Ör/ $x\sin y\,dx + (x^2+1)\cos y\,dy = 0$ diferansiyel denkleminin $y(1)=\frac{\pi}{2}$ başlangıç şartına uygun çözümünü bulunuz.

$$\frac{x\cancel{\sin y}\,dx}{(x^2+1)\sin y} + \frac{\cancel{(x^2+1)}\cos y\,dy}{\cancel{(x^2+1)}\sin y} = \frac{0}{(x^2+1)\sin y} \qquad (\sin y \neq 0)$$

$$\frac{1}{2}\int\frac{2\cdot x}{x^2+1}\,dx + \int\frac{\cos y}{\sin y}\,dy = \int 0$$

$$\frac{1}{2}\ln(x^2+1) + \ln|\sin y| = \ln c$$

$$\boxed{\sqrt{x^2+1}\cdot\sin y = c} \quad \text{G.Ç.}$$

$$\left.\begin{matrix} x=1 \\ y=\frac{\pi}{2}\end{matrix}\right\} \Rightarrow \sqrt{1+1}\cdot\sin\frac{\pi}{2} = c$$

$$\sqrt{2} = c$$

$$\Rightarrow \sqrt{x^2+1}\cdot\sin y = \sqrt{2} \Rightarrow \boxed{(\sin y)^2 = \frac{2}{x^2+1}}$$

Özel çözüm

Ör/ $\left(1+2e^{\frac{x}{y}}\right)dx + 2e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$ dif. denkleminin genel çözümünü bulunuz.

$\frac{dx}{dy} = g\left(\frac{x}{y}\right)$ gibi düşünülürse böyle bir durumda $\frac{x}{y}=u \Rightarrow x=u.y \Rightarrow dx = u\,dy + y\,du$

$$\left(1+2e^{\frac{x}{y}}\right)dx = 2e^{\frac{x}{y}}\left(\frac{x}{y}-1\right)dy \Rightarrow \frac{dx}{dy} = \frac{2e^{\frac{x}{y}}\left(\frac{x}{y}-1\right)}{1+2e^{\frac{x}{y}}} = g\left(\frac{x}{y}\right)$$

$$\frac{x}{y}=u \Rightarrow x=u.y \Rightarrow dx = u\,dy + y\,du \Rightarrow (1+2e^u)(u\,dy + y\,du) + 2e^u(1-u)\,dy = 0$$

$$(1+2e^u)(u\,dy+y\,du)+2e^u(1-u)\,dy=0$$

$$\Rightarrow \left(u+\cancel{2ue^u}-\cancel{2ue^u}+2e^u\right)dy+y(1+2e^u)\,du=0$$

$$\frac{(u+2e^u)\,dy+y(1+2e^u)\,du=0}{y\cdot(u+2e^u)\qquad y\cdot(u+2e^u)\qquad y\cdot(u+2e^u)} \qquad (y\neq 0)$$

$$\int\frac{dy}{y}+\int\frac{1+2e^u}{u+2e^u}\,du=\int 0$$

$$\ln|y|+\ln(u+2e^u)=\ln c$$

$$y\cdot(u+2e^u)=c$$

$$y\cdot\left(\frac{x}{y}+2e^{\frac{x}{y}}\right)=c$$

$$\boxed{x+2ye^{\frac{x}{y}}=c} \quad \text{G.Ç.}$$

3°) Değişkenlerine Ayrılabilir Hale Getirilebilen Denklemler

1. $\frac{dy}{dx} = f(ax+by+c)$ formundaki denklemlerdir.

Burada $ax+by+c$ ifadesi x ve y'ye göre lineerdir. Denklem $ax+by+c=u$ dönüşümü yapılarak çözülür.

$$ax+by+c=u \Rightarrow a\,dx+b\,dy=du \longrightarrow dy=\frac{du-a\,dx}{b}$$

$$a+b\frac{dy}{dx}=\frac{du}{dx} \Rightarrow \frac{dy}{dx}=\left(\frac{du}{dx}-a\right)\cdot\frac{1}{b}$$

Ör/ $\frac{dy}{dx}=\tan^2(x+y)$ dif. denkleminin genel çözümünü bulunuz.

$$x+y=u \Rightarrow dx+dy=du \Rightarrow dy=du-dx$$

$$1+y'=u' \Rightarrow y'=u'-1$$

$$\Rightarrow u'-1=\tan^2 u \Rightarrow u'=1+\tan^2 u \Rightarrow \frac{du}{dx}=1+\tan^2 u \Rightarrow \int\frac{du}{1+\tan^2 u}=\int dx$$

$$\Rightarrow \int\frac{du}{\sec^2 u}=\int dx \Rightarrow \int\cos^2 u\,du=\int dx \Rightarrow \int\frac{1+\cos 2u}{2}du=\int dx$$

$$\frac{u}{2}+\frac{1}{4}\sin 2u=x+C$$

$$\frac{u}{2}+\frac{1}{4}\sin 2u = x + C \Rightarrow \boxed{\frac{x+y}{2}+\frac{1}{4}\sin 2(x+y) = x+C}$$ G.Ç.

