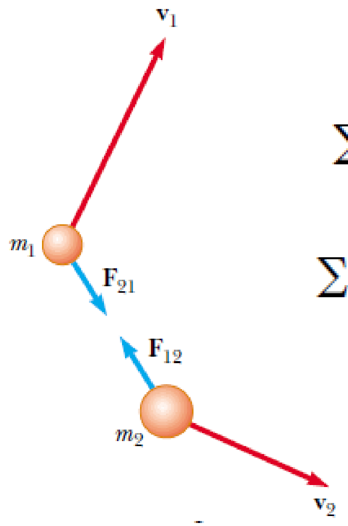
Linear Momentum and Its Conservation

The **linear momentum** of a particle or an object that can be modeled as a particle of mass m moving with a velocity $\mathbf{v}$ is defined to be the product of the mass and velocity:

$$\mathbf{p} \equiv m\mathbf{v}$$

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z$$



$$\mathbf{F}_{12} = -\mathbf{F}_{21}.$$

$$\sum \mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}$$

$$\sum \mathbf{F} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}$$

$$\mathbf{F}_{21} + \mathbf{F}_{12} = 0$$

$$m_1\mathbf{a}_1 + m_2\mathbf{a}_2 = 0$$

$$m_1\frac{d\mathbf{v}_1}{dt} + m_2\frac{d\mathbf{v}_2}{dt} = 0$$

$$\frac{d(m_1\mathbf{v}_1)}{dt} + \frac{d(m_2\mathbf{v}_2)}{dt} = 0$$

$$\frac{d}{dt}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2) = 0$$

then $m_1\mathbf{v}_1 + m_2\mathbf{v}_2$ is constant

Conservation of momentum:

$$\frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0$$

$$\mathbf{p}_{\text{tot}} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant}$$

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f}$$

$$p_{ix} = p_{fx} \qquad p_{iy} = p_{fy} \qquad p_{iz} = p_{fz}$$

The total momentum of an isolated system at all times equals its initial momentum.

Impulse and Momentum

$$\sum \mathbf{F} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}$$

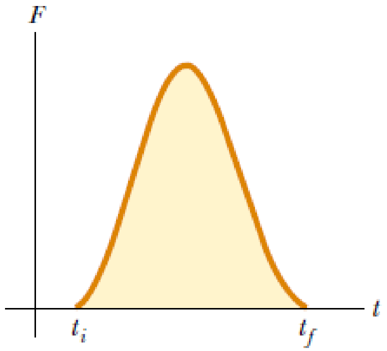
dp = F*dt*

$$\Delta \mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F}\, dt$$

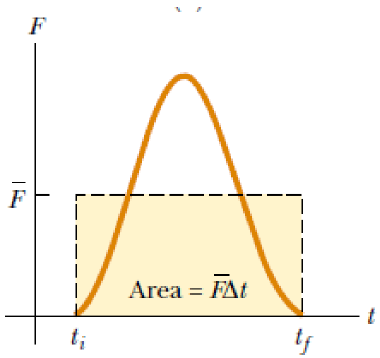
$$\mathbf{I} \equiv \int_{t_i}^{t_f} \mathbf{F}\, dt$$

The impulse of the force $\mathbf{F}$ acting on a particle equals the change in the momentum of the particle.

if the force is varying average force can be used



$$\mathbf{I} \equiv \overline{\mathbf{F}}\, \Delta t$$



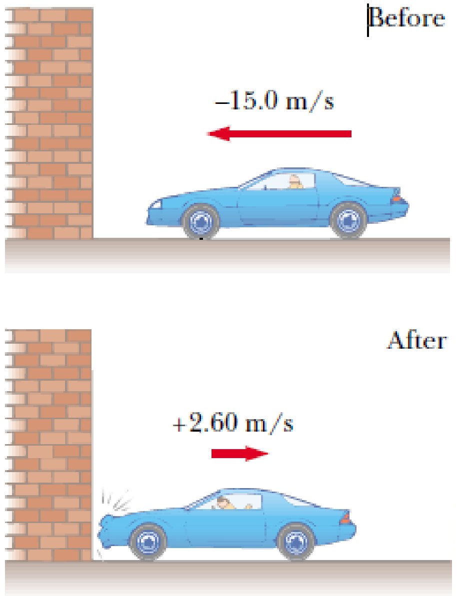
Ex/ In a particular crash test, a car of mass

1500 kg collides with a wall, as shown in Figure.

