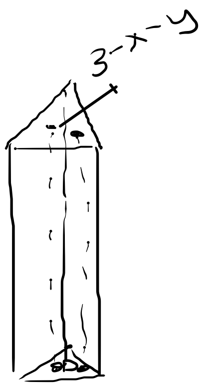
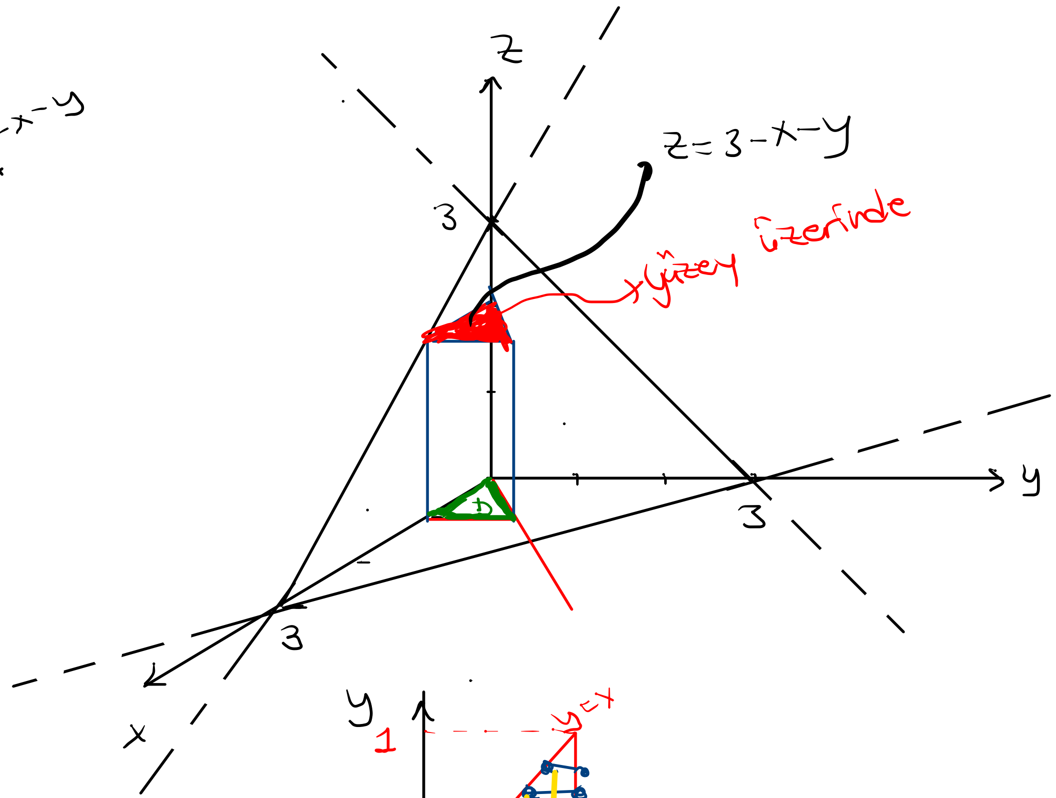
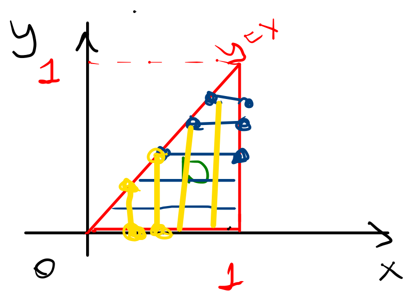
Ör/ Üçgen tabanı xy-düzleminde olan ve x-ekseni, $y=x$, $x=1$ doğruları ile sınırlanan, tepesi $z=3-x-y$ düzleminde bulunan prizmanın hacmini bulunuz.







