{2}(100\text{ kg})(2.0\text{ m})^2 + (60\text{ kg})(2.0\text{ m})^2}{\frac{1}{2}(100\text{ kg})(2.0\text{ m})^2 + (60\text{ kg})(0.50\text{ m})^2}\right)(2.0\text{ rad/s})$$

$$= \left(\frac{440\text{ kg}\cdot\text{m}^2}{215\text{ kg}\cdot\text{m}^2}\right)(2.0\text{ rad/s}) = \boxed{4.1\text{ rad/s}}$$

Ex/ A 2 kg disk traveling at 3 m/s strikes a 1 kg stick of length 4 m that is lying flat on nearly frictionless ice, as shown in Figure. Assume that the collision is elastic and that the disk does not deviate from its original line of motion. Find the translational speed of the disk, the translational speed of the stick, and the angular speed of the stick after the collision. The moment of inertia of the stick about its center of mass is 1.33 kg $\cdot$m^2.



$$p_i = p_f$$

$$m_d v_{di} = m_d v_{df} + m_s v_s$$

$$(2.0\ \text{kg})(3.0\ \text{m/s}) = (2.0\ \text{kg})\, v_{df} + (1.0\ \text{kg})\, v_s$$

$$6.0\ \text{kg}\cdot\text{m/s} - (2.0\ \text{kg})\, v_{df} = (1.0\ \text{kg})\, v_s$$

$$L_i = L_f$$

$$-r m_d v_{di} = -r m_d v_{df} + I\omega$$

$$\begin{aligned} -(2.0\ \text{m})(2.0\ \text{kg})(3.0\ \text{m/s}) &= -(2.0\ \text{m})(2.0\ \text{kg})\, v_{df} \\ &\quad + (1.33\ \text{kg}\cdot\text{m}^2)\,\omega \\ -12\ \text{kg}\cdot\text{m}^2/\text{s} &= -(4.0\ \text{kg}\cdot\text{m})\, v_{df} \\ &\quad + (1.33\ \text{kg}\cdot\text{m}^2)\,\omega \end{aligned}$$

$$-9.0\ \text{rad/s} + (3.0\ \text{rad/m})\, v_{df} = \omega$$

$$K_i = K_f$$

$$\tfrac{1}{2} m_d v_{di}^2 = \tfrac{1}{2} m_d v_{df}^2 + \tfrac{1}{2} m_s v_s^2 + \tfrac{1}{2} I\omega^2$$

$$\begin{aligned} \tfrac{1}{2}(2.0\ \text{kg})(3.0\ \text{m/s})^2 &= \tfrac{1}{2}(2.0\ \text{kg})\, v_{df}^2 + \tfrac{1}{2}(1.0\ \text{kg})\, v_s^2 \\ &\quad + \tfrac{1}{2}(1.33\ \text{kg}\cdot\text{m}^2)\,\omega^2 \end{aligned}$$

$$18\ \text{m}^2/\text{s}^2 = 2.0\, v_{df}^2 + v_s^2 + (1.33\ \text{m}^2)\,\omega^2$$

$$v_{df} = 2.3\ \text{m/s}, \quad v_s = 1.3\ \text{m/s},$$

$$\omega = -2.0\ \text{rad/s}.$$