The initial and final velocities of the car are

$$\mathbf{v}_i = -15.0\hat{\mathbf{i}} \text{ m/s and } \mathbf{v}_f = 2.60\hat{\mathbf{i}} \text{ m/s},$$

respectively. If the collision lasts for 0.150 s, find the impulse caused by the collision and the average force exerted on the car.



$$\mathbf{p}_i = m\mathbf{v}_i = (1\,500 \text{ kg})(-15.0\hat{\mathbf{i}} \text{ m/s})$$
$$= -2.25 \times 10^4\hat{\mathbf{i}} \text{ kg}\cdot\text{m/s}$$

$$\mathbf{p}_f = m\mathbf{v}_f = (1\,500 \text{ kg})(2.60\hat{\mathbf{i}} \text{ m/s})$$
$$= 0.39 \times 10^4\hat{\mathbf{i}} \text{ kg}\cdot\text{m/s}$$

$$\mathbf{I} = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = 0.39 \times 10^4\hat{\mathbf{i}} \text{ kg}\cdot\text{m/s} - (-2.25 \times 10^4\hat{\mathbf{i}} \text{ kg}\cdot\text{m/s})$$

$$\mathbf{I} = 2.64 \times 10^4\hat{\mathbf{i}} \text{ kg}\cdot\text{m/s}$$

$$\overline{\mathbf{F}} = \frac{\Delta\mathbf{p}}{\Delta t} = \frac{2.64 \times 10^4\hat{\mathbf{i}} \text{ kg}\cdot\text{m/s}}{0.150 \text{ s}} = 1.76 \times 10^5\hat{\mathbf{i}} \text{ N}$$

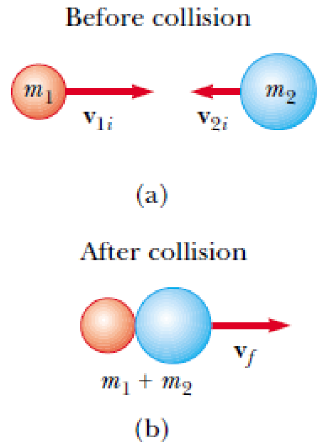
Collisions in One Dimension

An elastic collision between two objects is one in which the total kinetic energy (as well as total momentum) of the system is the same before and after the collision.

An inelastic collision is one in which the total kinetic energy of the system is not the same before and after the collision (even though the momentum of the system is conserved). Inelastic collisions are of two types. When the colliding objects stick together after the collision, the collision is called perfectly inelastic. When the colliding objects do not stick together, but some kinetic energy is lost the collision is called inelastic.

*Momentum of the system is conserved in **all collisions**, but kinetic energy of the system is conserved only in **elastic collisions.***

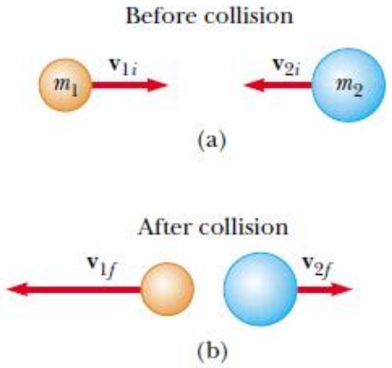
Perfectly Inelastic Collisions



$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = (m_1 + m_2)\mathbf{v}_f$$

$$\mathbf{v}_f = \frac{m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}}{m_1 + m_2}$$

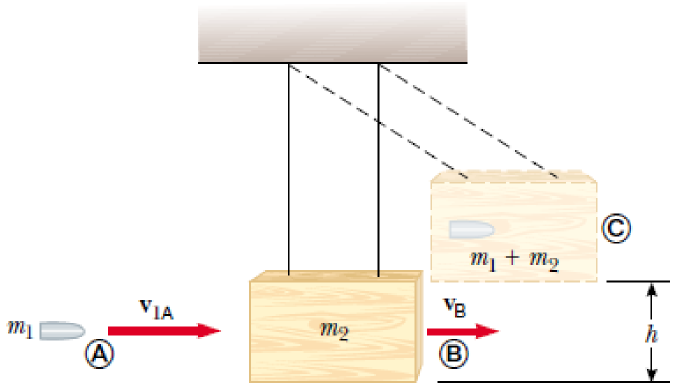
Elastic Collisions



$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\frac{1}{2} m_1 v_{1i}^{\ 2} + \frac{1}{2} m_2 v_{2i}^{\ 2} = \frac{1}{2} m_1 v_{1f}^{\ 2} + \frac{1}{2} m_2 v_{2f}^{\ 2}$$

Ex/ A bullet of mass m_1 is fired into a large block of wood of mass m_2 suspended from some light wires. The bullet embeds in the block, and the entire system swings through a height h. How can we determine the speed of the bullet from a measurement of h?



