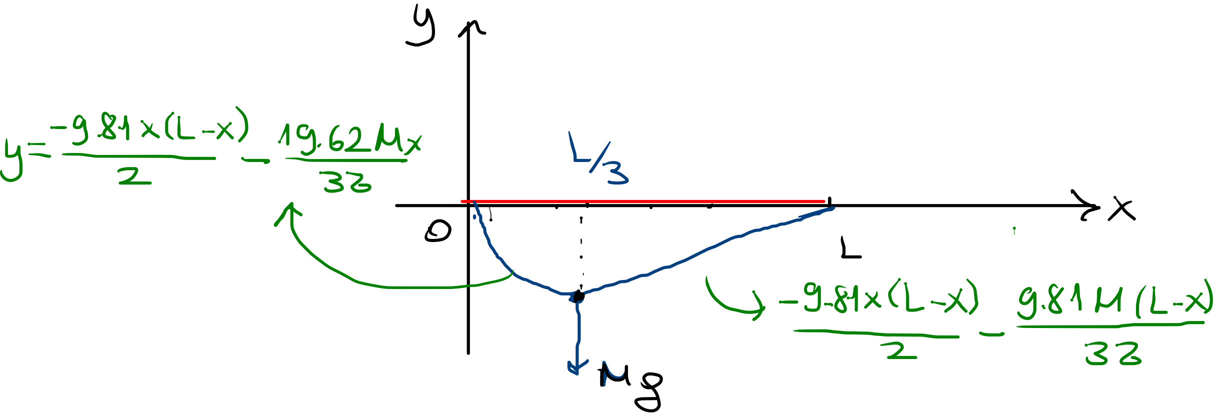
Green fonksiyonları kullanılarak sınır değer problemlerinin çözümleri

Homojen sınır şartlarıyla ilgili problemler

1. L uzunluğundaki bir teli, x-ekseni üzerinde uçlarından $x=0$, $x=L$ noktalarına bağlayarak gerelim. $x=\frac{L}{3}$ noktasından bu gergin tele bir M kütlesi astığımızda ağırlık hesaba katılmak suretiyle telin denge durumundan sapma miktarını bulalım.

$g = 9.81$



$$-\tau\frac{d^2y}{dx^2}=F(x) \qquad 0<x<L$$

$$y(0)=0=y(L)$$

$$g(x;T)=\begin{cases}\dfrac{x(L-T)}{L\tau} & 0\leq x\leq T\\[2mm] \dfrac{T(L-x)}{L\tau} & T\leq x\leq L\end{cases}$$

$$y(x)=\int_0^L g(x;T)F(T)\,dT+\sum_{j=1}^{n}F_j\cdot g(x;x_j)$$

$$g\left(x;\tfrac{L}{3}\right)=\begin{cases}\dfrac{x\cdot\left(L-\frac{L}{3}\right)}{L\tau} & 0\leq x\leq \frac{L}{3}\\[2mm] \dfrac{\frac{L}{3}(L-x)}{L\tau} & \frac{L}{3}\leq x\leq L\end{cases}$$

$$=\begin{cases}\dfrac{2x}{3\tau} & 0\leq x\leq \frac{L}{3}\\[2mm] \dfrac{L-x}{3\tau} & \frac{L}{3}\leq x\leq L\end{cases}$$

$$y(x)=\frac{-9.81\,x(L-x)}{2}-\frac{9.81\,M}{3\tau}\begin{cases}2x & 0\leq x\leq \frac{L}{3}\\ L-x & \frac{L}{3}\leq x\leq L\end{cases}$$

2. $\dfrac{d^2y}{dx^2}+4y=F(x) \qquad 0<x<3$

$y(0)=0=y'(3)$

sınır değer problemini

a) $F(x)=2x$

b) $F(x)=H(x-1)-H(x-2)$

için çözünüz.

$$\frac{d^2y}{dx^2}+4y=0 \Rightarrow r^2+4=0 \qquad r_{1,2}=\mp 2i$$

$$y=\begin{cases} A\cos 2x+B\sin 2x & 0<x<T \\ C\cos 2x+D\sin 2x & T<x<3 \end{cases}$$

$$y=\begin{cases} B_1\sin 2(x+\theta) & 0<x<T \\ C_1\cos 2(x+\theta) & T<x<3 \end{cases}$$

$$y'=\begin{cases} 2B_1\cos 2(x+\theta) & 0<x<T \\ -2C_1\sin 2(x+\theta) & T<x<3 \end{cases}$$