$$z+x+y=3$$

$$\frac{x}{3}+\frac{y}{3}+\frac{z}{3}=1$$

$$V=\iint_D (3-x-y)\, dA$$

$$=\int_0^1\int_y^1 (3-x-y)\,dx\,dy \qquad \text{(y'ye göre düzgün)}$$

$$=\int_0^1\left[\int_0^x (3-x-y)\,dy\right]dx \qquad \text{(x'e göre düzgün)}$$

$$=\int_0^1 \left(\left(3y-xy-\frac{y^2}{2}\right)\Big|_0^x\right)dx$$

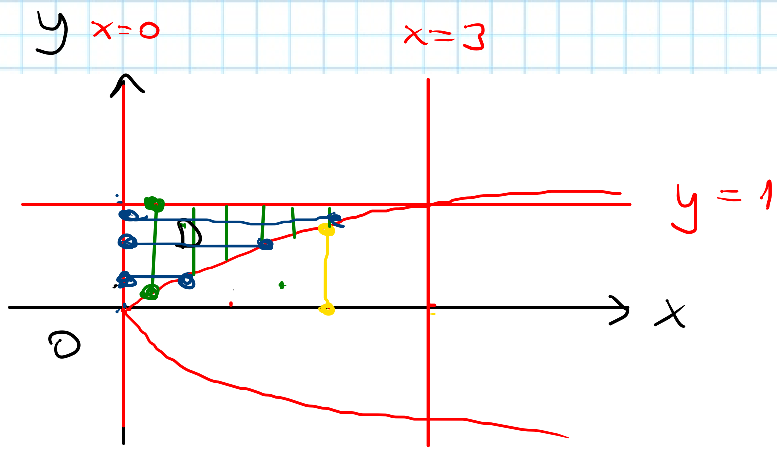
$$=\int_0^1 \left(3x-x^2-\frac{x^2}{2}\right)dx = 3\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^3}{6}\Big|_0^1$$

$$=\frac{3}{2}-\frac{1}{3}-\frac{1}{6}=\frac{9-2-1}{6}=\frac{6}{6}=1 \text{ br}^3$$

Ör/ $I=\int_0^3\left[\int_{\sqrt{\frac{x}{3}}}^1 e^{y^3}\,dy\right]dx$ integralini hesaplayınız.

$$D: \quad 0\le x\le 3, \qquad \sqrt{\tfrac{x}{3}}\le y\le 1$$

$$x=3y^2$$



$$I=\int_0^1\int_0^{3y^2} e^{y^3}\,dx\,dy$$

$$=\int_0^1\left[\left(x\cdot e^{y^3}\right)\Big|_0^{3y^2}\right]dy$$

$$=\int_0^1 3y^2e^{y^3}\,dy$$

$$y^3=t$$

$$3y^2\,dy=dt$$

$$y=0\Rightarrow t=0$$

$$y=1\Rightarrow t=1$$

$$=\int_0^1 e^t\,dt$$

$$=e^t\Big|_0^1$$

$$=e-1$$

İki katlı integrallerde değişken dönüşümü

1) Kartezyen koordinatlardan kartezyen koordinatlara değişken dönüşümü

$I = \iint_D f(x,y)\,\underbrace{dA}_{\substack{dx\,dy \\ \text{veya } dy\,dx}}$ integralinde $x = x(u,v)$, $y = y(u,v)$ değişken dönüşümü yapıldığında

D bölgesi bir D' bölgesine dönüşür ve integralin şekli $\underbrace{\left(\begin{array}{l}\text{Fonksiyonel Determinant} \\ \text{Jakobien} \downarrow\end{array}\right)}$

$$I = \iint_{D'} f\left[x(u,v), y(u,v)\right] |J| \cdot \underbrace{dA'}_{\substack{du\,dv \\ \text{veya } dv\,du}}$$

olur. Burada $J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \dfrac{D(x,y)}{D(u,v)}$

şeklindedir.

$$\frac{1}{J} = \frac{D(u,v)}{D(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