Velocity of the system just after the collision

$$v_B = \frac{m_1 v_{1A}}{m_1 + m_2}$$

Energy conservation after collision for the motion between B and C

$$K_B + U_B = K_C + U_C$$

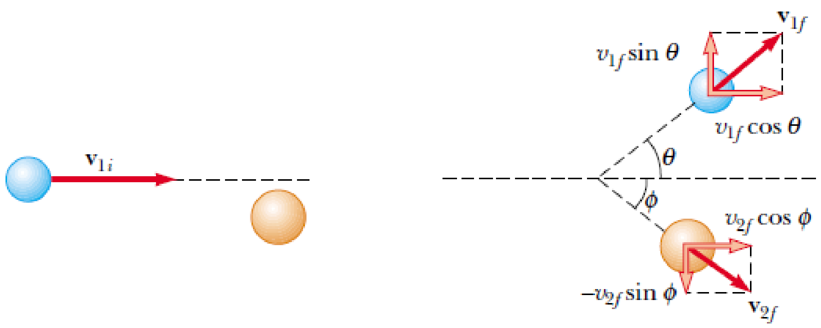
$$K_B = \frac{1}{2}(m_1 + m_2)v_B{}^2$$

$$K_B = \frac{m_1{}^2 v_{1A}{}^2}{2(m_1 + m_2)}$$

$$\frac{m_1{}^2 v_{1A}{}^2}{2(m_1 + m_2)} + 0 = 0 + (m_1 + m_2)gh$$

$$v_{1A} = \left(\frac{m_1 + m_2}{m_1}\right)\sqrt{2gh}$$

Two-Dimensional Collisions



(a) Before the collision (b) After the collision

$$m_1v_{1ix} + m_2v_{2ix} = m_1v_{1fx} + m_2v_{2fx}$$

$$m_1v_{1iy} + m_2v_{2iy} = m_1v_{1fy} + m_2v_{2fy}$$

x-axis $$m_1v_{1i} = m_1v_{1f}\cos\theta + m_2v_{2f}\cos\phi$$

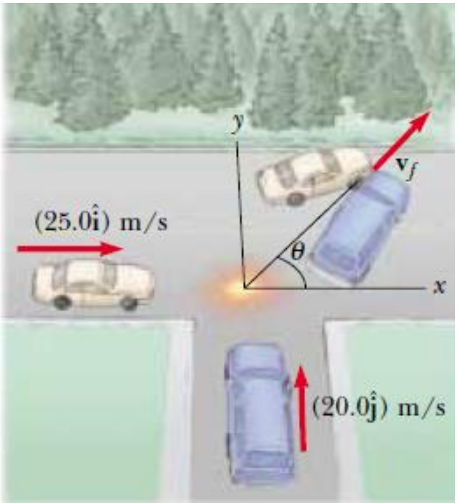
y-axis $$0 = m_1v_{1f}\sin\theta - m_2v_{2f}\sin\phi$$

İf the collision is elastic, kinetic energy is also conserved

$$\tfrac{1}{2}m_1{v_{1i}}^2 = \tfrac{1}{2}m_1{v_{1f}}^2 + \tfrac{1}{2}m_2{v_{2f}}^2$$

Ex/ A bullet of mass m_1 is fired into a large block of wood of mass m_2 suspended from some light wires. The bullet embeds in the block, and the entire system swings through a height h. How can we determine the speed of the bullet from a measurement of h?

