13 Mayıs 2015
Çarşamba

*An important identity;

If f is scalar form, then $\text{Curl}\,\text{grad} f = 0$ or $\nabla \times \nabla f = 0$.

$$\nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = (f_{zy} - f_{yz})\vec{i} + (f_{xz} - f_{zx})\vec{j} + (f_{yx} - f_{xy})\vec{k} = 0.$$

$$\left\{ \begin{array}{l} \text{If the second partial derivatives are continuous;} \\ \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} \quad (\text{Schwarz's Theorem}) \end{array} \right\}$$

Conservative Fields and Stoke's Theorem

Theorem: Curl $\vec{F} = 0$ Related to Closed-Loop Property

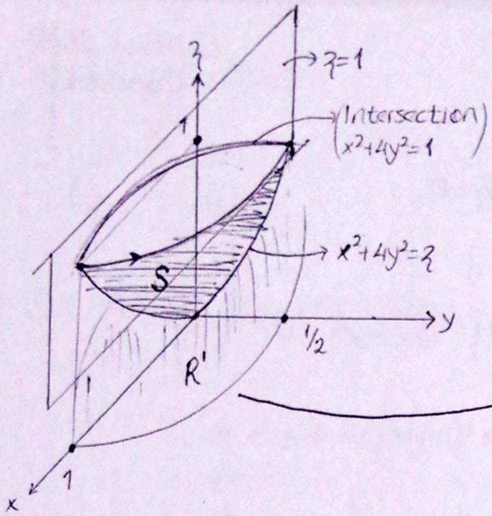
If $\nabla \times \vec{F} = 0$ at every point of simply connected, open region D in space, then any piecewise smooth closed path C in D;

$$\oint_C \vec{F} \cdot d\vec{r} = 0.$$

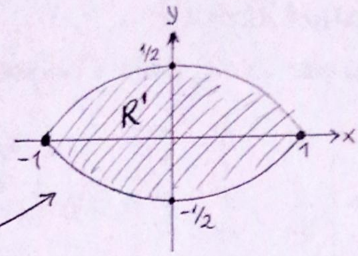
HW: Let surface S be the elliptical paraboloid $z = x^2 + 4y^2$ lying beneath the plane $z = 1$. We define the orientation of S by taking the inner normal vector $\vec{n}$ to the surface which is normal having a positive $\vec{k}$ component. Find the flux of the curl $\nabla \times \vec{F}$ across the S in the direction $\vec{n}$ for the vector field $\vec{F} = y\vec{i} - xz\vec{j} + xz^2\vec{k}$.

$$\underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{1} = \underbrace{\iint_S (\nabla \times \vec{F})\vec{n}\, d\sigma}_{2}$$

(Stoke's Theorem)



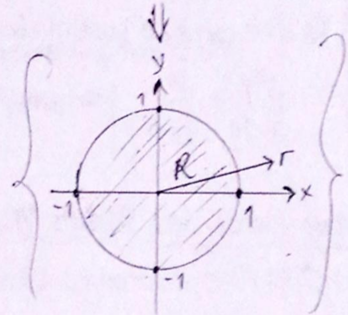
$\Rightarrow$



$$\frac{x^2}{1^2}+\frac{y^2}{(1/2)^2}=1$$

$\Downarrow$

$$x = ar\cos\theta \qquad x = r\cos\theta$$
$$y = br\sin\theta \quad \begin{pmatrix} a=1 \\ b=1/2 \end{pmatrix} \Rightarrow \quad y = \frac{r}{2}\sin\theta \quad , \quad 0 \le r \le 1$$
$$J = abr \qquad J = \frac{r}{2} \qquad 0 \le \theta \le 2\pi \Rightarrow$$
$$dA = abr\,dr\,d\theta \qquad dA = \frac{r}{2}\,dr\,d\theta$$



$$z = x^2+4y^2 \Rightarrow G(x,y,z) = z - x^2 - 4y^2 = 0 \Rightarrow \vec{n} = \frac{\nabla G}{|\nabla G|} = \frac{-2x\vec{i} - 8y\vec{j} + \vec{k}}{\sqrt{4x^2+64y^2+1}}$$

$$\vec{n}\cdot\vec{k} = \frac{1}{\sqrt{4x^2+64y^2+1}}$$

$$d\sigma = \frac{dA}{|\vec{n}\cdot\vec{p}|} = \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|} = \sqrt{4x^2+64y^2+1}\,dx\,dy$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -xz & xz^2 \end{vmatrix} = x\vec{i} - z^2\vec{j} - (z+1)\vec{k}$$

$$(\nabla\times\vec{F})\cdot\vec{n} = \frac{-2x^2+8yz^2-(z+1)}{\sqrt{4x^2+64y^2+1}}$$

$$② : \iint_S (\nabla\times\vec{F})\vec{n}\,d\sigma = \iint_{Rxy} \frac{-2x^2+8yz^2-(z+1)}{\sqrt{4x^2+64y^2+1}}\cdot\sqrt{4x^2+64y^2+1}\,dx\,dy = \iint_{Rxy} \left(-2x^2+8y(x^2+4y^2)^2-(x^2+4y^2+1)\right)dx\,dy$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \left(-2r^2\cos^2\theta + 8\frac{r}{2}\sin\theta\,(r^2)^2 - (r^2+1)\right)\frac{r}{2}\,dr\,d\theta = -\pi.$$