Ör/ $\frac{dy}{dx} = (4x+y-2)^2$ dif. denkleminin genel çözümünü bulunuz.

$$4x+y-2=u \Rightarrow 4dx+dy=du \Rightarrow dy = du-4dx$$

$$4+y'=u' \Rightarrow y' = u'-4$$

$$u'-4 = u^2 \Rightarrow \frac{du}{dx} = u^2+4 \Rightarrow \int\frac{du}{u^2+4} = \int dx \Rightarrow \frac{1}{2}\arctan\frac{u}{2} = x+C$$

$$\boxed{\frac{1}{2}\arctan\left(\frac{4x+y-2}{2}\right) = x+C}$$

$$\arctan\left(\frac{4x+y-2}{2}\right) = 2x+2C$$

$$\frac{4x+y-2}{2} = \tan[2(x+c)] \Rightarrow \boxed{y = 2-4x+2\tan[2(x+C)]}$$

2. $a_1x+b_1y+c_1$ ve $a_2x+b_2y+c_2$ ifadeleri x ve y ye göre lineer ifadeler olmak üzere

$$(a_1x+b_1y+c_1)\,dx+(a_2x+b_2y+c_2)\,dy=0$$

diferansiyel denkleminde a_1x+b_1y ile a_2x+b_2y birbirleriyle lineer bağımlı ise denklem değişkenlerine ayrılabilir hale getirilebilen denklemdir. Başka deyişle;

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=0$ ise denklem değişkenlerine ayrılabilir hale getirilebilen denklemdir.

Böyle durumda $a_2x+b_2y=\lambda(a_1x+b_1y)$ şeklinde olacağından $a_1x+b_1y=u$ dönüşümü yapılarak denklem çözülür.

Ör/ $(x+y+1)\,dx+(2x+2y+1)\,dy=0$ dif. denkleminin genel çözümünü bulunuz.

$$(x+y+1)\,dx+[2(x+y)+1]\,dy=0 \qquad \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}=0$$

$x+y=u \Rightarrow dx+dy=du \Rightarrow dy=du-dx$

$$(u+1)\,dx+(2u+1)(du-dx)=0 \Rightarrow (u+\cancel{1}-2u-\cancel{1})\,dx+(2u+1)\,du=0$$

$$-u\,dx+(2u+1)\,du=0$$

$$\frac{-\cancel{u}\,dx}{-\cancel{u}} + \frac{(2u+1)\,du}{-u} = \frac{0}{-u} \quad (u \neq 0) \Rightarrow \int dx - \int \left(2 + \frac{1}{u}\right) du = \int 0$$

$$x - 2u - \ln|u| = c$$

$$x - 2(x+y) - \ln|x+y| = c \Rightarrow -x - 2y - \ln|x+y| = c$$

Ör/

$$\frac{dy}{dx} = \frac{\overset{a_1}{2}x + \overset{b_1}{4}y + 2}{\underset{a_2}{4}x + \underset{b_2}{8}y - 3}$$

$$\begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} = 0$$

$$\left.\begin{aligned} 2x + 4y &= u \\ 2 + 4y' &= u' \\ y' &= \frac{u'-2}{4} \end{aligned}\right\} \Rightarrow \frac{u'-2}{4} = \frac{u+2}{2u-3} \Rightarrow \frac{u'}{4} = \frac{u+2}{2u-3} + \frac{1}{2}$$

$$\Rightarrow u' = 4\left(\frac{u+2}{2u-3}\right) + 2$$

$$\Rightarrow \frac{du}{dx} = \frac{4u+8+4u-6}{2u-3} \Rightarrow \frac{du}{dx} = \frac{8u+2}{2u-3}$$

$$\begin{array}{r|l} \cancel{2u} - 3 & 8u+2 \\ -\cancel{2u} \mp \frac{1}{2} & \frac{1}{4} \\ \hline -\frac{7}{2} & \end{array}$$

$$\Rightarrow \int \frac{2u-3}{8u+2}\,du = \int dx \Rightarrow \int \left[\frac{1}{4} - \frac{7/2}{8u+2}\right] du = \int dx \Rightarrow \frac{u}{4} - \frac{7}{2} \cdot \frac{1}{8} \ln(8u+2) = x + c$$

$$\frac{u}{4}-\frac{7}{2}\cdot\frac{1}{8}\ln(8u+2)=x+c \Rightarrow \boxed{\frac{2x+4y}{4}-\frac{7}{16}\ln\left[8(2x+4y)+2\right]=x+C}$$ G.Ç.