Ex/ A 1500-kg car traveling east with a speed of 25 m/s collides at an intersection with a 2500-kg van traveling north at a speed of 20 m/s, as shown in Figure. Find the direction and magnitude of the velocity of the wreckage after the collision, assuming that the vehicles undergo a perfectly inelastic collision (that is, they stick together).



$$\sum p_{xi} = (1\,500 \text{ kg})(25.0 \text{ m/s}) = 3.75 \times 10^4 \text{ kg}\cdot\text{m/s}$$

$$\sum p_{xf} = (4\,000 \text{ kg})v_f \cos\theta$$

$$3.75 \times 10^4 \text{ kg}\cdot\text{m/s} = (4\,000 \text{ kg})v_f \cos\theta$$

$$\sum p_{yi} = \sum p_{yf}$$

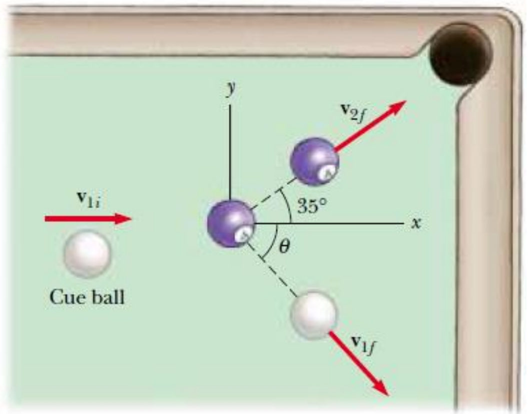
$$5.00 \times 10^4 \text{ kg}\cdot\text{m/s} = (4\,000 \text{ kg})v_f \sin\theta$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{5.00 \times 10^4}{3.75 \times 10^4} = 1.33$$

$$\theta = \boxed{53.1°}$$

$$v_f = \frac{5.00 \times 10^4 \text{ kg}\cdot\text{m/s}}{(4\,000 \text{ kg})\sin 53.1°} = \boxed{15.6 \text{ m/s}}$$

Ex/ In a game of billiards, a player wishes to sink a target ball in the corner pocket, as shown in Figure. If the angle to the corner pocket is 35°, at what angle θ is the cue ball deflected? Assume that friction and rotational motion are unimportant and that the collision is elastic. Also assume that all billiard balls have the same mass *m*.



Conservation of kinetic energy;

$$\frac{1}{2}m_1 v_{1i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

$$m_1 = m_2 = m,$$

$$\boxed{v_{1i}^2 = v_{1f}^2 + v_{2f}^2} \quad \textbf{(1)}$$

conservation of momentum

$$\boxed{m_1\mathbf{v}_{1i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f}} \quad \textbf{(2)}$$

$$m_1 = m_2 = m,$$

$$v_{1i}^2 = (\mathbf{v}_{1f} + \mathbf{v}_{2f}) \cdot (\mathbf{v}_{1f} + \mathbf{v}_{2f}) = v_{1f}^2 + v_{2f}^2 + 2\mathbf{v}_{1f} \cdot \mathbf{v}_{2f}$$

the angle between $\mathbf{v}_{1f}$ and $\mathbf{v}_{2f}$ is $\theta + 35°$,

$$\mathbf{v}_{1f} \cdot \mathbf{v}_{2f} = v_{1f} v_{2f} \cos(\theta + 35°),$$

$$\boxed{v_{1i}^2 = v_{1f}^2 + v_{2f}^2 + 2v_{1f}v_{2f}\cos(\theta + 35°)} \quad \textbf{(3)}$$

Subtracting Equation (1) from Equation (3)

$$0 = 2v_{1f}v_{2f}\cos(\theta + 35°)$$

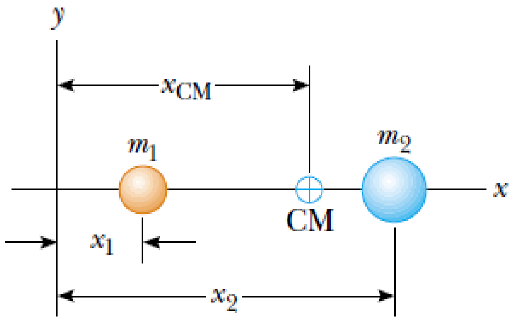
$$0 = \cos(\theta + 35°)$$

$$\theta + 35° = 90° \quad \text{or} \quad \theta = 55°$$

The Center of Mass

Consider a mechanical system consisting of a pair of particles that have different masses and are connected by a light, rigid rod. The position of the center of mass of a system can be described as being the *average position* of the system's mass. The center of mass of the system is located somewhere on the line joining the two particles and is closer to the particle having the larger mass.

