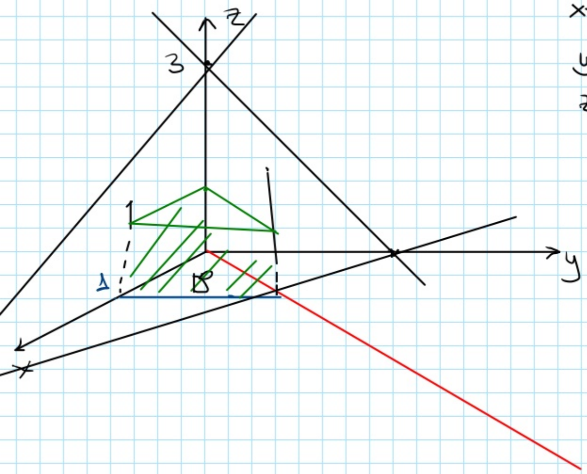
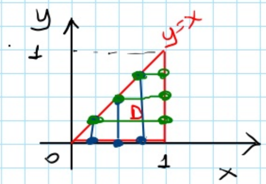
Ör/ Üçgen tabanı xoy düzleminde olan ve x-ekseni, $y=x$, $x=1$ doğruları ile sınırlanan, tepesi $z=3-x-y$ düzleminde bulunan prizmanın hacmini bulunuz.



$x=0 \Rightarrow z=3-y$

$y=0 \Rightarrow z=3-x$

$z=0 \Rightarrow x+y=3$



$$V=\iint_D (3-x-y)\,dA=\int_0^1\int_0^x (3-x-y)\,dy\,dx \quad (x'e\ göre)$$

$$=\int_0^1\int_y^1 (3-x-y)\,dx\,dy \quad (y'ye\ göre)$$

$$=\int_0^1 \left(3x-\frac{x^2}{2}-xy\Big|_y^1\right)dy$$

$$=\int_0^1 \left[\left(3-\frac{1}{2}-y\right)-\left(3y-\frac{y^2}{2}-y^2\right)\right]dy$$

$$=\int_0^1 \left[\frac{5}{2}-4y+\frac{3y^2}{2}\right]dy=\frac{5y}{2}-2y^2+\frac{y^3}{2}\Big|_0^1=1\,br^3$$

$$=1\ br^3$$

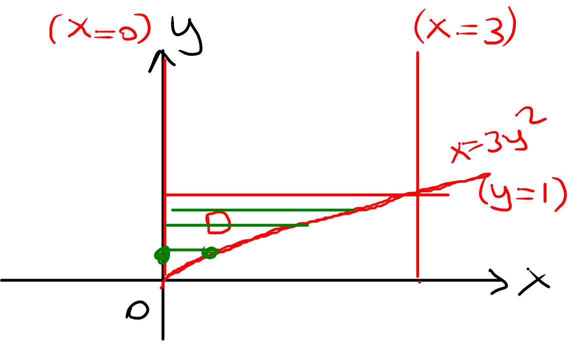
$$I = \int_0^3 \int_{\sqrt{\frac{x}{3}}}^{1} e^{y^3}\, dy\, dx$$ integralini hesaplayınız.

$0 \leq x \leq 3$

$1 \leq y \leq \sqrt{\frac{x}{3}}$

$y^2 = \frac{x}{3} \Rightarrow x = 3y^2$

$$\Rightarrow I = \int_0^1 \int_0^{3y^2} e^{y^3}\, dx\, dy = \int_0^1 \left(x \cdot e^{y^3} \Big|_0^{3y^2} \right) dy$$



$$= \int_0^1 3y^2 e^{y^3}\, dy = \int_0^1 e^t\, dt = e^t \Big|_0^1 = e - 1$$

$y^3 = t$

$3y^2 dy = dt$

$y = 0 \Rightarrow t = 0$

$y = 1 \Rightarrow t = 1$

İki katlı integrallerde değişken dönüşümü

1) Kartezyen koordinatlardan kartezyen koordinatlara değişken dönüşümü

$I = \iint_D f(x,y)\underbrace{dA}_{\substack{dx\,dy \\ \text{veya } dy\,dx}}$ integralinde $x = x(u,v)$, $y = y(u,v)$ değişken dönüşümü yapıldığında

D bölgesi bir D' bölgesine dönüşür ve integralin şekli

$$I = \iint_{D'} f\left[x(u,v), y(u,v)\right] \, |J| \, . \underbrace{dA'}_{\substack{du\,dv \\ \text{veya } dv\,du}}$$

olur. Burada

(Fonksiyonel Determinant)
$\underbrace{\text{Jakobien}}$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{D(x,y)}{D(u,v)}$$

şeklindedir.