4°) Homojen Hale Getirilebilen Diferansiyel Denklemler.

$a_1x+b_1y+c_1$ ve $a_2x+b_2y+c_2$ ifadeleri x ve y'ye göre Lineer ifadeler olmak üzere

$$(a_1x+b_1y+c_1)dx+(a_2x+b_2y+c_2)dy=0$$

diferansiyel denkleminde a_1x+b_1y, a_2x+b_2y ifadeleri birbirlerinden lineer bağımsız ise; Yani

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$ ise dif. denklem homojen hale getirilebilen dif. denklemdir.

Çözüm için h ve k sabitler olmak üzere X ve Y yeni değişkenler ise

$$\left.\begin{aligned} x&=X+h \Rightarrow dx=dX \\ y&=Y+k \Rightarrow dy=dY \end{aligned}\right\}$$ dönüşümleriyle denklem homojen dif. denklem haline gelir.

$$\left[a_1(X+h)+b_1(Y+k)+c_1\right]dX+\left[a_2(X+h)+b_2(Y+k)+c_2\right]dY=0$$

$$\left[a_1X+b_1Y+a_1h+b_1k+c_1\right]dX+\left[a_2X+b_2Y+a_2h+b_2k+c_2\right]dY=0$$

- $a_1h+b_1k+c_1$ ve $a_2h+b_2k+c_2$ sayıları aynı anda sıfır olduğunda denklem homojen denklemdir.

$$\left.\begin{array}{l} a_1 h + b_1 k + c_1 = 0 \\ a_2 h + b_2 k + c_2 = 0 \end{array}\right\} \Rightarrow \underbrace{(a_1 X + b_1 Y)}_{P(x,y)} dX + \underbrace{(a_2 X + b_2 Y)}_{Q(x,y)} dY = 0$$

Homojen dif. denk.

- h ve k bu denklemler-den birbirlerine bağlı olmaksızın hesaplanır

$P(\lambda X, \lambda Y) = \lambda (a_1 X + b_1 Y) = \lambda P(x,y)$

$Q(\lambda X, \lambda Y) = \lambda (a_2 X + b_2 Y) = \lambda Q(x,y)$

$$\frac{Y}{X} = u \Rightarrow Y = uX \Rightarrow dY = u\,dX + X\,du$$

$$(a_1 X + b_1 u X)\, dX + (a_2 X + b_2 Y)(u\, dX + X\, du) = 0$$

$F(X, Y, c) = 0$ çözüm bulunur.

$$\left.\begin{array}{l} x = X + h \\ y = Y + k \end{array}\right\} \Rightarrow \begin{array}{l} X = x - h \\ Y = y - k \end{array} \Rightarrow F[x-h, y-k, c] = 0$$

şeklinde istenilen genel çözüm elde edilir.

5) $(x+y-3)\,dx+(-x+y+1)\,dy=0$ dif. denkleminin genel çözümünü bulunuz.

$$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1-(-1)=1+1=2\neq 0$$ Denklem homojen hale getirilebilen denklemdir.

$$\left.\begin{array}{l} x=X+h \Rightarrow dx=dX \\ y=Y+k \Rightarrow dy=dY \end{array}\right\} \Rightarrow \left[(X+h)+(Y+k)-3\right]dX+\left[-(X+h)+(Y+k)+1\right]dY=0$$

$$\left[X+Y+(h+k-3)\right]dX+\left[-X+Y+(-h+k+1)\right]dY=0$$

$$\left.\begin{array}{l} h+k-3=0 \\ -h+k+1=0 \end{array}\right\} \Rightarrow \left[X+Y\right]dX+\left[-X+Y\right]dY$$ Homojen dif. denk.

