5) Runge-Kutta Method of Order Four (RK4):

(The classical fourth-order Runge-Kutta method)

The Taylor series method has the difficulty of requiring the computation of the derivatives of $\underbrace{f(x,y)}_{y'}$. For example if we wish to use the Taylor method of order 4 on the IV

$$y' = f(x,y), \quad y(x_0) = y_0,$$

we need to find y'', y''' and $y^{(IV)}$ by differentiating $\underbrace{f(x,y)}_{y'}$

$\left[y_{n+1} = y_n + h f(x_n,y_n) + \frac{h^2}{2} f'(x_n,y_n) + \frac{h^3}{3!} f''(x_n,y_n) + \frac{h^4}{4!} f'''(x_n,y_n)\right]$

The Runge-Kutta methods avoid this difficulty, although they do imitate the Taylor series method by means of clever combinations of values of $f(x,y)$.

The Runge-Kutta methods are formulated as follows:

$$y_{n+1} = y_n + h\,(w_1 k_1 + w_2 k_2 + \dots + w_m k_m),$$

where

- $w_1 + w_2 + \dots + w_m = 1$
- $k_1, k_2, \dots, k_m$ are (recursive) evaluations of the slopes $f(x,y)$
- Need to determine $w_1, w_2, \dots, w_m$ (and other parameters)

$$w_1 k_1 + w_2 k_2 + \dots + w_m k_m \approx h f(x_n,y_n) + \frac{h^2}{2!} f'(x_n,y_n) + \dots + \frac{h^m}{m!} f^{(m-1)}(x_n,y_n)$$

The most commonly used the fourth-order Runge-Kutta method: the classical (standard) fourth-order Runge-Kutta method:

$$y_{n+1} = y_n + \frac{1}{6} h\,(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{cases} k_1 = f(x_n, y_n) \\ k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_1\right) \\ k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_2\right) \\ k_4 = f\left(\underbrace{x_n + h}_{x_{n+1}}, y_n + h k_3\right) \end{cases}$$

$\left(w_1 = \frac{1}{6}, w_2 = w_3 = \frac{1}{3}, w_4 = \frac{1}{6}\right)$

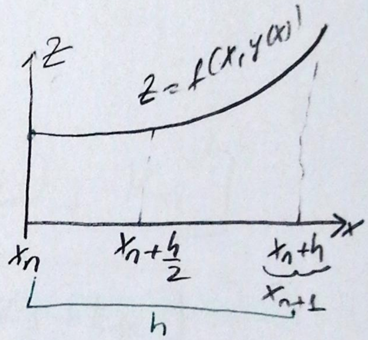
$$\boxed{\begin{array}{l} x_{n+1} = x_n + h \\ n = 0, 1, 2, \dots \end{array}}$$

Geometric interpretation of the RK4:

Let us consider the IVP $\frac{dy}{dx} = f(x, y(x))$, $y(x_0) = y_0$.
If the exact solution is denoted by $y(x)$, then using fundamental theorem of calculus we can write

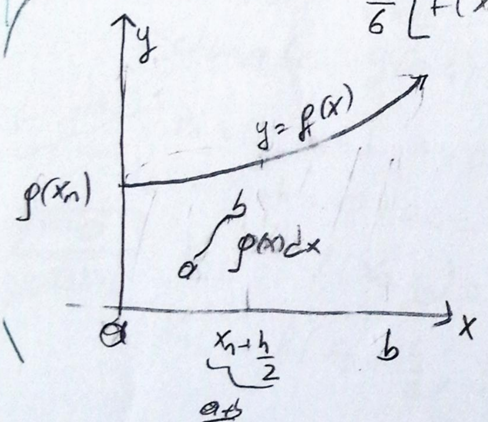
$$\int_{x_n}^{x_{n+1}} \frac{dy}{dx}\, dx = \int_{x_n}^{x_{n+1}} f(x, y(x))\, dx$$