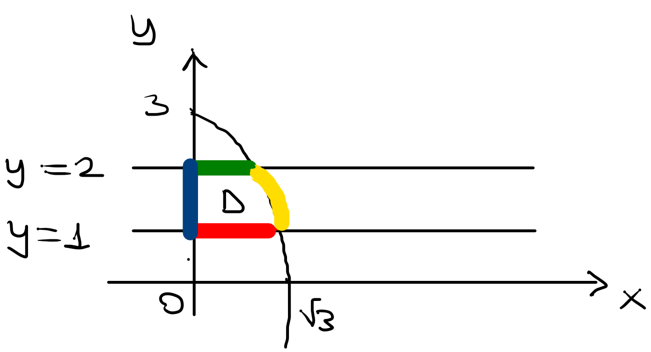
$$\frac{1}{J} = \frac{D(u,v)}{D(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$J \cdot \frac{1}{J} = 1$$

Ör/ $I=\int_1^2\int_0^{\sqrt{3-y}}\frac{x}{y}\,dx\,dy$ integralinde $x^2=u-v$, $y=v$ değişken dönüşümünü yaparak integralini yazınız.

$D:\quad 1\le y\le 2$

$\quad 0\le x\le\sqrt{3-y}$



$y=1 \Rightarrow v=1$

$y=2 \Rightarrow v=2$

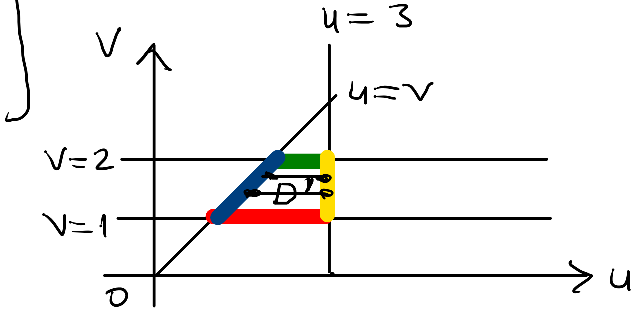
$x=0 \Rightarrow u-v=0 \Rightarrow u=v$

$x=\sqrt{3-y} \Rightarrow x^2=3-y \Rightarrow u-v=3-v$

$\Rightarrow u=3$

$D':\quad 1\le v\le 2$

$\quad v\le u\le 3$



$x^2=u-v \Rightarrow u=x^2+y$

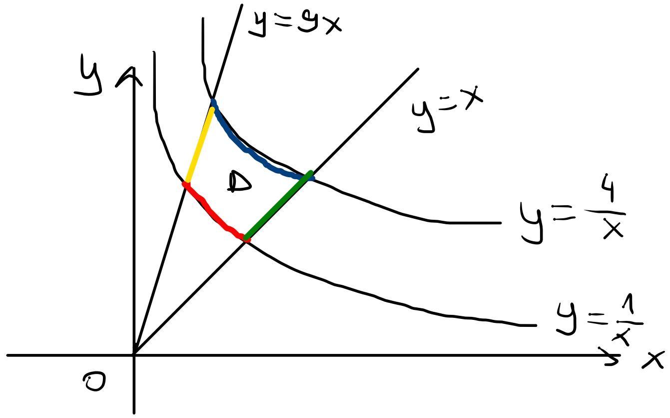
$y=v \qquad\quad v=y$

$$J=\frac{D(x,y)}{D(u,v)}=\frac{1}{\frac{D(u,v)}{D(x,y)}}=\frac{1}{\begin{vmatrix}2x & 1\\ 0 & 1\end{vmatrix}}=\frac{1}{2x}$$

$$\Rightarrow I=\iint_{D'}\frac{\cancel{x}}{y}\cdot\frac{1}{2\cancel{x}}\cdot dA'=\frac{1}{2}\int_1^2\int_v^3\frac{1}{v}\,du\,dv=\frac{1}{2}\int_1^2\left(\frac{u}{v}\Big|_v^3\right)dv=\frac{1}{2}\int_1^2\left(\frac{3}{v}-1\right)dv=\frac{1}{2}\left[3\ln|v|-v\Big|_1^2\right]$$

$$=\frac{1}{2}\left[(3\ln 2-2)-(\cancelto{0}{3\ln 1}-1)\right]=\frac{3}{2}\ln 2-\frac{1}{2}$$

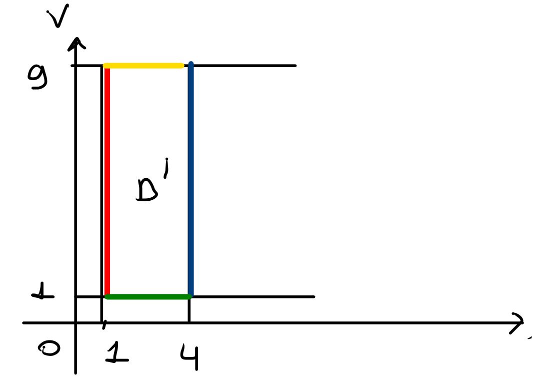
Örn: D: $x \cdot y = 1$, $xy = 4$, $\frac{y}{x} = 1$, $\frac{y}{x} = 9$ eğrileri ile sınırlı bölgenin 1. dörtte bir bölgede kalan kısmının alanını bulunuz.



