ENGINEERING SYSTEM MODELLING AND SIMULATION

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Classical Solutions of Ordinary Linear Differential Equations

Integrating Factor Method

The integrating factor method provides a solution to any first-order linear differential equation. Consider

$$a_1(x)\frac{dy}{dx} + a_o(x)y = r(x)$$

This equation can be put into the form

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad P(x) = a_0(x)/a_1(x) \qquad Q(x) = r(x)/a_1(x)$$

The solution is $$y = e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}\,dx + C\right]$$

The integration constant *C* is obtained using the initial condition.

Example

$$mC_p \frac{dT}{dt} + hAT = \dot{q}_{in} \quad \text{with} \quad T(0) = T_i$$

$$\dot{q}_{in} = \gamma e^{-\beta t}$$

$$\frac{dT}{dt} + \frac{hA}{mC_p} T = \frac{\gamma}{mC_p} e^{-\beta t}$$

$$\frac{dT}{dt} + aT = b\, e^{-\beta t}$$

$$a = (hA/mC_p) \qquad b = (\gamma/mC_p)$$

$$P(t) = a \qquad Q(t) = b\, e^{-\beta t}$$

So

$$e^{-\int P(t)dt} = e^{-\int a\,dt} = e^{-at} \qquad e^{\int P(t)dt} = e^{\int a\,dt} = e^{at}$$

Then,

$$T = e^{-at}\left[\int b e^{-\beta t} e^{at}\, dt + C\right] = e^{-at}\left[\int b e^{(a-\beta)t}\, dt + C\right]$$

$$T = e^{-at}\left[\frac{b}{a-\beta} e^{(a-\beta)t} + C\right]$$

Applying the initial condition

$$C = T_i - (b/a - \beta)$$

$$T = e^{-at}\left[T_i + \frac{b}{a-\beta}\left(e^{(a-\beta)t} - 1\right)\right]$$

Classical Solutions of Ordinary Linear Differential Equations

Characteristic Equation

The *characteristic equation method* is the technique for solving homogeneous *n*th-order linear differential equations with constant (time invariant) coefficients.
This method provides the solution for a *constant-coefficient linear homogeneous* differential equation. Consider

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = 0$$

a_n, a_{n-1}, a_1, and a_0 are *constant coefficients*.

The *fundamental* step of this technique is to assume a solution of the form $y = e^{rt}$. Using this assumption,

$$\frac{d^n y}{dt^n} = r^n e^{rt}\ ;\cdots;\quad \frac{d^2 y}{dt^2} = r^2 e^{rt}\ ;\quad \frac{dy}{dt} = r\, e^{rt}$$

Substituting into equation,

$$a_n r^n e^{rt} + a_{n-1} r^{n-1} e^{rt} + \ldots + a_1 r\, e^{rt} + a_0\, e^{rt} = 0$$

and dividing both sides by e^{rt} gives

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0$$

This equation is called the *characteristic equation*; it yields the following n roots: r_n, r_{n-1}, …,r_1. Because there is more than one root, we rewrite the initial assumption as

$$y = \sum_{i=1}^{n} C_i e^{r_i t}$$

or,

$$y = C_n e^{r_n t} + C_{n-1} e^{r_{n-1} t} + \ldots + C_1 e^{r_1 t}$$

The constants C_n, C_{n-1}, …, and C_1 are evaluated using the initial conditions.

This method is fairly simple; the most difficult step is obtaining the roots of the characteristic equation.

Qualitative Characteristic of System Response

The characteristic equation method is particularly powerful because the *roots of the equation* completely describe *the qualitative response (behavior) of the system*. Most of the important information about the system response can be obtained from these roots.

The relevant questions about the response are the following:

- Is the response stable? That is, will the response remain bounded when forced by a bounded input?

- Is the response monotonic or oscillatory?

- If monotonic and stable, how long will it take for the transients to die out?

- If oscillatory, what is the period of oscillation and how long will it take for the oscillations to die out?

A response is stable if it remains bounded when forced by a bounded input. An *unstable response* is one that when forced by a bounded response, it continues moving up or down without stopping and reaching a final value; a stable response reaches a final value.
The bounded input must be one that reaches a final value.

Consider a second-order differential equation; all findings apply to any *n*th-order differential equation,

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0$$

from which the following characteristic equation develops:

$$a_2 r^2 + a_1 r + a_0 = 0$$

and from the quadratic equation we obtain the roots

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

There are three possible cases depending on the value of the term under the square root:

1. $a_1^2 - 4a_2a_0 > 0$ yielding two real roots r_1 and r_2
2. $a_1^2 - 4a_2a_0 = 0$ yielding a single repeated real root
3. $a_1^2 - 4a_2a_0 < 0$ yielding two complex roots at α+ iβ

For case 1 (two real roots), $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

For case 2 (a single repeated root), both roots are at r = –(a_1 /2a_2). The first term of the solution is y_1 = C_1 e^{rt}.

The second term of the solution is the same (except with coefficient) multiplied by the independent variable, $y_2 = C_2 t\, e^{rt}$, then $y = C_1 e^{rt} + C_2 t\, e^{rt}$

For case 3 (two complex roots) the roots are at $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where $\alpha = a_1/2a_2$ and