$y(0)=0 \Rightarrow B_1\sin 2\theta=0 \Rightarrow \sin 2\theta=0 \Rightarrow 2\theta=0 \Rightarrow \theta=0$

$y'(3)=0 \Rightarrow -2C_1\sin 2(3+\theta)=0 \Rightarrow \sin 2(3+\theta)=0 \Rightarrow 2(3+\theta)=0 \Rightarrow 2\theta=-6 \Rightarrow \theta=-3$

$$y = \begin{cases} B_1 \sin 2x & 0 \le x < T \\ c_1 \cos 2(x-3) & T < x \le 3 \end{cases}$$

$$\left.\begin{array}{l} B_1 = 1 \\ c_1 = 1 \end{array}\right\} \Rightarrow \quad \begin{array}{ll} u(x) = \sin 2x & v(x) = \cos 2(x-3) \\ u' = 2\cos 2x & v' = -2\sin 2(x-3) \end{array}$$

$$g(x;T) = \left[u(T)\,v(x)\,H(x-T) + u(x)\,v(T)\,H(T-x)\right] \cdot \frac{1}{J(u,v)}$$

$$J(u,v) = \underbrace{p(x)}_{=1} \cdot \left[uv' - vu'\right] = -\left[2\cos 2x \cdot \cos 2(x-3) + 2\sin 2(x-3) \cdot \sin 2x\right]$$

$$= -2\cos(2x - 2x + 6)$$

$$= -2\cos 6$$

$$g(x;T) = \frac{-1}{2\cos 6}\left[\sin 2T \cos 2(x-3)\, H(x-T) + \sin 2x \cdot \cos 2(T-3)\, H(T-x)\right] = \begin{cases} -\dfrac{\sin 2x \cos 2(T-3)}{2\cos 6} & 0 \le x < T \\ -\dfrac{\sin 2T \cos 2(x-3)}{2\cos 6} & T < x \le 3 \end{cases}$$

$$y(x) = \int_0^3 g(x;T) \cdot F(T)\, dT = \int_0^3 \left\{ \frac{-1}{2\cos 6}\left[\sin 2T \cos 2(x-3)\, H(x-T) + \sin 2x \cos 2(T-3)\, H(T-x)\right]\right\} \cdot 2T\, dT$$

$$y(x)=\int_0^3 g(x;T).F(T)\,dT=\int_0^3\left\{\frac{-1}{2\cos 6}\left[\sin 2T\cos 2(x-3)H(x-T)+\sin 2x\cos 2(T-3)H(T-x)\right]\right\}.2T\,dT$$

$$=-\frac{1}{2\cos 6}\int_0^x \sin 2T\cos 2(x-3).\,2T\,dT-\frac{1}{2\cos 6}\int_x^3 \sin 2x\cos 2(T-3)\,.2T\,dT$$

$$=\frac{-\cos 2(x-3)}{\cos 6}\int_0^x T.\sin 2T\,dT-\frac{\sin 2x}{\cos 6}\int_x^3 T.\cos 2(T-3)\,dT$$

$$=-\frac{\cos 2(x-3)}{\cos 6}\left[-\frac{T}{2}\cos 2T\Big|_0^x+\frac{1}{2}\int_0^x \cos 2T\,dT\right]-\frac{\sin 2x}{\cos 6}\left[\frac{T}{2}\sin 2(T-3)\Big|_x^3-\frac{1}{2}\int_x^3 \sin 2(T-3)\,dT\right]$$

$$=-\frac{\cos 2(x-3)}{\cos 6}\left[\frac{-x}{2}\cos 2x+0+\left(\frac{1}{2}\cdot\frac{1}{2}\sin 2T\Big|_0^x\right)\right]-\frac{\sin 2x}{\cos 6}\left[\frac{3}{2}.0-\frac{x}{2}\sin 2(x-3)+\frac{1}{4}\cos 2(T-3)\Big|_x^3\right]$$

$$=-\frac{\cos 2(x-3)}{\cos 6}\left[-\frac{x\cos 2x}{2}+\frac{\sin 2x}{4}\right]-\frac{\sin 2x}{\cos 6}\left[-\frac{x\sin 2(x-3)}{2}+\frac{1}{4}-\frac{1}{4}\cos 2(x-3)\right]$$

$$=\frac{x\cos 2(x-3)\cos 2x}{2\cos 6}-\cancel{\frac{\cos 2(x-3)\sin 2x}{4\cos 6}}+\frac{x\sin 2x\sin 2(x-3)}{2\cos 6}-\frac{1}{4}\frac{\sin 2x}{\cos 6}+\cancel{\frac{\sin 2x\cos 2(x-3)}{4\cos 6}}$$

$$=\frac{x}{2\cos 6}\cos 2(\cancel{x}-3-\cancel{x})-\frac{\sin 2x}{4\cos 6}=\frac{x}{2}-\frac{\sin 2x}{4\cos 6}$$