① : $C: \begin{cases} x^2+4y^2=z,\ z=1 \\ x=r\cos\theta \quad\quad x=\cos\theta \quad\quad dx=-\sin\theta d\theta \\ y=\frac{r}{2}\sin\theta \quad (r=1) \Rightarrow y=\frac{1}{2}\sin\theta \Rightarrow dy=\frac{1}{2}\cos\theta d\theta \ , \ 0\leq\theta\leq 2\pi \\ z=1 \quad\quad z=1 \quad\quad dz=0 \end{cases}$

$$\oint_C \vec{F}\,d\vec{r}=\oint_C P\,dx+Q\,dy+R\,dz=\oint_C y\,dx-xz\,dy+\cancel{xz^2dz}=\oint_C y\,dx-x\,dy$$

$$=\int_{\theta=0}^{2\pi}\frac{1}{2}\sin\theta(-\sin\theta d\theta)-(\cos\theta)\left(\frac{1}{2}\cos\theta d\theta\right)=-\pi$$

① = ② . Verified. ✓

The Divergence Theorem

Divergence theorem states that the net outward flux of a vector field across a closed surface in space can be calculated by integrated the divergence of the field over the region enclosed by the surface.

Divergence in Three Dimensions

The divergence of a vector field,

$$\vec{F}=P(x,y,z)\vec{i}+Q(x,y,z)\vec{j}+R(x,y,z)\vec{k}$$

is the scalar function;

$$\nabla.\vec{F}=\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)(P\vec{i}+Q\vec{j}+R\vec{k})=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}.$$

If $\vec{F}$ is the velocity field of a flowing gas, the value of div$\vec{F}$ at a point $(x.y.z)$, is the rate at which the gas is compressing or expanding at $(x,y.z)$. The divergence is the flux per unit volume or flux density at the point.
The divergence theorem says that under suitable conditions, the outward flux of a vector field across a closed surface equals the triple integral of the divergence of the field over the region enclosed by the surface.

15 Mayıs 2015
Cuma

Divergence Theorem

Let $\vec{F}$ be a vector field whose components have continuous first partial derivatives and let S be a piecewise smooth oriented closed surface. The flux of $\vec{F}$ across S in the direction of surface's outward unit normal field $\vec{n}$ equals the integral of $\nabla.\vec{F}$ over the region D enclosed by the surface.

$$\underbrace{\iint_S \vec{F}.\vec{n}.ds}_{\text{Outward Flux}} = \underbrace{\iiint_D (\nabla.\vec{F}).dV}_{\text{Divergence Integral}} \quad \text{(DIVERGENCE THEOREM)}$$

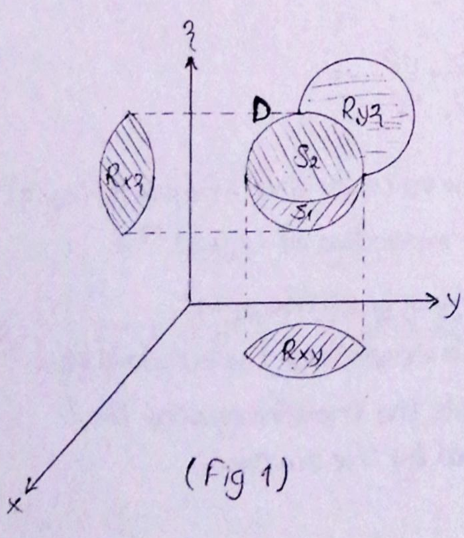
Proof of the Divergence Theorem

To prove the divergence theorem, we take the components of $\vec{F}$ to have continuous first partial derivatives. We first assume that D is a convex region with no holes or bubbles, such as a solid ball, cube or ellipsoid and that S is a piecewise smooth surface. In addition, we assume that any line perpendicular to the xy-plane at an interior point of the region Rxy that is the projection of D on the xy-plane intersects the surface S in exactly two points producing surfaces.

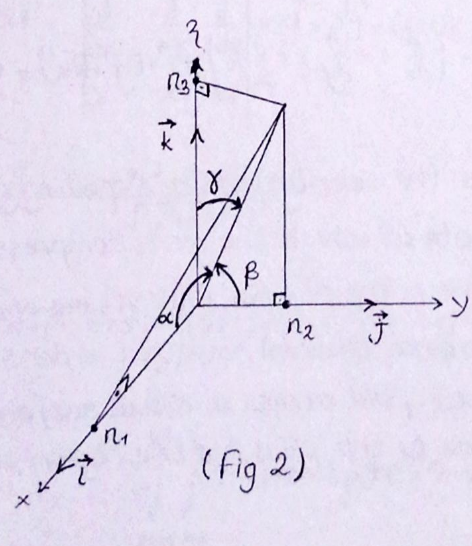
$S_1 : z = f_1(x,y)$, (x,y) in Rxy

$S_2 : z = f_2(x,y)$, (x,y) in Rxy

with $f_1 < f_2$. We make similar assumptions about the projection of D onto the other coordinate planes. (Fig.1)