$$\frac{Y}{X}=u \Rightarrow Y=uX \Rightarrow dY=u\,dX+X\,du$$

$$\Rightarrow \left[X+uX\right]dX+\left[-X+uX\right]\left[u\,dX+X\,du\right]=0$$

$$\left[X+\cancel{uX}-\cancel{uX}+u^2X\right]dX+\left[-X^2+uX^2\right]du=0$$

$$X(1+u^2)\,dX+X^2(u-1)\,du=0$$

$$\frac{\cancel{X}\,(\cancel{1+u^2})\,dX}{X^{\cancel{2}}\,(\cancel{1+u^2})} + \frac{\cancel{X^2}\,(u-1)\,du}{\cancel{X^2}\,(1+u^2)} = \frac{0}{X^2(1+u^2)} \qquad (X \neq 0)$$

$$\int \frac{dX}{X} + \int \frac{u-1}{1+u^2}\,du = \int 0 \Rightarrow \ln|X| + \frac{1}{2}\ln(u^2+1) - \arctan u = C$$

$$\ln|X| + \frac{1}{2}\ln\left(\frac{Y^2}{X^2}+1\right) - \arctan\left(\frac{Y}{X}\right) = C$$

$$\boxed{\ln|x-2| + \frac{1}{2}\ln\left(\frac{(y-1)^2}{(x-2)^2}+1\right) - \arctan\left(\frac{y-1}{x-2}\right) = C}$$

G.G.

$$\begin{array}{r} h+k-3=0 \\ -h+k+1=0 \\ \hline 2k-2=0 \end{array}$$

$$k=1 \Rightarrow h=2$$

$$\left.\begin{array}{l} \Rightarrow x = X+2 \Rightarrow X = x-2 \\ \quad y = Y+1 \Rightarrow Y = y-1 \end{array}\right\}$$

05) $\dfrac{dy}{dx} = \dfrac{3x - y + 1}{3y - x + 5}$ dif. denkleminin genel çözümünü bulunuz.

a_1, b_1, b_2, a_2

$$\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 9 - 1 = 8 \neq 0$$ Homojen hale getirilebilen denk.

$$\left.\begin{aligned} x &= X + h \Rightarrow dx = dX \\ y &= Y + k \Rightarrow dy = dY \end{aligned}\right\} \Rightarrow \frac{dY}{dX} = \frac{3[X+h] - [Y+k] + 1}{3[Y+k] - [X+h] + 5}$$

$$\Rightarrow \frac{dY}{dX} = \frac{3X - Y + (3h - k + 1)}{3Y - X + (3k - h + 5)}$$

$$\left.\begin{aligned} 3h - k + 1 &= 0 \\ 3k - h + 5 &= 0 \end{aligned}\right\} \Rightarrow \frac{dY}{dX} = \frac{3X - Y}{3Y - X} = \frac{\cancel{X}\left(3 - \frac{Y}{X}\right)}{\cancel{X}\left(3\frac{Y}{X} - 1\right)} = g\left(\frac{Y}{X}\right)$$ Homojen dif. denk.

$$\frac{Y}{X} = u \Rightarrow Y = uX \Rightarrow \frac{dY}{dX} = \frac{du}{dX} \cdot X + u \Rightarrow \frac{du}{dX} \cdot X + u = \frac{3 - u}{3u - 1}$$

$$\frac{du}{dX}\cdot X + u = \frac{3-u}{3u-1} \Rightarrow \frac{du}{dX} = \frac{3-u}{3u-1} - u \Rightarrow \frac{du}{dX} = \frac{3-\cancel{u}-3u^2\cancel{+u}}{3u-1}$$

$$\Rightarrow \frac{du}{dX} = \frac{3(1-u^2)}{3u-1}$$

$$\Rightarrow \int \frac{3u-1}{1-u^2}\,du = \int 3\,dX$$

$$\Rightarrow \int \left[\frac{1}{1-u} - \frac{2}{1+u}\right] du = 3\int dX$$

$$\Rightarrow -\ln|1-u| - 2\ln|1+u| = 3X + C$$

$$\Rightarrow -\ln\left|1-\frac{Y}{X}\right| - 2\ln\left|1+\frac{Y}{X}\right| = 3X + C$$

$$\frac{3u-1}{\underset{(1-u)(1+u)}{1-u^2}} = \frac{A}{1-u} + \frac{B}{1+u}$$

$$A = 1$$

$$B = -2$$

$$3h - k + 1 = 0 \quad /3$$
$$3k - h + 5 = 0$$

$$9h - \cancel{3k} + 3 = 0$$
$$\cancel{3k} - h + 5 = 0$$

$$8h + 8 = 0$$

$$\left.\begin{aligned} h &= -1 \\ k &= -2 \end{aligned}\right\} \cdot \quad \left.\begin{aligned} x &= X + h \Rightarrow X = x - h \Rightarrow X = x + 1 \\ y &= Y + k \Rightarrow Y = y - k \Rightarrow Y = y + 2 \end{aligned}\right\} \Rightarrow \boxed{-\ln\left|1-\frac{y+2}{x+1}\right| - 2\ln\left|1+\frac{y+2}{x+1}\right| = 3(x+1) + C}$$

G.G.