The overall motion of a mechanical system can be described in terms of a special point called the center of mass of the system.



$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

We can extend this concept to a system of many particles with masses m_i in three dimensions.

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\sum_i m_i x_i}{M}$$

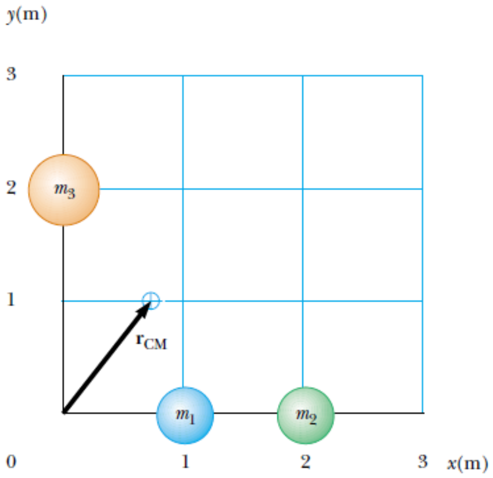
$$y_{CM} = \frac{\sum_i m_i y_i}{M} \quad \text{and} \quad z_{CM} = \frac{\sum_i m_i z_i}{M}$$

$$\mathbf{r}_{CM} = x_{CM}\hat{\mathbf{i}} + y_{CM}\hat{\mathbf{j}} + z_{CM}\hat{\mathbf{k}} = \frac{\sum_i m_i x_i \hat{\mathbf{i}} + \sum_i m_i y_i \hat{\mathbf{j}} + \sum_i m_i z_i \hat{\mathbf{k}}}{M}$$

$$\mathbf{r}_{CM} = \frac{\sum_i m_i \mathbf{r}_i}{M}$$

$$\mathbf{r}_i = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$$

Ex/ A system consists of three particles located as shown in Figure. Find the center of mass of the system.



$$x_{CM} = \frac{\sum_i m_i x_i}{M} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.0 \text{ kg})(1.0 \text{ m}) + (1.0 \text{ kg})(2.0 \text{ m}) + (2.0 \text{ kg})(0)}{1.0 \text{ kg} + 1.0 \text{ kg} + 2.0 \text{ kg}}$$

$$= \frac{3.0 \text{ kg}\cdot\text{m}}{4.0 \text{ kg}} = 0.75 \text{ m}$$

We can extend this concept to a system of many particles with masses m_i in three dimensions.

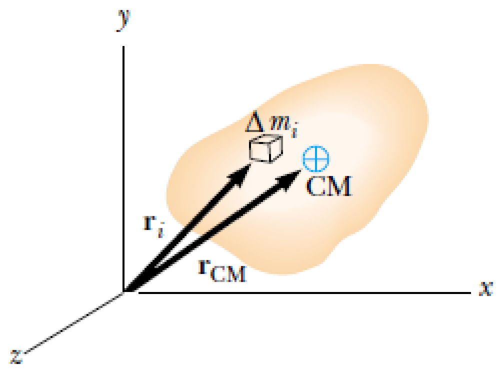
$$y_{CM} = \frac{\sum_i m_i y_i}{M} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.0 \text{ kg})(0) + (1.0 \text{ kg})(0) + (2.0 \text{ kg})(2.0 \text{ m})}{4.0 \text{ kg}}$$

$$= \frac{4.0 \text{ kg}\cdot\text{m}}{4.0 \text{ kg}} = 1.0 \text{ m}$$

$$\mathbf{r}_{CM} \equiv x_{CM}\hat{\mathbf{i}} + y_{CM}\hat{\mathbf{j}} = \boxed{(0.75\hat{\mathbf{i}} + 1.0\hat{\mathbf{j}}) \text{ m}}$$

- Center of mass of an extended object



$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{\sum_i x_i \Delta m_i}{M} = \frac{1}{M} \int x \, dm$$

$$y_{CM} = \frac{1}{M} \int y \, dm \qquad \text{and} \qquad z_{CM} = \frac{1}{M} \int z \, dm$$