$$\beta = \frac{\sqrt{4a_2 a_0 - a_1^2}}{2a_2}$$

then

$$y = C_1' e^{(\alpha+i\beta)t} + C_2' e^{(\alpha-i\beta)t}$$

$$y = e^{\alpha t}\left[C_1' e^{i\beta t} + C_2' e^{-i\beta t}\right]$$

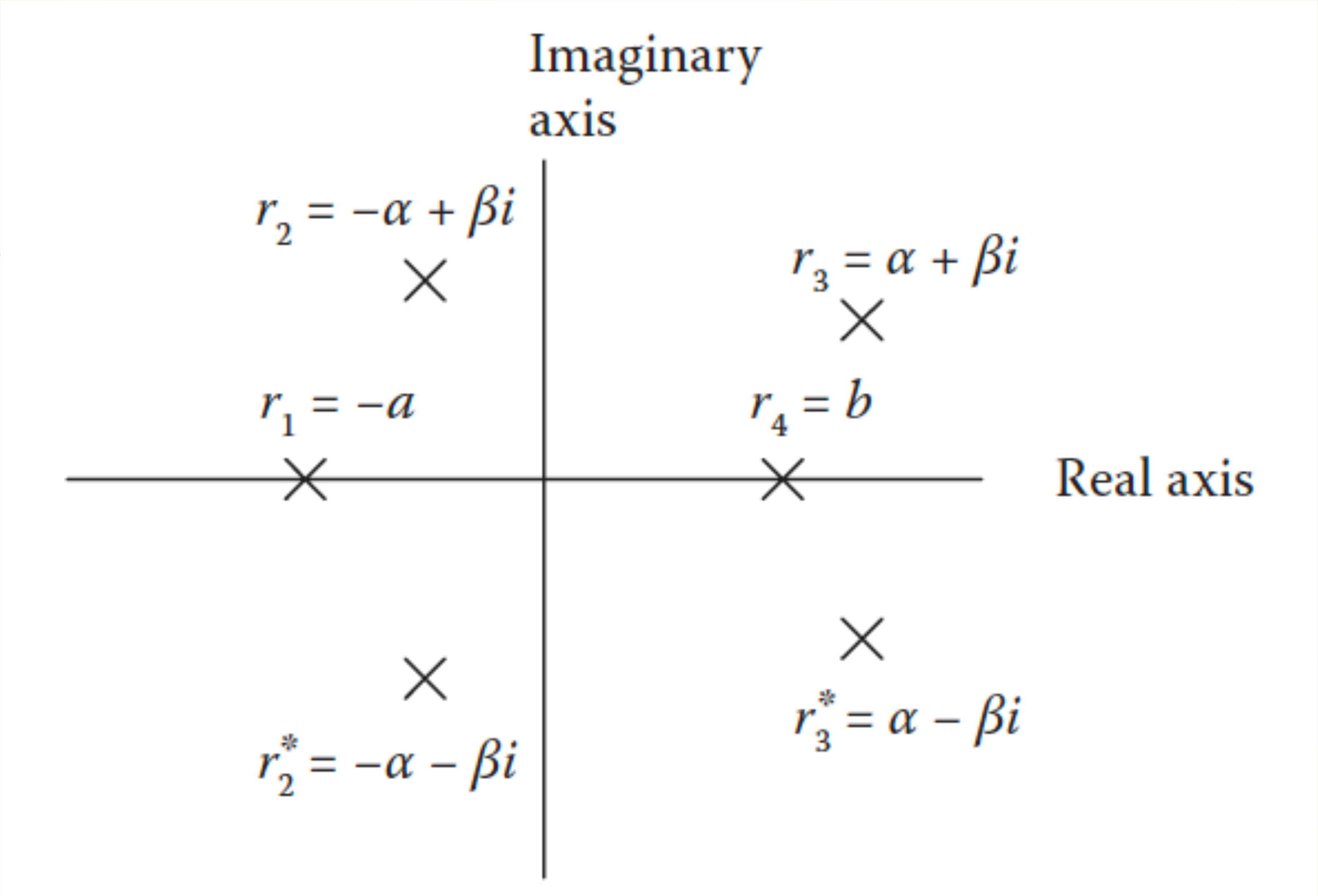
It is rather difficult to obtain a good qualitative indication of this response because of the complex exponential powers. A better expression, avoiding complex numbers, can be obtained using Euler's identity $e^{i\beta t} = \cos \beta t + i \sin \beta t$.

$$y = e^{\alpha t}[C_1 \cos \beta t + C_2 \sin \beta t]$$

The first two cases, result in exponential responses, and the third case, in an oscillatory response. The roots can be either real or complex; consider Figure "Roots of characteristic equation".

Locations 1 and 4 correspond to cases 1 and 2 when the roots are real, and locations 2 and 3 correspond to case 3 when the roots are complex (the asterisk denotes the complex conjugate).

Roots	Form of the solution
r_1	$y = C_1 e^{-at}$
r_2, r_2^*	$y = e^{-at}[C_1 \cos \beta t + C_2 \sin \beta t]$
r_3, r_3^*	$y = e^{at}[C_1 \cos \beta t + C_2 \sin \beta t]$
r_4	$y = C_1 e^{bt}$



Roots of characteristic equation

As the independent variable t increases, the solutions for the roots in locations 1 and 2 indicate that the response decays exponentially, owing to the negative exponent, with oscillations superimposed in location 2. However, the solutions for the roots in locations 3 and 4 indicate that the responses increase without bounds, owing to the positive exponent. Thus, the roots in locations 1 and 2 provide *stable responses*, and the roots in locations 3 and 4 provide *unstable responses*. The difference is in the location of the real part of the root.

For roots with negative real parts the response is stable; for roots with positive real parts the response is unstable. Furthermore, for real roots the response is *monotonic*, and for complex roots the response is *oscillatory* . Not very often, although the roots are real and thus the response is given by exponentials, the response is a bit oscillatory and not monotonic. This rare instance may only happen when there are multiple roots. Because for homogeneous differential equations there is no forcing function, *the qualitative behavior of the system does not depend on the type of forcing function, f(t), only on the characteristics of the system itself.*

We can also express these last statements as

$$\text{Root } \alpha \pm i\beta$$

Response is stable or unstable solely depending on the sign of α. If negative, the response is stable; if positive, the response is unstable

Response is monotonic or oscillatory solely depending on the numerical value of β. If $\beta = 0$, the roots are real and the response is monotonic; if $\beta \neq 0$, the roots are complex, or imaginary, and the response is oscillatory

Classical Solutions of Ordinary Linear Differential Equations

Undetermined Coefficients

The undetermined coefficient method is a technique for solving nonhomogeneous *n*th-order linear differential equations with constant coefficients.

The characteristic equation method only applies to homogeneous differential equations.

The method for obtaining *the general solution, y,* of a nonhomogeneous differential equation calls for dividing the solution into two parts, the *complementary solution* y_C and *the particular solution* y_P or

$$y = y_C + y_P$$

The form of the particular solution solely depends on the form of the forcing function, and this is why sometimes it is also called the *forced response*.

