Ör/ $y'' - (x-2)y' + 2y = 0$ dif. denkleminin $x=2$ noktası civarında çözümünü bulunuz.

$P(x) = 1 \neq 0$ Her nokta adi nokta.

$y = \sum_{n=0}^{\infty} a_n (x-2)^n$ şeklinde seri çözüm aranır.

$$y' = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} = \sum_{n=0}^{\infty} n a_n (x-2)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} = \sum_{n=1}^{\infty} n(n-1) a_n (x-2)^{n-2} = \sum_{n=0}^{\infty} n(n-1) a_n (x-2)^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} - (x-2) \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} + 2 \sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} - \sum_{n=1}^{\infty} n a_n (x-2)^n + 2 \sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} - \sum_{n=3}^{\infty} (n-2) a_{n-2} (x-2)^{n-2} + 2 \sum_{n=2}^{\infty} a_{n-2} (x-2)^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2} - \sum_{n=2}^{\infty} (n-2) a_{n-2} (x-2)^{n-2} + 2 \sum_{n=2}^{\infty} a_{n-2} (x-2)^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} - \sum_{n=2}^{\infty}(n-2)a_{n-2}(x-2)^{n-2} + 2\sum_{n=2}^{\infty} a_{n-2}(x-2)^{n-2} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty}\left[n(n-1)a_n - (n-2)a_{n-2} + 2a_{n-2}\right]\cdot(x-2)^{n-2} = 0$$

$$\Rightarrow n(n-1)a_n - (n-2)a_{n-2} + 2a_{n-2} = 0$$

$$\Rightarrow n(n-1)a_n - (n-4)a_{n-2} = 0 \quad \Rightarrow \quad a_n = \frac{n-4}{n(n-1)}a_{n-2} \quad (n \geq 2) \text{ için}$$

$$a_0 \neq 0, \quad a_1 \neq 0$$

$$a_2 = \frac{2-4}{2\cdot 1}\cdot a_0 = -a_0$$

$$a_3 = \frac{3-4}{3.2}\cdot a_1 = \frac{-1}{6}a_1$$

$$a_4 = 0$$

$$a_5 = \frac{5-4}{5.4}a_3 = \frac{1}{20}a_3 = \frac{1}{20}\cdot\left(-\frac{1}{6}a_1\right) = \frac{-1}{120}a_1$$

$$a_6 = \frac{6-4}{6.5}a_4 = 0$$

$$a_7 = \frac{7-4}{7.6}a_5 = \frac{1}{14}a_5 = \frac{1}{14}\cdot\left(-\frac{1}{120}\right)a_1 = \frac{-1}{1680}a_1$$

$$\vdots$$

$$y=\sum_{n=0}^{\infty} a_n (x-2)^n$$

$$= a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + a_4(x-2)^4 + \cdots$$

$$= a_0 + a_1(x-2) + (-a_0)(x-2)^2 + \left(\frac{-1}{6}a_1\right)(x-2)^3 + 0.(x-2)^4 + \left(\frac{-1}{120}\right)a_1(x-2)^5 + 0.(x-2)^6 + \cdots$$

$$= a_0\left[1-(x-2)^2\right] + a_1\left[(x-2) - \frac{(x-2)^3}{6} - \frac{(x-2)^5}{120} + \cdots\right]$$

Ör $y''-xy=0$ dif. denkleminin $x=0$ noktası civarında seri çözümünü elde ediniz.

$P(x)=1\neq 0$ Her nokta adi noktadır.

$$y=\sum_{n=0}^{\infty} a_n x^n$$

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=3}^{\infty} a_{n-3} x^{n-2} = 0$$

$$2.(2-1)a_2 x^0 + \sum_{n=3}^{\infty}\left[n(n-1)a_n - a_{n-3}\right]x^{n-2} = 0$$

$$2\cdot(2-1)\,a_2 x^0 + \sum_{n=3}^{\infty}\left[n(n-1)a_n - a_{n-3}\right]x^{n-2} = 0 \qquad (a_1 \neq 0 \quad a_0 \neq 0)$$