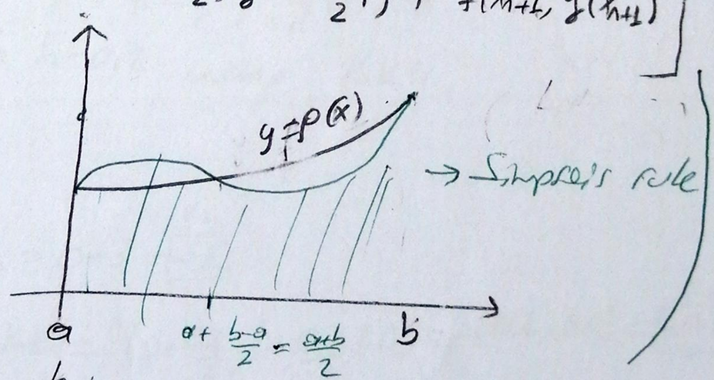
$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y(x))\, dx$$



and then using Simpson's rule with step size $\frac{h}{2}$ for the right-hand side, we obtain

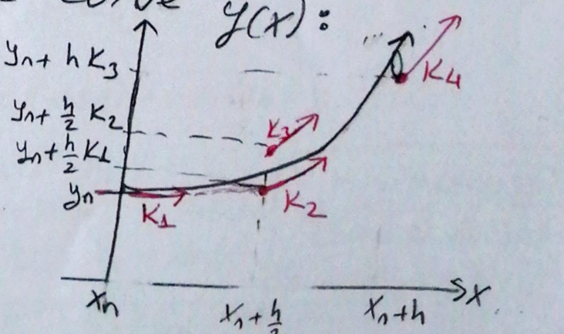
$$y(x_{n+1}) - y(x_n) \approx \frac{h}{6}\left[f(x_n, y(x_n)) + 4 f\left(x_n + \frac{h}{2}, y\left(x_n + \frac{h}{2}\right)\right) + f(x_{n+1}, y(x_{n+1}))\right]$$





$$\int_a^b f(x)\, dx \approx \frac{b-a}{6}\left[f(a) + 4 f\left(\frac{a+b}{2}\right) + f(b)\right]$$

On the other hand, the values K_1, K_2, K_3 and K_4 are approximations for slopes of the curve $y(x)$:



K_1: slope at the left of the interval

K_2: slope at the midpoint of the interval using y and K_1

K_3: again the slope at the midpoint but now using y and K_2

K_4: slope at the right of the interval using y and K_3

Hence, we can choose

$$f(x_n, y(x_n)) = K_1$$

$$f\left(x_n + \frac{h}{2}, y\left(x_n + \frac{h}{2}\right)\right) = \frac{K_2 + K_3}{2}$$

$$f(x_{n+1}, y(x_{n+1})) = K_4$$

Then the equation becomes

$$y(x_{n+1}) \simeq y_{n+1} = y_n + \frac{h}{6}\left[K_1 + \frac{4(K_2+K_3)}{2} + K_4\right]$$

$$\boxed{y_{n+1} = y_n + \frac{h}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]}$$

Ex: Consider the IVP: $y' = x - y$, $y(0) = 1$.
Estimate $y(0.4)$ with $h = 0.2$ using RK4.

solution

$x_0 = 0$, $y_0 = 1$, $h = 0.2$

$x_1 = x_0 + h = 0 + 0.2 = 0.2$

$\boxed{n=0}$ $y(0.2) \simeq y_1$

$$K_1 = f(x_0, y_0) = x_0 - y_0 = 0 - 1 = \boxed{-1}$$

$$K_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_1\right) = f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}(-1)\right) = f(0.1, 0.9) = \boxed{-0.8}$$

$$K_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} \cdot K_2\right) = f(0+0.1, 1 + 0.1 \cdot (-0.8)) = f(0.1, 0.92) = \boxed{-0.82}$$

$$K_4 = f(x_0 + h, y_0 + h \cdot K_3) = f(0+0.2, 1 + 0.2(-0.82)) = f(0.2, 0.836) = \boxed{-0.636}$$

$$y(0.2) \simeq y_1 = y_0 + \frac{h}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$= 1 + \frac{0.2}{6}\left(-1 + 2 \cdot (-0.8) + 2 \cdot (-0.82) + (-0.636)\right]$$

$$\boxed{= 0.837467}$$

$\boxed{n=1}$ $y(0.4) \simeq y_2$

$x_1 = 0.2$, $y_1 = 0.837467$, $h = 0.2$, $x_2 = 0.4$

$$K_1 = f(x_1, y_1) = x_1 - y_1 = 0.2 - 0.837467 = \boxed{-0.637467}$$

$$K_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}K_1\right) = f\left(0.2 + \frac{0.2}{2}, 0.837467 + \frac{0.2}{2}(-0.637467)\right) = \boxed{-0.4737}$$