Because the particular solution only depends on the forcing function, it does not have anything to do with the system itself. The complementary solution is the one related to the system, including the initial conditions.

Consider

$$a_2\frac{d^2y}{dt^2}+a_1\frac{dy}{dt}+a_0y=f(t)$$

$$a_2\frac{d^2(y_C+y_P)}{dt^2}+a_1\frac{d(y_C+y_P)}{dt}+a_0(y_C+y_P)=f(t)$$

$$a_2\frac{d^2y_C}{dt^2}+a_1\frac{dy_C}{dt}+a_0y_C+a_2\frac{d^2y_P}{dt^2}+a_1\frac{dy_P}{dt}+a_0y_P=f(t)$$

Because y_P is the solution that depends on the forcing function

$$a_2\frac{d^2y_P}{dt^2}+a_1\frac{dy_P}{dt}+a_0y_P=f(t)$$

Then

$$a_2 \frac{d^2 y_C}{dt^2} + a_1 \frac{dy_C}{dt} + a_0 y_C = 0$$

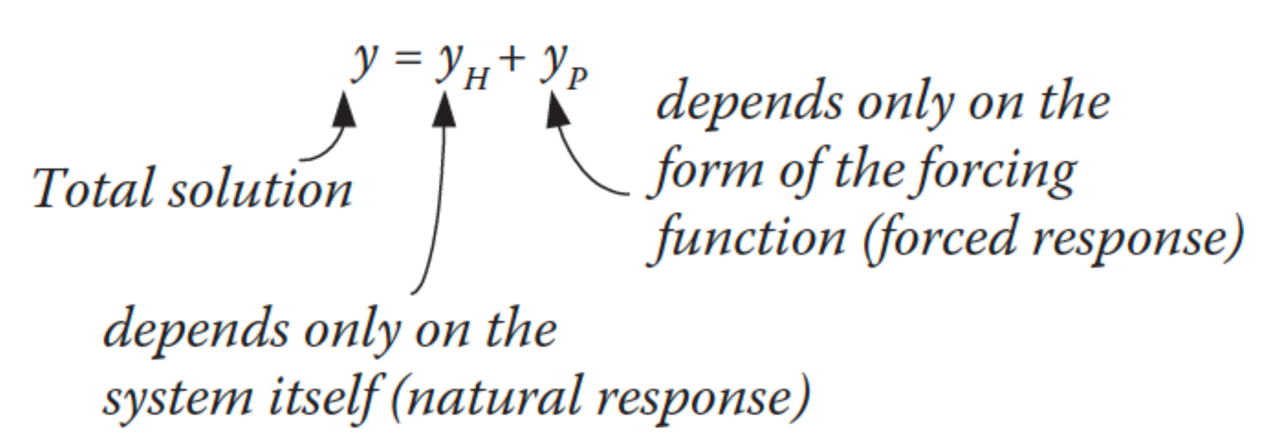
So, the solution of the complementary part is just the solution of the corresponding homogeneous equation, which hereinafter we refer to as y_H or

$y = y_H + y_P$

Thus, the general solution is the summation of a "solution of the corresponding homogeneous equation" plus a "particular solution of the nonhomogeneous equation."

The form of the particular solution depends only on the form of the forcing function.

The solution of homogeneous differential equations is independent of the type of forcing function affecting the system. Thus, it is only dependent on the system itself, and it is why sometimes it is called the *natural response*.



Using the characteristic equation method, we obtained the homogeneous solution. For obtaining the particular solution we use the method of undetermined coefficients; this method consists of the following:

• On the basis of the forcing function select a particular solution (also referred to as a “trial solution”).

Form of the Forcing Function, $f(t)$	Form of the Particular Solution
$a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$	$A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$
$(a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0) e^{qt}$	$(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0) e^{qt}$
$(a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0) \cos pt + (b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0) \sin pt$	$(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0) \cos pt + (B_n t^n + B_{n-1} t^{n-1} + \cdots + B_1 t + B_0) \sin pt$
$a e^{qt} \cos pt + b e^{qt} \sin pt$	$A e^{qt} \cos pt + B e^{qt} \sin pt$

- If the forcing function involves a sine or cosine, the particular solution should contain *both* a sine and a cosine
- If any part of the particular solution is a solution of the homogeneous equation, multiply that particular solution by the independent variable. Repeat if necessary, that is, if after multiplied by the independent variable the result is still a solution of the homogeneous equation, multiply again by the independent variable

Example

Obtain the solution of

$$y'' + y' - 12y = 12t - 72t^2 \text{ with } y'(0) = 0 \; y(0) = 3$$

$$y = y_H + y_P$$

$$y''_H + y'_H - 12y_H = 0$$

Assume $y_H = e^{rt}$

$$r^2 + r - 12 = 0$$

$$(r+4)(r-3) = 0 \quad \Rightarrow \quad r_1 = 3 \; ; \; r_2 = -4$$

One root, r_1, is located in the positive real axis, and the other root, r_2, is in the negative real axis, thus indicating an *unstable response*. The fact that both roots are real (no imaginary parts) indicates that the response is *monotonic*.

$$y_H = C_1\, e^{3t} + C_2\, e^{-4t}$$

The forcing function $12t - 72t^2$ is a polynomial in the independent variable t, so a particular solution can be

$$y_P = A_2 t^2 + A_1 t + A_0$$

$$y_P' = 2A_2 t + A_1 \qquad y_P'' = 2A_2$$

$$2A_2 + 2A_2t + A_1 - 12(A_2t^2 + A_1t + A_0) = 12t - 72t^2$$

$$(2A_2 + A_1 - 12A_0) + (2A_2 - 12A_1)t - 12A_2t^2 = 12t - 72t^2$$

This last equation shows two polynomials, one on each side of the equal sign.

