Değişken Dönüşümü

$z=f(x,y)$ fonksiyonu ve bu fonksiyonun herhangi mertebeden türevlerini içeren

$$F\left(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial x\partial y},\frac{\partial^2 z}{\partial y^2},\dots\right)=0$$

şeklindeki bir denklemde $x=x(u,v)$, $y=y(u,v)$ değişken dönüşümü yaparak denklemi bu yeni değişkenler cinsinden

$$F\left(u,v,z,\frac{\partial z}{\partial u},\frac{\partial z}{\partial v},\frac{\partial^2 z}{\partial u^2},\frac{\partial^2 z}{\partial u\partial v},\frac{\partial^2 z}{\partial v^2},\dots\right)=0$$

şeklinde ifade etmek isteyebiliriz. Bu bileşik fonksiyonun türevlerinin alınması problemidir.

1. yol Verilen denklemdeki fonksiyon

$$\left.\begin{aligned} z&=f(x,y)\\ x&=x(u,v)\\ y&=y(u,v)\end{aligned}\right\}\Rightarrow z=f\big(x(u,v),y(u,v)\big)=F(u,v)$$

şeklinde bir bileşik fonksiyondur. Fonksiyonun u ve v değişkenlerine kısmi türevleri alınırsa;

$$\frac{\partial z}{\partial u} = \boxed{\frac{\partial z}{\partial x}} \cdot \frac{\partial x}{\partial u} + \boxed{\frac{\partial z}{\partial y}} \cdot \frac{\partial y}{\partial u} \qquad (\star)$$

$$\frac{\partial z}{\partial v} = \boxed{\frac{\partial z}{\partial x}} \cdot \frac{\partial x}{\partial v} + \boxed{\frac{\partial z}{\partial y}} \cdot \frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial x} = \frac{\begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}} = \frac{\dfrac{D(z,y)}{D(u,v)}}{\dfrac{D(x,y)}{D(u,v)}} \qquad \frac{\partial z}{\partial y} = \frac{\dfrac{D(x,z)}{D(u,v)}}{\dfrac{D(x,y)}{D(u,v)}}$$

İkinci mertebe türevler için de $(\star)$ sistemi üzerinden tekrar türevler alınır;

$\frac{\partial^2 z}{\partial u^2}, \frac{\partial^2 z}{\partial u \partial v}, \frac{\partial^2 z}{\partial v^2}$ türevleri hesaplanarak üç bilinmeyenli üç denklem sisteminden $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ türevleri hesaplanır.

2. yol

$$\left.\begin{aligned} z &= f(x,y) \\ x &= x(u,v) \\ y &= y(u,v) \end{aligned}\right\} \quad z = f\left[x(u,v),\, y(u,v)\right] = F(u,v)$$

bileşik fonksiyonu, ters dönüşüm denklemleri gözönüne alınarak yazılır. Yani

$$\left.\begin{aligned} z &= f(u,v) \\ u &= u(x,y) \\ v &= v(x,y) \end{aligned}\right\} \quad z = f\left[u(x,y),\, v(x,y)\right] = F(x,y)$$

olarak düşünülerek x'e ve y'ye göre türevler alınır:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \qquad (**)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

2. mertebe türevler için de benzer şekilde (**) üzerinden x ve y'ye göre türevler alınır.

Ör

$x^2 \cdot \frac{\partial^2 z}{\partial x^2} - y^2 \cdot \frac{\partial^2 z}{\partial y^2} = 0$ denkleminde $u = xy$ $\quad v = \frac{x}{y}$ değişken dönüşümünü yaparak denklemin yeni değişkenlere göre alacağı şekli bulunuz.

1.-yol

$$\left.\begin{aligned} z &= f(x,y) \\ x &= x(u,v) \\ y &= y(u,v) \end{aligned}\right\} \quad z = F(u,v)$$

$$u \cdot v = x^2 \Rightarrow x = \sqrt{uv}$$

$$\frac{u}{v} = y^2 \Rightarrow y = \sqrt{\frac{u}{v}}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{v}{2\sqrt{uv}} + \frac{\partial z}{\partial y} \cdot \frac{1/v}{2\sqrt{\frac{u}{v}}}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \Rightarrow \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{u}{2\sqrt{uv}} + \frac{\partial z}{\partial y} \cdot \frac{\left(-u/v^2\right)}{2\sqrt{\frac{u}{v}}}$$

İkinci mertebe türevler de hesaplanacak.