$$\Rightarrow a_2 = 0 \qquad n(n-1)a_n - a_{n-3} = 0 \Rightarrow a_n = \frac{a_{n-3}}{n(n-1)} \qquad n \geq 3$$

$$a_3 = \frac{a_0}{3\cdot 2} = \frac{a_0}{6}$$

$$a_4 = \frac{a_1}{4\cdot 3} = \frac{a_1}{12}$$

$$a_5 = \frac{a_2}{5\cdot 4} = 0$$

$$a_6 = \frac{a_3}{6\cdot 5} = \frac{a_3}{30} = \frac{a_0}{180}$$

$$a_7 = \frac{a_4}{7\cdot 6} = \frac{a_4}{42} = \frac{a_1}{12\cdot 42}$$

$$\vdots$$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + a_1 x + 0\cdot x^2 + \frac{a_0}{6}\cdot x^3 + \frac{a_1}{12}\cdot x^4 + 0\cdot x^5 + \frac{a_0}{180}\cdot x^6 + \cdots = a_0\left[1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right] + a_1\left[x + \frac{x^4}{12} + \frac{x^7}{12\cdot 42} + \cdots\right]$$

LAPLACE DÖNÜŞÜMÜ

İkinci taraflı bir Lineer diferansiyel denklemin çözümü, denklemin sağ tarafında bulunan fonksiyonun sürekli olması koşuluyla şimdiye kadar verdiğimiz yöntemlerle bulunabilir. Sağ taraftaki fonksiyonun sürekliliği bozulduğunda ise Laplace dönüşümü yardımıyla diferansiyel denklemin çözümü bulunabilecektir.

Lineer diferansiyel denklemlerin çözümünde kullanılan yöntemlerden biri integral dönüşümdür.

Bir integral dönüşüm

$$F(s)=\int_{\alpha}^{\beta} k(s,t)\, f(t)\, dt$$

formundadır. İntegral dönüşüm ile verilen bir $f(t)$ fonksiyonu diğer bir $F(s)$ fonksiyonuna dönüştürülür.

$F(s)$'e $f(t)$'nin integral dönüşümü, $k(s,t)$'ye integral dönüşümün çekirdeği denir.

$k(s,t)$ çekirdeği değiştikçe integral dönüşüm farklı adlar alır.

Eğer $k(s,t)=e^{-st}$ $\alpha=0$ $\beta=\infty$ alınırsa bu integral dönüşüme Laplace dönüşümü denir.

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = F(s) \quad (t \geq 0) \qquad k(x;t)=\begin{cases} 0 & t<0 \\ e^{-st} & t\geq 0 \end{cases}$$

$\rightarrow$ $f(t)$'nin Laplace dönüşümü denir.

Bazı elemanter fonksiyonların Laplace Dönüşümleri

1) $f(t)=c$ (sabit)

$$\mathcal{L}\{c\} = \int_0^\infty e^{-st}.c\,dt = c.\int_0^\infty e^{-st}dt = c.\lim_{A\to\infty}\int_0^A e^{-st}dt = c.\lim_{A\to\infty}\left[-\frac{1}{s}e^{-st}\Big|_0^A\right]$$

$$= -\frac{c}{s}\lim_{A\to\infty}\left[e^{-At}-1\right]$$

$$= -\frac{c}{s}.(-1) = \frac{c}{s}$$

2) $f(t)=t$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}.t\,dt = \lim_{A\to\infty}\int_0^A t.e^{-st}dt = \lim_{A\to\infty}\left[-\frac{t}{s}e^{-st}\Big|_0^A - \int_0^A -\frac{1}{s}e^{-st}dt\right]$$

$t=u$ $\quad e^{-st}dt=dv$

$dt=du$ $\quad -\frac{1}{s}e^{-st}=v$

$$= \lim_{A\to\infty}\left[-\frac{A}{s}e^{-As} - 0 + \frac{1}{s}.\left(-\frac{1}{s}\right)\left(e^{-st}\Big|_0^A\right)\right]$$

$$= -\lim_{A\to\infty}\left[\underbrace{\frac{A}{s}e^{-As}}_{0} + \frac{1}{s^2}\left(e^{-As}-1\right)\right] = \frac{1}{s^2}$$

$$\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\, dt = \frac{1.2}{s^3}$$

$$\mathcal{L}\{t^3\} = \int_0^\infty t^3 e^{-st}\, dt = \frac{1.2.3}{s^4}$$

$$\vdots \qquad\qquad \vdots$$

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}\, dt = \frac{n!}{s^{n+1}}$$