Equating the coefficients of equal terms in these polynomials, we obtain A_2, A_1, and A_0.

$$12A_2 = 72 \Rightarrow A_2 = 6$$

$$2A_2 - 12A_1 = 12$$

$$2(6) - 12A_1 = 12 \Rightarrow A_1 = 0$$

$$2A_2 + A_1 - 12A_0 = 0$$

$$2(6) + 0 - 12A_1 = 0 \Rightarrow A_0 = 1$$

Thus,

$$y_p = 1 + 6t^2$$

$$y = C_1e^{3t} + C_2e^{-4t} + 6t^2 + 1$$

And applying the initial conditions gives $C_1 = 8/7$ and $C_2 = 6/7$. Therefore

$$y = \underbrace{\frac{8}{7}e^{3t} + \frac{6}{7}e^{-4t}}_{\text{Due to the system itself}} + \underbrace{6t^2 + 1}_{\text{Due to the forcing function}}$$

Example

Obtain the solution of

$$y'' + y' - 12y = e^{2t} \text{ with } y'(0) = 0 \ y(0) = 3$$

The corresponding homogeneous equation is the same as in the previous example. This time the forcing function is an exponential, so the particular solution can be

$$y_P = A_0 e^{2t}$$

$$y'_P = 2A_0 e^{2t} \qquad y''_P = 4A_0 e^{2t}$$

$$4A_0\,e^{2t} + 2A_0e^{2t} - 12A_0\,e^{2t} = e^{2t}$$

$$4A_0 + 2A_0 - 12A_0 = 1 \qquad\qquad A_0 = -\frac{1}{6}$$

$$y_P = -\frac{1}{6}e^{2t}$$

$$y = \underbrace{1.857e^{3t} + 1.309e^{-4t}}_{\text{Due to the system itself and initial conditions}} \underbrace{- 0.167e^{2t}}_{\text{Due to the forcing function}}$$

Example

Obtain the solution of

$$y'' + y' - 12y = e^{3t} \text{ with } y'(0) = 0 \; y(0) = 3$$

The corresponding homogeneous equation is the same as in the previous two examples. The particular solution, $y_P = A_0e^{3t}$, does not work because it is part of the homogeneous solution. So the suggestion is to multiply y_P by the independent variable t,

$$y_P = A_0 t\, e^{3t}$$

$$y_P' = A_0 e^{3t} + 3A_0 t e^{3t} \qquad y_P'' = 6A_0 e^{3t} + 9A_0 t e^{3t}$$

Substituting into the differential equation gives $A_0 = 1/7$; thus,

$$y_P = \frac{1}{7} t e^{3t}$$

Using the initial conditions gives
$C_1 = 1.694$ and $C_2 = 1.306$.

Therefore,

$$y = 1.694\, e^{3t} + 1.306\, e^{-4t} + 0.143t\, e^{3t}$$

Example

The following differential equation describes an undamped mass-spring system:

$$x'' + 16x = 4 \sin \omega t$$

We start by finding the solution to the corresponding homogeneous equation,

$$x_H'' + 16x_H = 0$$

Assuming $x_H = e^{rt}$, we get

$$r^2 + 16 = 0 \;\Rightarrow\; r_1 = 4i \;;\; r_2 = -4i$$

And using the previous treatment,

$$x_H = C_1 \cos 4t + C_2 \sin 4t$$

The frequency of this homogeneous response or natural response is 4 radians/time. We call this the *natural frequency* and denote it by ω_n.

For the particular solution, let us assume first that $\omega \neq 4$. In this case, the particular solution can be

$$x_P = A \cos \omega t + B \sin \omega t$$

$$x_P' = -A\omega \sin\omega t + B\omega \cos\omega t \qquad x_P'' = -A\omega^2 \cos\omega t - B\omega^2 \sin\omega t$$

$$-A\omega^2 \cos \omega t - B\omega^2 \sin \omega t + 16A \cos \omega t + 16B \sin \omega t = 4 \sin \omega t$$

$$(16A - A\omega^2)\cos \omega t + (16B - B\omega^2)\sin \omega t = 4 \sin \omega t$$

Equating the coefficients of equal terms in this equation,

$$16A - A\omega^2 = 0 \quad \Rightarrow \quad A = 0$$

$$16B - B\omega^2 = 4 \qquad B = \frac{4}{16-\omega^2}$$

Finally

$$x = C_1 \cos 4t + C_2 \sin 4t + \frac{4}{16-\omega^2} \sin \omega t$$

x becomes large as $\omega \to 4$ or $\omega \to \omega_n$.

Let us now assume that $\omega = 4$. In this case, the suggestion for the particular solution is in itself a solution of the homogeneous part and will not work. The procedure then is to multiply the suggestion the independent variable. The particular solution is then

$$x_p = t(A \cos 4t + B \sin 4t)$$

$$x_p' = A\cos 4t + B\sin 4t + t(-4A\sin 4t + 4B\cos 4t)$$

$$x_p'' = -8A\sin 4t + 8B\cos 4t + t(-16A\cos 4t + 16B\sin 4t)$$

$$-8A \sin 4t + 8B \cos 4t + t(-16A \cos 4t + 16B \sin 4t)$$
$$+16t(A \cos 4t + B \sin 4t) = 4 \sin 4t$$

$$-8A \sin 4t + 8B \cos 4t = 4 \sin 4t$$

Equating the coefficients of equal terms gives A = −1/2 and B = 0. Therefore, the general solution is

$$x = C_1 \cos 4t + C_2 \sin 4t - \frac{1}{2} t \cos 4t$$

Note that x becomes unbounded as t increases.

Multiple Forcing Functions

Sometimes multiple forcing functions may affect the system at the same time. For example, consider

$$y'' + y' - 12y = 12t - 72t^2 + e^{2t} \text{ with } y'(0) = 0 \; y(0) = 3$$

We can write this differential equation explicitly showing two forcing functions

$$y'' + y' - 12y = f_1(t) + f_2(t)$$

where

$$f_1(t) = 12t - 72t^2$$

$$f_2(t) = e^{2t}$$

$$y = y_H + y_{P_1} + y_{P_2}$$

y_{P1} is the particular solution due the first forcing function $f_1(t)$ and y_{P2} is the particular solution due the second forcing function $f_2(t)$.

$$y = C_1 e^{3t} + C_2 e^{-4t} + 1 + 6t^2 - \frac{1}{6}e^{2t}$$

Using the initial conditions, $C_1 = 27/21$ and $C_2 = 37/42$

$$y = \frac{27}{21}e^{3t} + \frac{37}{42}e^{-4t} + 1 + 6t^2 - \frac{1}{6}e^{2t}$$

Classical Solutions of Ordinary Linear Differential Equations

Response of First- and Second-Order Systems

Most of the models are composed of first- or second order differential equations; this is often the case for physical/industrial models.