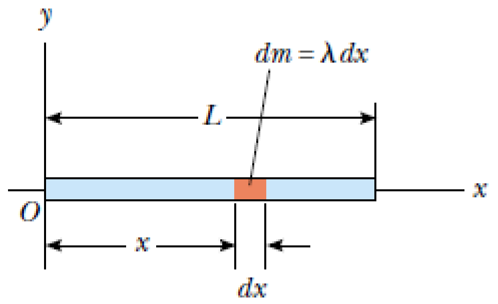
$$\boxed{\mathbf{r}_{CM} = \frac{1}{M} \int \mathbf{r} \, dm}$$

- *Mass densities for extended objects*

dimensions	*for uniform object*	*in general*
1	$\lambda = \frac{M}{L}$	$\lambda = \frac{dm}{dl}$
2	$\sigma = \frac{M}{A}$	$\sigma = \frac{dm}{dA}$
3	$\rho = \frac{M}{V}$	$\rho = \frac{dm}{dV}$

Ex/

(A) Show that the center of mass of a rod of mass M and length L lies midway between its ends, assuming the rod has a uniform mass per unit length.



for uniform one dimensional rod $\lambda = M/L$

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{\lambda}{M}\frac{x^2}{2}\bigg|_0^L = \frac{\lambda L^2}{2M}$$

$$x_{CM} = \frac{L^2}{2M}\left(\frac{M}{L}\right) = \boxed{\frac{L}{2}}$$

(B) Suppose a rod is *nonuniform* such that its mass per unit length varies linearly with x according to the expression $\lambda = \alpha x$, where α is a constant. Find the x coordinate of the center of mass as a fraction of L.

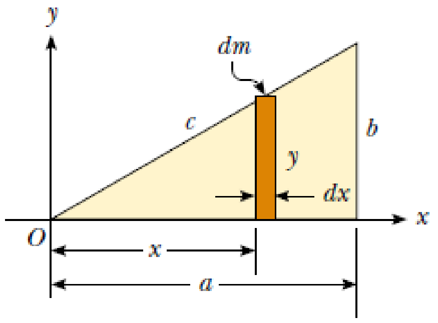
$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{1}{M}\int_0^L x\alpha x\,dx$$

$$= \frac{\alpha}{M}\int_0^L x^2\,dx = \frac{\alpha L^3}{3M}$$

$$M = \int dm = \int_0^L \lambda\,dx = \int_0^L \alpha x\,dx = \frac{\alpha L^2}{2}$$

$$x_{CM} = \frac{\alpha L^3}{3\alpha L^2/2} = \boxed{\tfrac{2}{3}L}$$

Ex/ Find the x-coordinate of center of mass of the triangle shaped uniform plate.



$$x_{\mathrm{CM}} = \frac{1}{M}\int x\,dm :$$

$$\sigma = \frac{dm}{dA} \qquad dA = y\,dx$$

Plate is uniform $\qquad \sigma = \dfrac{M}{A}$

$$dm = \frac{2My}{ab}\,dx \qquad\qquad y = (b/a)\,x.$$

$$x_{\mathrm{CM}} = \frac{2}{ab}\int_0^a x\left(\frac{b}{a}x\right)dx = \frac{2}{a^2}\int_0^a x^2\,dx = \frac{2}{a^2}\left[\frac{x^3}{3}\right]_0^a$$

$$= \boxed{\tfrac{2}{3}a}$$

Assuming *M* remains constant for a system of particles, that is, no particles enter or leave the system, we obtain the following expression for the velocity of the center of mass of the system

$$\mathbf{v}_{\mathrm{CM}} = \frac{d\mathbf{r}_{\mathrm{CM}}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\mathbf{r}_i}{dt} = \frac{\sum_i m_i \mathbf{v}_i}{M}$$

Momentum $$M\mathbf{v}_{\mathrm{CM}} = \sum_i m_i \mathbf{v}_i = \sum_i \mathbf{p}_i = \mathbf{p}_{\mathrm{tot}}$$

acceleration

$$\mathbf{a}_{\mathrm{CM}} = \frac{d\mathbf{v}_{\mathrm{CM}}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\mathbf{v}_i}{dt} = \frac{1}{M}\sum_i m_i \mathbf{a}_i$$

$$M\mathbf{a}_{\mathrm{CM}} = \sum m_i \mathbf{a}_i = \sum_i \mathbf{F}_i$$

$$\sum \mathbf{F}_{\mathrm{ext}} = M\mathbf{a}_{\mathrm{CM}}$$