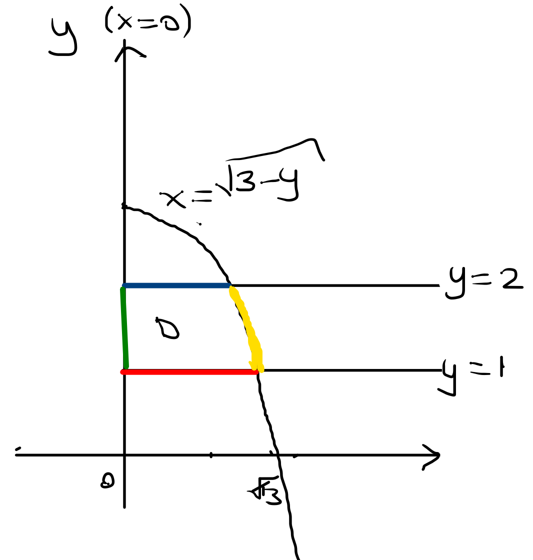
$$\underbrace{\frac{D(x,y)}{D(u,v)}}^{J} = \frac{1}{\dfrac{D(u,v)}{D(x,y)}} \rightarrow \frac{1}{J}$$

$$J \cdot \frac{1}{J} = 1$$

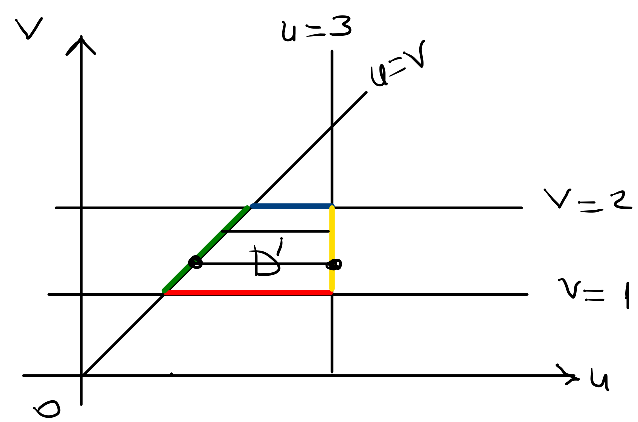
Ör/ $I=\displaystyle\int_1^2\int_0^{\sqrt{3-y}} \frac{x}{y}\,dx\,dy$ integralinde $x^2=u-v$, $y=v$ değişken dönüşümünü yaparak integralini yazınız.

$$\left.\begin{aligned} u&=x^2+y\\ v&=y\end{aligned}\right\} \qquad \begin{aligned} x&=\sqrt{u-v}\\ y&=v\end{aligned} \qquad \frac{D(x,y)}{D(u,v)}=\begin{vmatrix}\frac{1}{2\sqrt{u-v}} & \frac{-1}{2\sqrt{u-v}}\\ 0 & 1\end{vmatrix}=\frac{1}{2\sqrt{u-v}}$$

$$D:\quad \begin{aligned} &1\le y\le 2\\ &0\le x\le\sqrt{3-y}\end{aligned}$$



$$\left.\begin{aligned} &y=1\Rightarrow v=1\\ &y=2\Rightarrow v=2\\ &x=0\Rightarrow u-v=0\Rightarrow u=v\\ &x=\sqrt{3-y}\Rightarrow x^2=3-y\\ &\quad\Rightarrow u-v=3-v\\ &\quad\Rightarrow u=3\end{aligned}\right\}\Rightarrow D':\quad \begin{aligned}&1\le v\le 2\\ &v\le u\le 3\end{aligned}$$



$$J=\frac{D(x,y)}{D(u,v)}=\frac{1}{\dfrac{D(u,v)}{D(x,y)}}=\frac{1}{\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}}=\frac{1}{\begin{vmatrix}2x & 1\\ 0 & 1\end{vmatrix}}=\frac{1}{2x}$$

$$I=\iint_{D'}\frac{x}{y}\,|J|\,dA'$$

$$=\iint_{D'}\frac{\cancel{x}}{y}\cdot\frac{1}{2\cancel{x}}\cdot dA'=\int_1^2\int_v^3\frac{1}{2v}\,du\,dv$$

$$x = x(u,v)$$
$$y = y(u,v)$$

$$J = \frac{D(x,y)}{D(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

$$J = \frac{D(x,y)}{D(u,v)} = \frac{1}{\frac{D(u,v)}{D(x,y)}} = \frac{1}{\frac{1}{J}}$$

$$\frac{D(u,v)}{D(x,y)} = \frac{1}{J}$$

$$x = x(u,v,t)$$
$$y = y(u,v,t)$$
$$z = z(u,v,t)$$
$$\vdots$$

$$J = \frac{D(x,y,z)}{D(u,v,t)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial t} \end{vmatrix}$$

Ör/ D: $x.y=1$, $xy=4$, $\frac{y}{x}=1$, $\frac{y}{x}=9$ eğrileri ile sınırlı bölgenin 1. dörtte bir bölgede kalan kısmının alanını bulunuz.