1. için

$T=u \Rightarrow dT=du$

$\sin 2T\,dT=dv$

$-\frac{1}{2}\cos 2T=v$

2. için

$T=u \Rightarrow dT=du$

$\cos 2(T-3)\,dT=dv$

$\frac{1}{2}\sin 2(T-3)=v$

b) $F(x) = H(x-1) - H(x-2)$

$$H(T-1) = \begin{cases} 1 & T>1 \\ 0 & T<1 \end{cases}$$

$$H(T-2) = \begin{cases} 1 & T>2 \\ 0 & T<2 \end{cases}$$

$$g(x;T) = \begin{cases} -\dfrac{\sin 2x \cos 2(T-3)}{2\cos 6} & 0 \le x < T \\ -\dfrac{\sin 2T \cos 2(x-3)}{2\cos 6} & T < x \le 3 \end{cases}$$

$$y(x) = \int_0^3 g(x;T)\left[H(T-1) - H(T-2)\right] dT$$

$$= \int_0^3 g(x;T)\, H(T-1)\, dT - \int_0^3 g(x;T) H(T-2)\, dT$$

$$= \int_1^3 g(x;T)\, dT - \int_2^3 g(x;T)\, dT$$

$$= \int_1^2 g(x;T)\, dT$$

$0 < x \le 1$

$x \le 1$ ise

$$y(x) = \int_1^2 -\frac{\sin 2x \cos 2(T-3)}{2\cos 6} dT = -\frac{\sin 2x}{2\cos 6} \int_1^2 \cos 2(T-3) dT = -\frac{\sin 2x}{4\cos 6} \sin 2(T-3) \Big|_1^2$$

$$= \frac{\sin 2x}{4\cos 6} \sin 2 - \frac{\sin 2x}{4\cos 6} \sin 4$$

$1 \le x \le 2$

$$y(x) = \int_1^2 g(x;\tau)\, d\tau = \int_1^x g(x;\tau) d\tau + \int_x^2 g(x;\tau)\, d\tau$$

$$= \int_1^x -\frac{\sin 2\tau \cos 2(x-3)}{2\cos 6}\, d\tau + \int_x^2 -\frac{\sin 2x \cos 2(\tau-3)}{2\cos 6} d\tau$$

$$= -\frac{\cos 2(x-3)}{2\cos 6} \int_1^x \sin 2\tau\, d\tau - \frac{\sin 2x}{2\cos 6} \int_x^2 \cos 2(\tau-3)\, d\tau$$

$$= \frac{\cos 2(x-3)}{4\cos 6} \cos 2\tau \Big|_1^x - \frac{\sin 2x}{4\cos 6} \sin 2(\tau-3)\Big|_x^2$$

$$= \frac{\cos 2(x-3)}{4\cos 6} (\cos 2x - \cos 2) - \frac{\sin 2x}{4\cos 6} (-\sin 2 - \sin 2(x-3))$$

$$= \frac{\cos 2x \cos 2(x-3)}{4\cos 6} - \frac{\cos 2 \cos 2(x-3)}{4\cos 6} + \frac{\sin 2 \sin 2x}{4\cos 6} + \frac{\sin 2x \sin 2(x-3)}{4\cos 6}$$

$$= \frac{\cos(\cancel{2x} - \cancel{2x} + 6)}{4\cos 6} - \frac{\cos 2 \cos 2(x-3)}{4\cos 6} + \frac{\sin 2 \sin 2x}{4\cos 6}$$

$$= \frac{1}{4} + \frac{1}{4\cos 6} \left[\sin 2 \sin 2x - \cos 2 \cos 2(x-3) \right]$$

$2 \leq x \leq 3$

$$y(x) = \int_1^2 g(x;T)\, dT$$

$$= \int_1^2 g(x;T)\, dT = \int_1^2 -\frac{\sin 2T \cos 2(x-3)}{2\cos 6}\, dT$$

$$= \frac{-\cos 2(x-3)}{2\cos 6} \int_1^2 \sin 2T\, dT$$

$$= \frac{\cos 2(x-3)}{4\cos 6} \cdot \cos 2T \Big|_1^2 = \frac{\cos 2(x-3)}{4\cos 6} \left[\cos 4 - \cos 2\right]$$