ZOR YOL

2-yol

$$\left.\begin{aligned} z &= f(u,v) \\ u &= u(x,y) \\ v &= v(x,y) \end{aligned}\right\} \quad z = F(x,y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \cdot y + \frac{\partial z}{\partial v} \cdot \frac{1}{y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} \cdot x + \frac{\partial z}{\partial v} \cdot \left(-\frac{x}{y^2}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = \left[\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v \partial u}\cdot\frac{\partial v}{\partial x}\right]\cdot y + \left[\frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}\right]\cdot\frac{1}{y}$$

$$= y^2 \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial v \partial u} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \left[\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v \partial u}\cdot\frac{\partial v}{\partial y}\right]\cdot x + \left[\frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial y}\right]\cdot\left(-\frac{x}{y^2}\right) + \frac{2x}{y^3}\cdot\frac{\partial z}{\partial v}$$

$$= x^2\frac{\partial^2 z}{\partial u^2} - 2\frac{x^2}{y^2}\frac{\partial^2 z}{\partial v \partial u} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2} + \frac{2x}{y^3}\frac{\partial z}{\partial v}$$

$$x^2\cdot\frac{\partial^2 z}{\partial x^2} - y^2\cdot\frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow x^2\cdot\left[y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial v \partial u} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}\right] - y^2\left[x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial v \partial u} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2} + \frac{2x}{y^3}\frac{\partial z}{\partial v}\right] = 0$$

$$\Rightarrow \cancel{x^2 y^2\frac{\partial^2 z}{\partial u^2}} + 2x^2\frac{\partial^2 z}{\partial v \partial u} + \cancel{\frac{x^2}{y^2}\frac{\partial^2 z}{\partial v^2}} - \cancel{x^2 y^2\frac{\partial^2 z}{\partial u^2}} + 2x^2\frac{\partial^2 z}{\partial v \partial u} - \cancel{\frac{x^2}{y^2}\frac{\partial^2 z}{\partial v^2}} - 2\frac{x}{y}\frac{\partial z}{\partial v} = 0$$

$$4x^2\frac{\partial^2 z}{\partial u \partial v} - 2\frac{x}{y}\frac{\partial z}{\partial v} = 0$$

$$\frac{\partial^2 z}{\partial u \partial v} - \frac{1}{2xy}\frac{\partial z}{\partial v} = 0 \Rightarrow \boxed{\frac{\partial^2 z}{\partial u \partial v} - \frac{1}{2u}\frac{\partial z}{\partial v} = 0}$$

Ör/ $u\cdot v\dfrac{\partial^2 z}{\partial u^2}=(1-v^2)\dfrac{\partial^2 z}{\partial u\partial v}$ denkleminde $u=y\cos x$ $v=\sin x$ değişken dönüşümünü yaparak yeni değişkenlere göre denklemin alacağı şekli belirleyiniz.

$$\left.\begin{aligned} z&=f(u,v)\\ u&=u(x,y)\\ v&=v(x,y)\end{aligned}\right\}\ z=f(x,y)$$

$$\left.\begin{aligned} z&=f(x,y)\\ x&=x(u,v)\\ y&=y(u,v)\end{aligned}\right\}\ z=F(u,v)$$

$u=y\cos x$

$v=\sin x$

u ve v'ye göre ~~türetelim~~.

2-yol

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

1-yol

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot(-y\sin x)+\frac{\partial z}{\partial v}\cdot\cos x$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\cos x+\frac{\partial z}{\partial v}\cdot 0$$

$$\Rightarrow \frac{\partial z}{\partial u}=\frac{1}{\cos x}\frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial x^2}=\left[\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial v}{\partial x}\right]\cdot(-y\sin x)+\frac{\partial z}{\partial u}(-y\cos x)$$

$$+\left[\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}\right]\cdot\cos x+\frac{\partial z}{\partial v}\cdot(-\sin x)$$

$$\frac{\partial^2 z}{\partial x^2}=y^2\sin^2 x\frac{\partial^2 z}{\partial u^2}-y\sin x\cos x\frac{\partial^2 z}{\partial u\partial v}-y\cos x\cdot\frac{\partial z}{\partial u}$$