$xy=u \quad \frac{y}{x}=v$

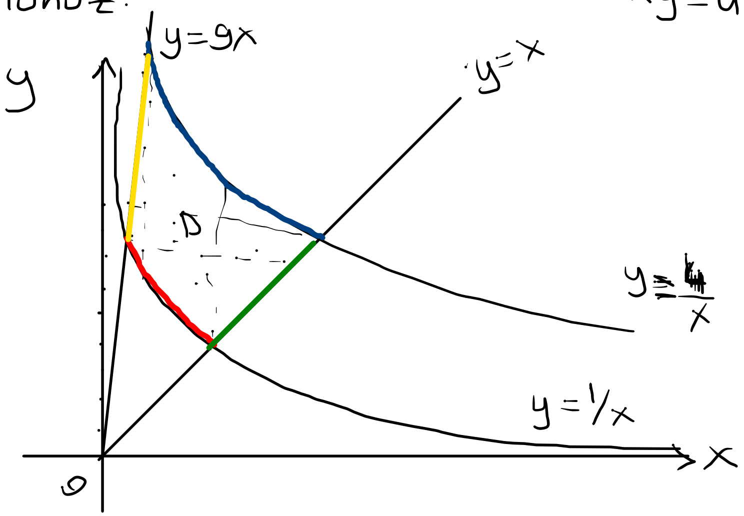
D:

$xy=1 \Rightarrow y=\frac{1}{x}$

$xy=4 \Rightarrow y=\frac{4}{x}$

$\frac{y}{x}=1 \Rightarrow y=x$

$\frac{y}{x}=9 \Rightarrow y=9x$



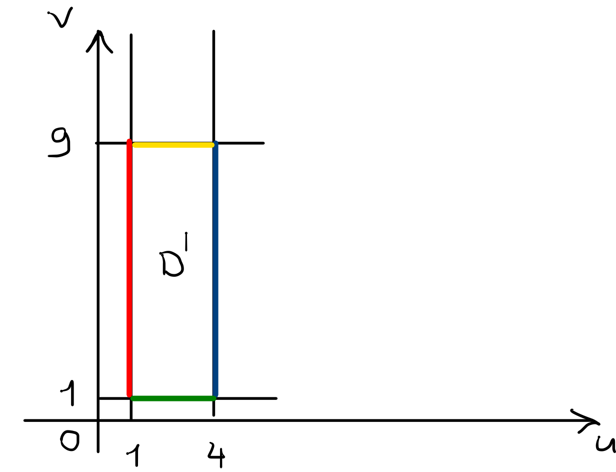
D':

$u=1$

$u=4$

$v=1$

$v=9$



$$A=\iint_D dA$$

$$J=\frac{D(x,y)}{D(u,v)}=\frac{1}{\frac{D(u,v)}{D(x,y)}}=\frac{1}{\begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x}\end{vmatrix}}=\frac{1}{\frac{y}{x}+\frac{y}{x}}$$

$$=\frac{1}{2\frac{y}{x}}=\frac{1}{2v}$$

$$A=\iint_{D'} |J|\cdot dA'$$

$$=\int_1^9\int_1^4 \frac{1}{2v}\cdot du\,dv=\int_1^9\left(\frac{u}{2v}\Big|_1^4\right)dv$$