$$D: \quad xy = 1 \Rightarrow y = \frac{1}{x}$$
$$xy = 4 \Rightarrow y = \frac{4}{x}$$
$$\frac{y}{x} = 1 \Rightarrow y = x$$
$$\frac{y}{x} = 9 \Rightarrow y = 9x$$

$$xy = u \qquad \frac{y}{x} = v$$

$$D': \quad u = 1, \quad u = 4, \quad v = 1, \quad v = 9$$



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix}} = \frac{1}{\frac{y}{x} + \frac{y}{x}} = \frac{1}{\frac{2y}{x}} = \frac{1}{2v}$$

$$A = \iint_D dA = \iint_{D'} |J| \cdot dA' = \int_1^9 \int_1^4 \frac{1}{2v}\, du\, dv = \int_1^9 \left(\frac{u}{2v} \Big|_1^4 \right) dv = \int_1^9 \left(\frac{4}{2v} - \frac{1}{2v} \right) dv = \frac{3}{2} \int_1^9 \frac{dv}{v}$$

$$= \frac{3}{2} \ln|v| \Big|_1^9 = \frac{3}{2}(\ln 9 - \overset{0}{\cancel{\ln 1}})$$

$$= \frac{3}{2} \ln 9 = \frac{3}{2} \ln 3^2 = 3 \ln 3 \ br^2$$

Ör

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \left(\frac{2x-y}{2}\right) dx\, dy$$ integralinde $y=2v$, $x=\dfrac{u+v}{2}$ değişken dönüşümünü yaparak integrali yazınız.

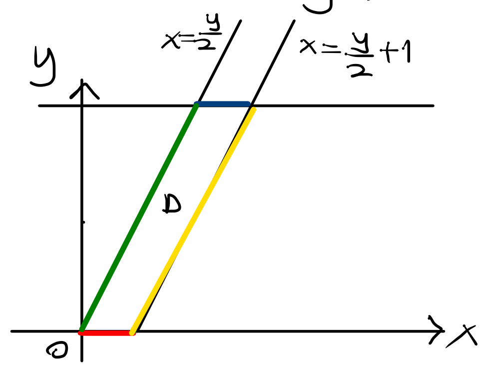
$2x = u+v \Rightarrow 2x-y = u+v-2v = u-v$

$D: 0 \le y \le 4$

$\frac{y}{2} \le x \le \frac{y}{2}+1$

$y = 2x$

$y = 2(x-1)$



$y=0 \Rightarrow 2v=0 \Rightarrow v=0$

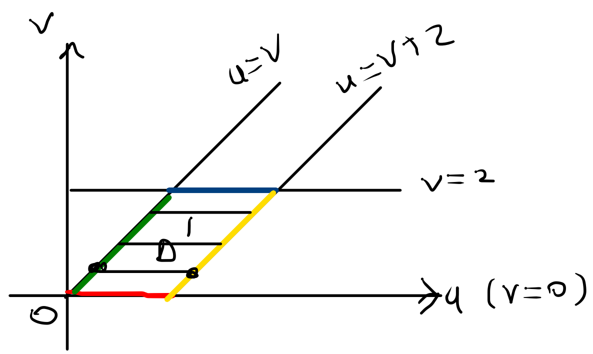
$y=4 \Rightarrow 2v=4 \Rightarrow v=2$

$x=\frac{y}{2} \Rightarrow y=2x \Rightarrow 2v = \cancel{2}\cdot\left(\frac{u+v}{\cancel{2}}\right) \Rightarrow u=v$

$x=\frac{y}{2}+1 \Rightarrow \frac{u+v}{2} = \frac{2v}{2}+1 \Rightarrow u+v = 2v+2 \Rightarrow u = v+2$

$D': 0 \le v \le 2$

$v \le u \le v+2$



$$J = \frac{D(x,y)}{D(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 0 & 2 \end{vmatrix} = 1$$

$$I = \int_0^2 \int_v^{v+2} \frac{u-v}{2} \cdot 1 \cdot du\,dv$$

Öd

$\int_{1}^{2}\int_{1/y}^{y} \sqrt{\frac{y}{x}} \cdot e^{\sqrt{x \cdot y}} \, dx\,dy$ integralini uygun bir değişken dönüşümü yaparak hesaplayınız.