First-Order Systems

For example;

$$mC\frac{dT}{dt} = \dot{q}_{\text{in}} - hA(T - T_A)$$

The response of these systems to two different types of input, *a step function*, and *a sine wave*.

Consider the linear first-order differential equation with constant coefficients:

$$a_1 \frac{dy(t)}{dt} + a_0 y(t) = bx(t) \quad \text{with} \quad y(0) = y_0$$

The equation has three coefficients, a_1, a_0, and b, but, without loss of generality, we can divide the equation by one of the three so that we can characterize the equation by just two parameters.

It is often customary to divide by the coefficient of the dependent variable, a_0, provided it is not zero. Such an operation results in the following equation, which we shall call the *standard form* of the linear first-order differential equation with constant coefficients.

$$\tau \frac{dy(t)}{dt} + y(t) = Kx(t)$$

$\tau = a_1/a_0$, often called the *time constant*; with unit of time
$K = b/a_0$, often called the *system gain*; with units of the dependent variable over units of the forcing function.

Starting from a steady-state operation, meaning that

$$\left.\frac{dy}{dt}\right|_{t=0} = 0$$

with a forcing function of x(0) then, y(0) = *K* x(0).
τ must have dimension of time, and *K* must have dimension of y over dimension of x.

Any linear first-order differential equation can be transformed into the standard form, as long as the dependent variable y(t) appears in the equation.

For example;

$$\tau \frac{dT}{dt} + T = K_1 \dot{q}_{in} + K_2 T_A$$

$$\tau = \frac{MC}{hA} = \frac{(1.75\ \mathrm{kg})(450\ \mathrm{J/kg \cdot {}^\circ C})}{(20\ \mathrm{J/s \cdot m^2 \cdot {}^\circ C})(0.05\ \mathrm{m^2})} = 787.5\ \mathrm{s}$$

$$K_1 = \frac{1}{hA} = \frac{1}{(20\ \mathrm{J/s \cdot m^2 \cdot {}^\circ C})(0.05\ \mathrm{m^2})} = 1.0\ \frac{{}^\circ \mathrm{C}}{\mathrm{J/s}}$$

K_2 = 1.0 dimensionless

Although equation $\tau\frac{dy(t)}{dt}+y(t)=Kx(t)$ can be solved by anti-differentiation, separation of variables or integrating factor, we choose to solve it here by the characteristic equation and undetermined coefficient methods. Being a nonhomogeneous equation, we first solve for the corresponding homogeneous equation

$$\tau\frac{dy_H(t)}{dt}+y_H(t)=0 \qquad \tau r+1=0$$

and the root is $r=-\frac{1}{\tau}$ $\qquad y_H(t)=Ce^{-\frac{t}{\tau}}$

The particular solution depends on the forcing function.

Step Function Input

Suppose that the forcing function x(t) changes from its initial value of x(0) to its final value of x_F = x(0) + D at time = 0, that is, x(t) = x(0) + Du(t), a step change of D units of magnitude. In this case,

$$y_P(t) = A_0 \qquad \frac{dy_P(t)}{dt} = 0$$

Substituting into equation $\tau \frac{dy(t)}{dt} + y(t) = Kx(t)$

$$\tau(0) + A_0 = Kx_F \text{ or } y_P(t) = A_0 = Kx_F$$

Finally,

$$y(t) = y_H(t) + y_P(t) = C\,e^{-\frac{t}{\tau}} + Kx_F$$

Using the initial condition, we obtain C

$$y_0 = C(1) + Kx_F \quad C = y_0 - Kx_F$$

$$y(t) = y_0\,e^{-\frac{t}{\tau}} + Kx_F\left(1 - e^{-\frac{t}{\tau}}\right)$$

Because $x_F = x(0) + D$,

$$y(t) = y_0\, e^{-\frac{t}{\tau}} + K(x(0) + D)\left(1 - e^{-\frac{t}{\tau}}\right)$$

$$y(t) = y_0\, e^{-\frac{t}{\tau}} + Kx(0) - Kx(0)e^{-\frac{t}{\tau}} + KD\left(1 - e^{-\frac{t}{\tau}}\right)$$