$$=\frac{3}{2}\int_1^9\frac{dv}{v}$$

$$=\frac{3}{2}\ln|v|\Big|_1^9$$

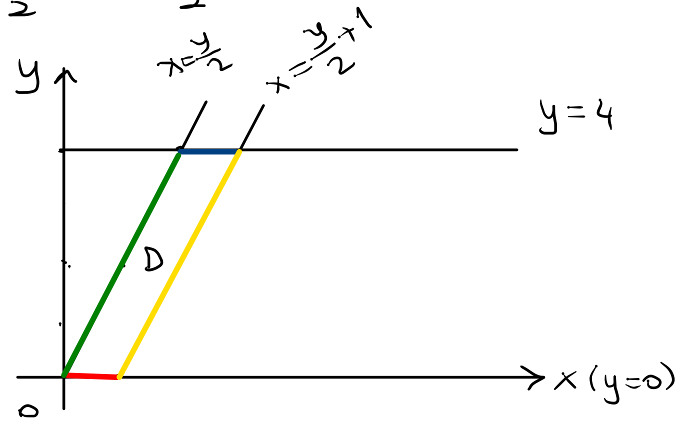
$$=\frac{3}{2}\left[\ln 9-\cancel{\ln 1}^{0}\right]=3\ln 3 \text{ br}^2$$

Ör/ $I=\int_0^4\int_{\frac{y}{2}}^{\frac{y}{2}+1}\left(\frac{2x-y}{2}\right)dx\,dy$ integralinde $y=2v$, $x=\frac{u+v}{2}$ değişken dönüşümünü yaparak integrali yazınız.

$x-\frac{y}{2}=\frac{u+v}{2}-v$

D: $0\le y\le 4$

$\frac{y}{2}\le x\le \frac{y}{2}+1$



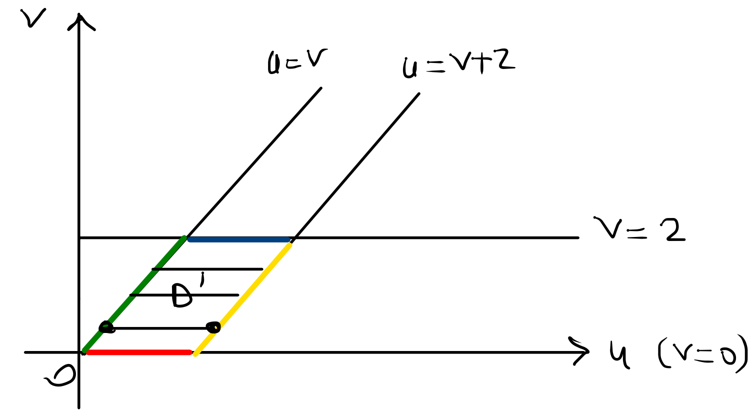
$y=0\Rightarrow 2v=0\Rightarrow v=0$

$y=4\Rightarrow 2v=4\Rightarrow v=2$

$x=\frac{y}{2}\Rightarrow \frac{u+v}{2}=\frac{2v}{2}\Rightarrow u=v$

$x=\frac{y}{2}+1\Rightarrow \frac{u+v}{2}=\frac{2v}{2}+1$

$\Rightarrow \frac{u+v}{2}=\frac{2v+2}{2}\Rightarrow u=v+2$



$$J=\frac{D(x,y)}{D(u,v)}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ 0 & 2\end{vmatrix}=1$$

$$I=\iint_{D'}\left(\frac{u-v}{2}\right)1\,dA'=\frac{1}{2}\int_0^2\int_v^{v+2}(u-v)\,du\,dv=\frac{1}{2}\int_0^2\left[\frac{u^2}{2}-uv\Big|_v^{v+2}\right]dv$$

$$=\frac{1}{2}\int_0^2\left\{\left[\frac{(v+2)^2}{2}-v(v+2)\right]-\left[\frac{v^2}{2}-v^2\right]\right\}dv=\frac{1}{2}\int_0^2\left[\cancel{\frac{v^2}{2}}+\cancel{2v}+2-\cancel{v^2}-\cancel{2v}-\cancel{\frac{v^2}{2}}+\cancel{v^2}\right]dv$$

$$=\frac{1}{\cancel{2}}\cdot\cancel{2}\int_0^2 dv=v\Big|_0^2=2$$

Öd/ $\displaystyle\int_{1}^{2}\int_{1/y}^{y}\sqrt{\frac{y}{x}}\cdot e^{\sqrt{x\cdot y}}\,dx\,dy$ integralini uygun bir değişken dönüşümü yaparak hesaplayınız.