3) $f(t) = e^{at} \qquad (s>a)$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}. e^{at}\, dt = \lim_{A\to\infty}\int_0^A e^{-(s-a)t}\, dt = \lim_{A\to\infty}\left(\frac{-1}{s-a} e^{-(s-a)t}\Big|_0^A\right) = \lim_{A\to\infty}\left\{\left(\frac{-1}{s-a}\right).\left[e^{-(s-a)A} - 1\right]\right\}$$

$$= \frac{1}{s-a}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

4) $f(t) = \sin at \qquad (s>a)$

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}. \sin at\, dt = \lim_{A\to\infty}\overbrace{\int_0^A e^{-st}. \sin at\, dt}^{I} = \lim_{A\to\infty}\left[-\frac{e^{-st}}{a}\cos at\Big|_0^A + \frac{s}{a}\int_0^A e^{-st}\cos at\, dt\right]$$

$e^{-st} = u$

$-se^{-st}dt = du$

$\sin at\, dt = dv$

$-\frac{1}{a}\cos at = v$

$e^{-st} = u$

$-se^{-st}dt = du$

$\cos at\, dt = dv$

$\frac{1}{a}\sin at = v$

$$\underbrace{\lim_{A\to\infty}\int_0^A e^{-st}\cdot \sin at\, dt}_{I} = \lim_{A\to\infty}\left[-\frac{e^{-st}}{a}\cos at \Big|_0^A + s\int_0^A e^{-st}\cos at\, dt\right]$$

$$= \lim_{A\to\infty}\left\{\left[-\frac{e^{-sA}}{a}\cos aA - \left(-\frac{1}{a}\right)\right] + s\cdot\left[\frac{e^{-st}}{a}\sin at\Big|_0^A - \frac{s}{a}\int_0^A e^{-st}\sin at\, dt\right]\right.$$

$$= \frac{1}{a} - \frac{s^2}{a^2}\underbrace{\lim_{A\to\infty}\int_0^A e^{-st}\sin at\, dt}_{I}$$

$$\Rightarrow I + \frac{s^2}{a^2}I = \frac{1}{a}$$

$$I\left(\frac{a^2+s^2}{a^2}\right) = \frac{1}{a} \Rightarrow I = \frac{a}{s^2+a^2} \Rightarrow \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

5) $f(t) = \cos at \Rightarrow \mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\, dt = \frac{s}{s^2+a^2} \Rightarrow \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$

6) $f(t) = \sinh at \Rightarrow \mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$

7) $f(t) = \cosh at \Rightarrow \mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$

Laplace Dönüşümünün Temel Özellikleri

1) Lineerlik Özelliği

$c_1, c_2, \ldots$ sabit olmak üzere

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t) + \ldots\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} + \ldots$$

Ör/ $$\mathcal{L}\{3t^2 - 4\sin 3t - 7e^{2t}\} = 3\mathcal{L}\{t^2\} - 4\mathcal{L}\{\sin 3t\} - 7\mathcal{L}\{e^{-2t}\}$$

$$= 3 \cdot \frac{2!}{s^3} - 4 \cdot \frac{3}{s^2+9} - 7\frac{1}{s+2} = \frac{6}{s^3} - \frac{12}{s^2+9} - \frac{7}{s+2}$$

2) Kaydırma özelliği

$$\mathcal{L}\{f(t)\} = F(s) \Rightarrow \mathcal{L}\{e^{at} \cdot f(t)\} = F(s-a)$$

Ör/ $$\mathcal{L}\{e^{-t} \cdot \cos 2t\} = ?$$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4} = F(s) \Rightarrow \mathcal{L}\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4} = F(s+1)$$