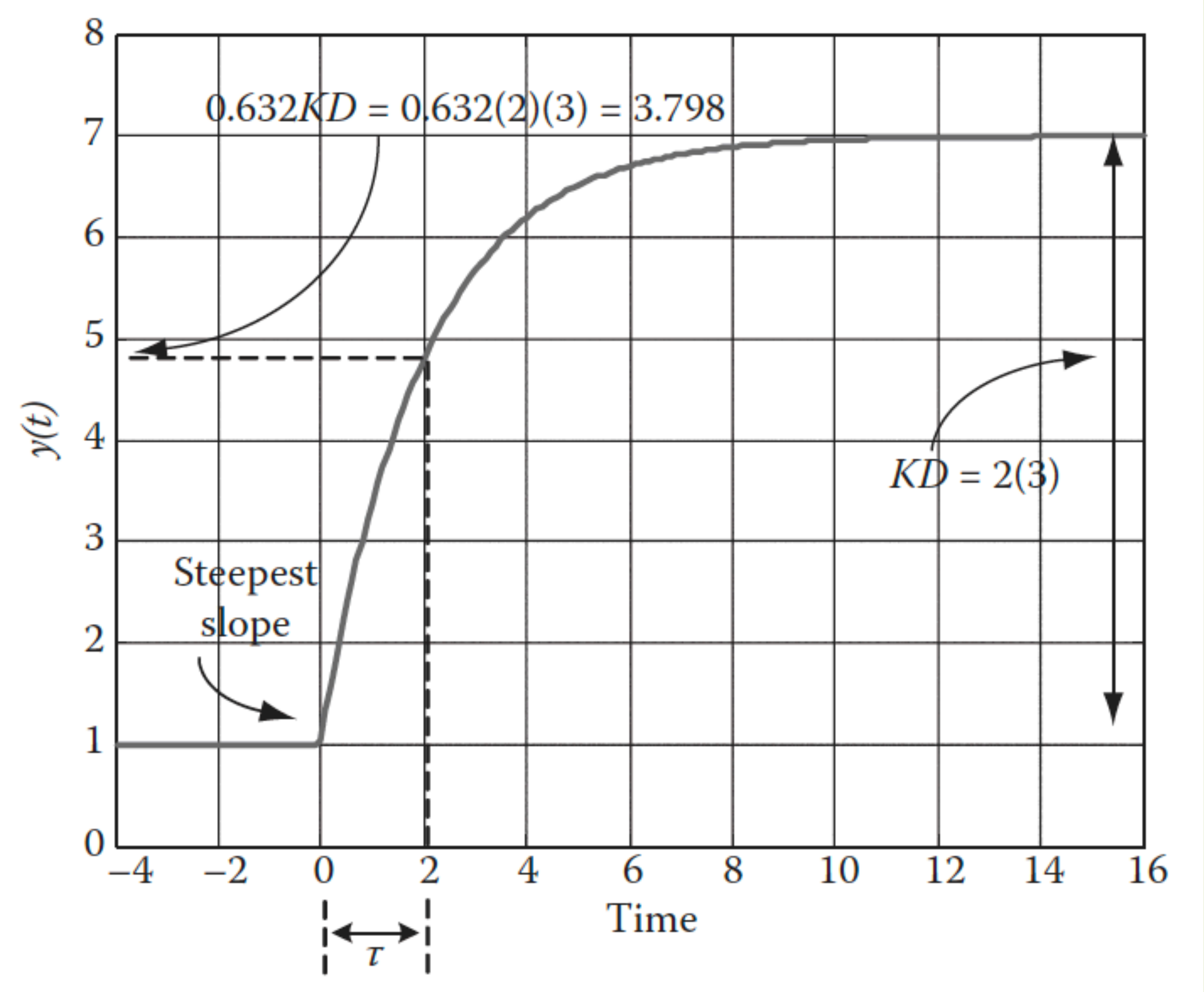
And finally, because y(0) = *K* x(0),

$$y(t) = y(0) + KD\left(1 - e^{-\frac{t}{\tau}}\right)$$

where D = x_F – x(0). Instead of writing D for the step change, many textbooks show the step response equation as

$$y(t) = y(0) + K(x_F - x(0))\left(1 - e^{-\frac{t}{\tau}}\right)$$

Obviously, the negative real root indicates that the system is stable and monotonic in its response. The above equation describes the unit step response of any first-order system. These equations are used in many engineering courses.



Response of a first-order system to a step change in input.

A graph of the response is very instructive; the above figure shows the response of the system when K = 2, D = 3, τ = 2, and y(0) = 1.

- The steepest slope in the response curve occurs at the beginning of the response; this is the typical response of first-order systems to a step change in forcing function or input variable.
- The total change in dependent variable, output variable, is given by KD, the system gain times the change in input; thus, we say that the gain K gives the change in output per unit change in input (or how sensitive the output is to a change in input); obviously, the units of K also show this meaning.

- 63.2% of the total change occurs in one time constant. Obviously, the response equation provides this number. When t = $\boldsymbol{\tau}$,

$$y(t=\tau)=y_0+KD\left(1-e^{-\frac{\tau}{\tau}}\right)=y_0+KD(1-e^{-1})$$

$$y(t=\tau)=y_0+KD(1-0.368)=y_0+0.632KD$$

Actually, this helps us in obtaining the significance of the time constant τ. The smaller the time constant, the less time it takes the system to reach 63.2% of its total change; thus, the faster responding the system is. Therefore, *the time constant is related to the speed of the system once it starts changing*. The table below tabulates the change in output versus t/τ. Note that the response starts at maximum rate of change right after the step is applied, and then the rate of change decreases so that the final change of KD is approached asymptotically.

First-Order Step Response

t/τ	Fraction Change in Output
0	0
1.0	0.632
2.0	0.865
3.0	0.950
4.0	0.982
5.0	0.993
≅ ≅ ≅	≅ ≅ ≅
∞	1.000

After one time constant the response reaches 63.2% of its final change, and in five time constants it reaches over 99% of the change. In other words, the response is essentially complete after five time constants; it is commonly accepted in most areas of engineering to *use 5τ as the time it takes to reach the new steady state*.

The qualitative response of all first-order differential equations to a step change in input is the same. The quantitative portion is the one that differs.

Sinusoidal Function Input

Suppose x(t) = B sinωt, in this case

$$y_P(t) = A_1 \cos \omega t + A_0 \sin \omega t$$

$$\frac{dy_P(t)}{dt} = -A_1\omega \sin \omega t + A_0\omega \cos \omega t$$

Substituting, $y_P(t)$ and $dy_P(t)/dt$ into equation $\tau \frac{dy(t)}{dt} + y(t) = Kx(t)$

$$\tau(-A_1\omega \sin \omega t + A_0\omega \cos \omega t) + A_1 \cos \omega t + A_0 \sin \omega t = KB \sin \omega t$$

$$(-\tau A_1\omega + A_0)\sin \omega t + (\tau A_0\omega + A_1)\cos \omega t = KB \sin \omega t$$

Equating equal terms,

$$-\tau A_1\omega + A_0 = KB$$

$$\tau A_0\omega + A_1 = 0$$

From these last two equations,

$$A_0 = \frac{KB}{1+(\tau\omega)^2} \qquad A_1 = \frac{-KB\tau\omega}{1+(\tau\omega)^2}$$

$$y_P(t) = \frac{-KB\tau\omega}{1+(\tau\omega)^2}\cos\omega t + \frac{KB}{1+(\tau\omega)^2}\sin\omega t$$

$$y(t) = y_H(t) + y_P(t) = Ce^{-\frac{t}{\tau}} - \frac{KB\tau\omega}{1+(\tau\omega)^2}\cos\omega t + \frac{KB}{1+(\tau\omega)^2}\sin\omega t$$