$$-y\sin x\cos x\cdot\frac{\partial^2 z}{\partial u\partial v}+\cos^2 x\frac{\partial^2 z}{\partial v^2}-\sin x\frac{\partial z}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = y^2 \sin^2 x \, \frac{\partial^2 z}{\partial u^2} - 2y \sin x \cos x \, \frac{\partial^2 z}{\partial u \partial v} + \cos^2 x \, \frac{\partial^2 z}{\partial v^2} - y \cos x \, \frac{\partial z}{\partial u} - \sin x \, \frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \cos x \implies \frac{\partial^2 z}{\partial y^2} = \left[\frac{\partial^2 z}{\partial u^2} \cdot \frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v \partial u} \cdot \frac{\partial v}{\partial y} \right] \cdot \cos x$$

$$\frac{\partial^2 z}{\partial y^2} = \cos^2 x \, \frac{\partial^2 z}{\partial u^2} \implies \boxed{\frac{\partial^2 z}{\partial u^2} = \frac{1}{\cos^2 x} \frac{\partial^2 z}{\partial y^2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \left[\frac{\partial^2 z}{\partial u^2} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v \partial u} \cdot \frac{\partial v}{\partial x} \right] \cdot \cos x - \sin x \cdot \frac{\partial z}{\partial u}$$

$$= -y \sin x \cdot \cos x \cdot \frac{\partial^2 z}{\partial u^2} + \cos^2 x \, \frac{\partial^2 z}{\partial v \partial u} - \sin x \cdot \frac{\partial z}{\partial u}$$

$$\cos^2 x \cdot \frac{\partial^2 z}{\partial v \partial u} = \frac{\partial^2 z}{\partial x \partial y} + y \cdot \sin x \cdot \cos x \cdot \frac{1}{\cos^2 x} \frac{\partial^2 z}{\partial y^2} + \sin x \cdot \frac{1}{\cos x} \frac{\partial z}{\partial y}$$

$$\boxed{\frac{\partial^2 z}{\partial v \partial u} = \frac{1}{\cos^2 x} \frac{\partial^2 z}{\partial x \partial y} + y \cdot \frac{\sin x}{\cos^3 x} \frac{\partial^2 z}{\partial y^2} + \frac{\sin x}{\cos^3 x} \frac{\partial z}{\partial y}}$$

$$u \cdot v \frac{\partial^2 z}{\partial u^2} = (1-v^2)\frac{\partial^2 z}{\partial u \partial v}$$

$$y \cos x \cdot \sin x \cdot \left[\frac{1}{\cos^2 x}\frac{\partial^2 z}{\partial y^2}\right] = \underbrace{(1-\sin^2 x)}_{\cos^2 x} \cdot \left[\frac{1}{\cos^2 x}\frac{\partial^2 z}{\partial x \partial y} + y \cdot \frac{\sin x}{\cos^3 x}\frac{\partial^2 z}{\partial y^2} + \frac{\sin x}{\cos^3 x}\frac{\partial z}{\partial y}\right]$$

$$\cancel{y \frac{\sin x}{\cos x} \cdot \frac{\partial^2 z}{\partial y^2}} = \frac{\partial^2 z}{\partial x \partial y} + \cancel{y \frac{\sin x}{\cos x}\frac{\partial^2 z}{\partial y^2}} + \frac{\sin x}{\cos x}\frac{\partial z}{\partial y}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} + \tan x \frac{\partial z}{\partial y} = 0$$

$r = \sqrt{x^2+y^2}$, $\theta = \arctan \frac{y}{x}$

Öd/ $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ denkleminde $x = r\cos\theta$ $y = r\sin\theta$ değişken dönüşümünü yaparak denklemin yeni şeklini bulunuz.

$$\left.\begin{aligned} z &= f(r,\theta) \\ r &= r(x,y) \\ \theta &= \theta(x,y) \end{aligned}\right\} z = F(x,y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r} \cdot \frac{x}{\sqrt{x^2+y^2}} + \frac{\partial z}{\partial \theta} \cdot \frac{-y/x^2}{1+\frac{y^2}{x^2}}$$

$$= \frac{\partial z}{\partial r} \cdot \frac{x}{\sqrt{x^2+y^2}} - \frac{y}{x^2+y^2}\frac{\partial z}{\partial \theta}$$