3) Türevin Laplace Dönüşümü

$\mathcal{L}\{f(t)\} = F(s)$ olsun. $F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$

$$\mathcal{L}\{f'(t)\} = \int_0^{\infty} e^{-st} f'(t)\,dt = \lim_{A\to\infty} \int_0^{A} e^{-st} f'(t)\,dt = \lim_{A\to\infty}\left[e^{-st} f(t)\Big|_0^{A} + s\int_0^{A} e^{-st} f(t)\,dt \right]$$

$e^{-st} = u \qquad f'(t)\,dt = dv$

$-se^{-st}\,dt = du \qquad f(t) = v$

$$= \lim_{A\to\infty}\left(e^{-sA} f(A) - f(0)\right) + s.\underbrace{\lim_{A\to\infty}\int_0^{A} e^{-st} f(t)\,dt}_{F(s)}$$

$$= -f(0) + sF(s)$$

$$\Rightarrow \mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2 f(0) - sf'(0) - f''(0)$$

$\vdots$

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Ör/ $f(t)=\cos 3t \Rightarrow \mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$$= s.\frac{s}{s^2+9} - 1 = \frac{s^2-s^2-9}{s^2+9} = \frac{-9}{s^2+9}$$

$$=$$

$f'(t) = -3\sin 3t = g(t) \Rightarrow \mathcal{L}\{g(t)\} = -3.\frac{3}{s^2+9} = \frac{-9}{s^2+9}$

4°) t^n ile çarpma

$$\mathcal{L}\{f(t)\} = F(s) \Rightarrow \mathcal{L}\{t^n f(t)\} = (-1)^n.\frac{d^n}{ds^n} F(s)$$

Ör/ $\mathcal{L}\{t^2 e^{2t}\} = (-1)^2.\frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{2}{(s-2)^3}$

$$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2} = F(s)$$

$$\frac{d}{ds}\left(\frac{1}{s-2}\right) = \frac{-1}{(s-2)^2}$$

$$\frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{d}{ds}\left(\frac{-1}{(s-2)^2}\right) = \frac{2}{(s-2)^3}$$

KISA SINAV

1) $y'' - \frac{3}{x} y' = -x^4 (y')^2$

y' nin bulunmadığı

$y' = p,\ y'' = p' \Rightarrow p' - \frac{3}{x} p = -x^4 p^2$ Bernoulli

$\Rightarrow \frac{p'}{p^2} - \frac{3}{xp} = -x^4$

$\frac{1}{p} = z \Rightarrow -\frac{p'}{p^2} = z' \Rightarrow \frac{p'}{p^2} = -z' \Rightarrow -z' - \frac{3}{x} z = -x^4 \Rightarrow z' + \frac{3}{x} z = x^4$

8) $y''' + my'' + ny' + py = 3x^2 + 7e^{3x}$ dif. denkleminin belirsiz katsayılar yöntemi ile çözümünde kullanılacak olan ikinci taraflı çözüm önerisi $y_2 = \underbrace{ax^3 + bx^2 + cx}_{x(ax^2+bx+c)} + \underbrace{kxe^{3x}}_{x.\,ke^{3x}} \Rightarrow m+n+p = ?$

$r(r-3)$

$r_1 = 0 \qquad r_2 = 3$

K.D: $r^3 + mr^2 + nr + p = 0$

$0 + m.0 + n.0 + p = 0 \Rightarrow p = 0$

$27 + 9m + 3n = 0 \Rightarrow 3m + n = -9$

HATALI

7) $y''' - 3y'' = 18 e^{3x} \sin x$ dif. denkleminin parametrelerin değişimi yöntemi ile çözülmesiyle elde edilen genel çözüm $y = c_1(x) + c_2(x) . x + c_3(x) . e^{3x}$ ise $c_3(x) = ?$

$$c_1' + c_2' x + c_3' e^{3x} = 0 \qquad (\text{keyfi})$$

$$c_2' + 3 c_3' e^{3x} = 0 \qquad (\text{keyfi})$$

$$9 c_3' e^{3x} = 18 e^{3x} \sin x$$

$$c_3' = 2 \sin x \Rightarrow c_3 = 2 \int \sin x \, dx + k_3$$

$$= -2 \cos x + k_3$$