Using the initial condition, we obtain C

$$C = y_0 + \frac{KB\tau\omega}{1+(\tau\omega)^2}$$

Therefore,

$$y(t) = \left(y_0 + \frac{KB\tau\omega}{1+(\tau\omega)^2}\right)e^{-\frac{t}{\tau}} - \frac{KB\tau\omega}{1+(\tau\omega)^2}\cos\omega t + \frac{KB}{1+(\tau\omega)^2}\sin\omega t$$

In many engineering fields the study of system dynamics is of prime importance, and the above equation is often used. Commonly, this equation is also expressed as

$$y(t)=\left(y_0+\frac{KB\tau\omega}{1+(\tau\omega)^2}\right)e^{-\frac{t}{\tau}}+\frac{KB}{\sqrt{1+(\tau\omega)^2}}\sin(\omega t+\theta)$$

where $\theta = -\tan^{-1}(\tau\omega)$. The above equation develops using the following trigonometric identity,

$$E\cos\omega t + F\sin\omega t = D\sin(\omega t+\theta)$$

where $D=\sqrt{E^2+F^2}$ and $\theta=\tan^{-1}(E/F)$.

Second-Order Systems

Models composed of second-order differential equations are also quite common. For example,

$$m\frac{d^2x}{dt^2}+P\frac{dx}{dt}+kx=f_A(t)$$

Here we present the response of second-order systems to the same two inputs, a step function, and a sine wave. Our objective is to learn how the parameters of second-order systems affect their response.

Consider the linear second-order differential equation:

$$a_2 \frac{d^2y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = bx(t) \quad \text{with} \quad \left.\frac{dy(t)}{dt}\right|_{t=0} = 0 \quad y(0) = y_0$$

The equation has four coefficients, a_2, a_1, a_0, and b, but, without loss of generality, we can divide the equation by one of the three (commonly by a_0) to characterize the equation by just three parameters as given in

$$\tau^2 \frac{d^2y(t)}{dt^2} + 2\zeta\tau \frac{dy(t)}{dt} + y(t) = Kx(t)$$

where

$$\tau = \sqrt{\frac{a_2}{a_0}}$$

:(assuming a_2 and a_0 have the same sign) is often called a *characteristic time*; time units.

$$\zeta = \frac{a_1}{2\sqrt{a_2 a_0}}$$

:is often called the *damping ratio*; dimensionless

$$K = \frac{b}{a_0}$$

: is often called the system gain; with units of the dependent variable over units of the forcing function.

We now write $m\frac{d^2x}{dt^2} + P\frac{dx}{dt} + kx = f_A(t)$

using this form,

$$\tau^2 \frac{d^2x}{dt^2} + 2\zeta\tau\frac{dx}{dt} + x = Kf_A(t)$$

with $\tau = \sqrt{m/k}$ s; $\zeta = \left(P/2\sqrt{mk}\right)$ and $K = \frac{1}{k}$ m/N.

The complete solution is $y(t) = y_H + y_P$

The corresponding homogeneous equation is

$$\tau^2 \frac{d^2y_H(t)}{dt^2} + 2\zeta\tau\frac{dy_H(t)}{dt} + y_H(t) = 0$$

$$\tau^2 r^2 + 2\zeta\tau r + 1 = 0$$

the roots are

$$r_1, r_2 = \frac{-2\zeta\tau \pm \sqrt{4\zeta^2\tau^2 - 4\tau^2}}{2\tau^2} = \frac{-\zeta \pm \sqrt{\zeta^2 - 1}}{\tau}$$

The equation shows the roots of the equation so the response of the system depends on the numerical value of ζ.

We can now see that the term “damping ratio” refers to the damping of oscillations; the behavior of the response is summarized as follows:

$\zeta>1$	The roots are negative real, thus a monotonic and stable response
$0<\zeta<1$	The roots are complex with negative real part, thus an oscillatory and stable response
$\zeta=0$	The roots are complex with zero real part, thus a sustained oscillations response
$\zeta=1$	The roots are real and repeated, thus a monotonic and stable response
$-1<\zeta<0$	The roots are complex with positive real part, thus a growing oscillations response
$\zeta\leqq-1$	The roots are positive real, thus a monotonic unstable response

Step Function Input

Suppose that the forcing function x(t) changes from its initial value of x(0) to its final value of x_F = x(0) + D at time = 0, that is, x(t) = x(0) + Du(t), a step change of D units of magnitude. In this case,