$K_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}K_2\right) =$ $\boxed{K_3 = -0.490095}$

$K_4 = f(x_1 + h, y_1 + hK_3) =$ $\boxed{K_4 = -0.339448}$

$\boxed{y(0.4) \simeq y_2 = 0.74066}$

Table: RK4 method for $y' = x - y$, $y(0) = 1$ with $h = 0.2$

n	x_n	Exact solution	RK4 Approximation	Error (RK4) $\lvert e_n \rvert$	Error (Heun) $\lvert e_n \rvert$
0	$x_0 = 0$	1.00	1.0000	0.0	0.0
1	$x_1 = 0.2$	0.837	0.8375	0.000005	0.0025
2	$x_2 = 0.4$	0.741	0.7406	0.000008	0.0042
3	$x_3 = 0.6$	0.698	0.6976	0.000010	0.0051
4	$x_4 = 0.8$	0.699	0.6987	0.000011	0.0055
5	$x_5 = 1$	0.736	0.7358	0.000012	0.0057

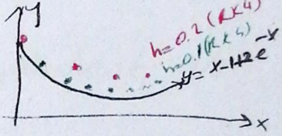
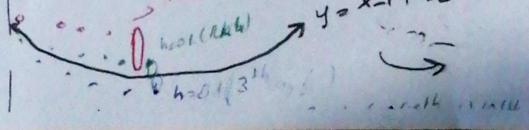


Table: RK4 method for $y' = x - y$, $y(0) = 1$ with $h = 0.1$

x_n	Exact solution	y_n (RK4)	Error (RK4) $\lvert e_n \rvert$	Error 4th Order Taylor	Error 3th order Taylor
0.0	1.0	1.0	0.0	0.0	0.0
0.1	0.9097	0.909675	1.64 E-7	1.64 E-7	0.0000082
0.2	0.8375	0.837462	2.97 E-7	2.97 E-7	0.000015
0.3	0.7816	0.781637	4.03 E-7	4.03 E-7	0.000020
0.4	0.7406	0.740641	4.86 E-7	4.86 E-7	0.000024
0.5	0.7131	0.713062	5.50 E-7	5.50 E-7	0.000027
0.6	0.6976	0.697624	5.97 E-7	5.97 E-7	0.000030
0.7	0.6932	0.693171	6.30 E-7	6.30 E-7	0.000031
0.8	0.6987	0.698659	6.51 E-7	6.51 E-7	0.000032
0.9	0.7131	0.713140	6.63 E-7	6.63 E-7	0.000033
1.0	0.7358	0.735760	6.66 E-7	6.66 E-7	0.000033

Note that doubling the value of n divides the error bound by 16 in the RK4 method!

Ex: $\underset{6.66 \times 10^{-7}}{e_{10}} \approx \dfrac{e_5 \rightarrow 1.2 \times 10^{-5}}{16}$



[illegible] error).

Let y_n be the approximate value of the exact solution $y = \phi(x)$ at $x = b$. The error can be estimated by

$$|\phi(x_n) - y_n| \leq C_1 h \quad \text{for Euler's method}$$

$$|\phi(x_n) - y_n| \leq C_2 h^2 \quad \text{for Heun's method}$$

$$|\phi(x_n) - y_n| \leq C_3 h^4 \quad \text{for Runge-Kutta method of order 4.}$$

The constants C_1, C_2 and C_3 are dependent on

- the length of the interval $[x_0, b]$
- the function $f(x,y)$ in the differential equation
- the initial condition $y(x_0) = y_0$