$$y_P(t) = A_0, \frac{dy_P(t)}{dt} = 0, \frac{d^2 y_P(t)}{dt^2} = 0$$

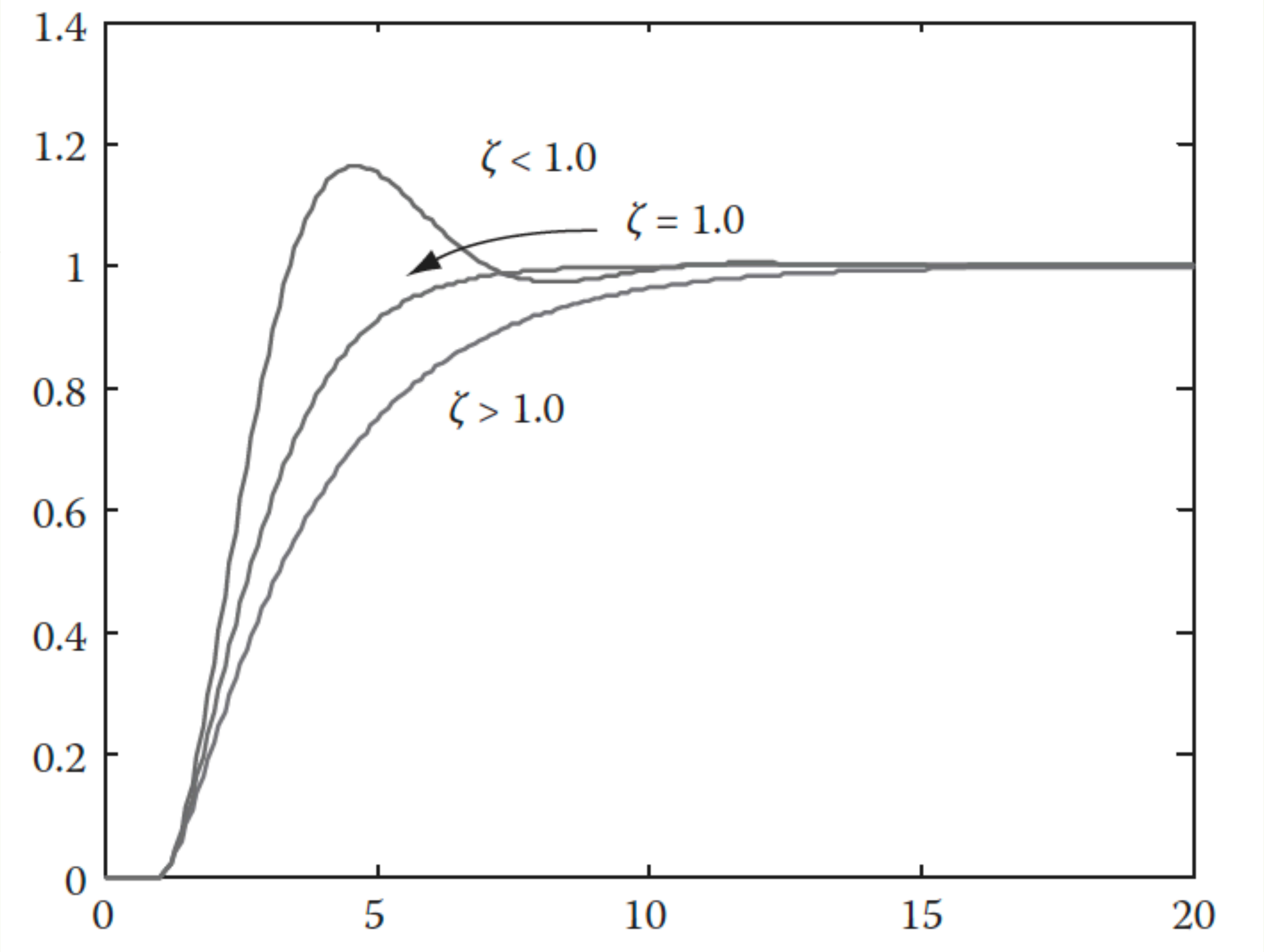
So $\tau^2(0) + 2\zeta\tau(0) + A_0 = Kx_F$ or $y_P(t) = A_0 = Kx_F$

Finally $y(t) = y_H + y_P = y_H + Kx_F$

To show the step responses graphically consider the following second order system:

$$\tau^2 \frac{d^2y}{dt^2} + 2\zeta\tau \frac{dy}{dt} + y = f(t)$$

and assume τ = 1. The initial conditions are y(0) = 0 and y′(0) = 0. The figure below, shows the system's response when f(t) changes from 0 to 1.



Response of second-order system to a change in forcing function

Systems that oscillate before reaching their final values are called *underdamped systems*; systems that do not oscillate before reaching their final values are called *overdamped systems*. There is another type of system called critically damped, which is the one with the fastest approach to its final value without oscillations.

We can summarize this as

Underdamped Systems: $\zeta < 1.0$ or $b_1^2 - 4a_2a_0 < 0$

Overdamped Systems: $\zeta > 1.0$ or $b_1^2 - 4a_2a_0 > 0$

Critically Damped Systems: $\zeta = 1.0$ or $b_1^2 - 4a_2a_0 = 0$

Sinusoidal Function Input

Suppose x(t) = B sinωt , in this case,

$$y_P(t) = A_1 \cos \omega t + A_0 \sin \omega t$$

$$\frac{dy_P(t)}{dt} = -A_1\omega \sin \omega t + A_0\omega \cos \omega t$$

$$\frac{d^2y_P(t)}{dt^2} = -A_1\omega^2 \cos \omega t - A_0\omega^2 \sin \omega t$$

Substituting $y_P(t)$, $[dy_P(t)/dt]$, and $[d^2y_P(t)/dt^2]$ into

$$\tau^2 \frac{d^2y(t)}{dt^2} + 2\zeta\tau \frac{dy(t)}{dt} + y(t) = Kx(t)$$

we get

$$\tau^2(-A_1\omega^2 \cos \omega t - A_0\omega^2 \sin \omega t) + 2\zeta\tau(-A_1\omega \sin \omega t + A_0\omega \cos \omega t) + A_1 \cos \omega t + A_0 \sin \omega t = Kx(t)$$

or

$$(-A_1\tau^2\omega^2 + 2\zeta\tau A_0\omega + A_1)\cos \omega t + (A_0 - A_0\tau^2\omega^2 - 2\zeta\tau A_1\omega)\sin \omega t = KB \sin \omega t$$

from matching terms

$$A_0 - A_0\tau^2\omega^2 - 2\zeta\tau A_1\omega = KB$$

$$-A_1\tau^2\omega^2 + 2\zeta\tau A_0\omega + A_1 = 0$$

$$A_0 = \frac{KB(1-\tau^2\omega^2)}{(\tau^2\omega^2-1)^2+4\tau^2\zeta^2\omega^2} \quad A_1 = \frac{-2KB\tau\zeta\omega}{(\tau^2\omega^2-1)^2+4\tau^2\zeta^2\omega^2}$$

$$y(t) = y_H + y_P = y_H + \frac{KB}{(\tau^2\omega^2-1)^2+4\tau^2\zeta^2\omega^2}\left(-2\tau\zeta\omega\cos\omega t + (1-\tau^2\omega^2)\sin\omega t\right)$$

$$y(t) = y_H + y_P = y_H + \frac{KB}{\sqrt{(\tau^2\omega^2-1)^2+4\tau^2\zeta^2\omega^2}}\sin(\omega t + \theta)$$

$$\theta = \tan^{-1}\left(\frac{2\tau\zeta\omega}{1-\omega^2\tau^2}\right)$$