(Fig 1)



(Fig 2)

The components of the unit normal vector

$$\vec{n} = n_1\vec{i} + n_2\vec{j} + n_3\vec{k}$$

are the cosines of the angles α, β and γ that normal vector $\vec{n}$ makes with $\vec{i}, \vec{j}, \vec{k}$. (Fig 2).

$$\left.\begin{aligned} &\vec{n}.\vec{i} = n_1 = \underbrace{|\vec{n}|}_{1}.\underbrace{|\vec{i}|}_{1}\cos\alpha = \cos\alpha \\ &\vec{n}.\vec{j} = n_2 = \cos\beta \\ &\vec{n}.\vec{k} = n_3 = \cos\gamma \end{aligned}\right\} \vec{n} = \cos\alpha\vec{i} + \cos\beta\vec{j} + \cos\gamma\vec{k}$$

$$\vec{F} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

$$\vec{F}.\vec{n} = P\cos\alpha + Q\cos\beta + R\cos\gamma$$

$$\Rightarrow \iint_S \vec{F}\vec{n}.ds = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds = \underbrace{\iint_S P\cos\alpha ds}_{I_3} + \underbrace{\iint_S Q\cos\beta ds}_{I_2} + \underbrace{\iint_S R\cos\gamma ds}_{I_1}$$

$$I_1 = \iint_S R\cos\gamma ds = \iint_{S_1} R\underbrace{\cos\gamma ds_1}_{dxdy} + \iint_{S_2} R\underbrace{\cos\gamma ds_2}_{dxdy} = -\iint_{R_{xy}} R(x,y,f_1(x,y))dxdy + \iint_{R_{xy}} R(x,y,f_2(x,y))dxdy$$

$$= \iint_{R_{xy}} [R(x,y,f_2(x,y)) - R(x,y,f_1(x,y))]dxdy$$

$$I_1 = \iiint_D \frac{\partial R}{\partial z}.dV = \iint_{\underbrace{R_{xy}}_{dA}} \left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial R}{\partial z}.dz \right) dA \quad \{dV = dzdA\}$$

$$\Rightarrow I_2 = \iint_S Q\cos\beta ds = \iint_{R_{xz}} \left(\int_{y=\varphi_1(x,z)}^{y=\varphi_2(x,z)} \frac{\partial Q}{\partial y}.dy \right) dA = \iiint_D \frac{\partial Q}{\partial y}.dV$$

$$I_3 = \iint_S P\cos\alpha ds = \iint_{R_{yz}} \left(\int_{x=G_1(y,z)}^{x=G_2(y,z)} \frac{\partial P}{\partial x}.dx \right) dA = \iiint_D \frac{\partial P}{\partial x}dV$$

$$\Rightarrow \iint_S \vec{F}.\vec{n}.ds = I_1+I_2+I_3 = \iiint_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dV = \iiint_D (\nabla.\vec{F})dV$$

*** Source :**

$div F(x_0,y_0) > 0$
Gas is expanding.

Well :

$div F(x_0,y_0) < 0$
Gas is compressing.

Examples

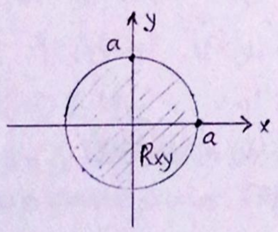
1) Evaluate both sides of divergence theorem formula for the expanding over field $\vec{F}=x\vec{i}+y\vec{j}+z\vec{k}$ over the sphere $x^2+y^2+z^2=a^2$.

2) Find the flux of $\vec{F}=xy\vec{i}+yz\vec{j}+xz\vec{k}$ outward through the surface of the cube cut from the first octant by the planes $x=1, y=1, z=1$.

3) Let $\vec{F}(x,y,z)=2x^2y\vec{i}-y^2\vec{j}+4xz^2\vec{k}$, S is the portion of the cylinder $y^2+z^2=9$ between the planes $x=0$ and $x=2$ in the first octant. Verify the divergence theorem.

Solutions

1)



$x = r\cos\theta$
$y = r\sin\theta$
$dA = r\,dr\,d\theta$
$0 \le r \le a$
$0 \le \theta \le 2\pi$

$$\phi(x,y,z) = x^2+y^2+z^2-a^2=0$$

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{2a} = \frac{x}{a}\vec{i}+\frac{y}{a}\vec{j}+\frac{z}{a}\vec{k} \Rightarrow |\vec{n}.\vec{k}| = \frac{z}{a} \Rightarrow \vec{F}.\vec{n} = \frac{x^2}{a}+\frac{y^2}{a}+\frac{z^2}{a}=a$$

$$ds = \frac{dA}{|\vec{n}.\vec{p}|} = \frac{dx\,dy}{|\vec{n}.\vec{k}|} = \frac{a}{z}dx\,dy = \frac{a}{\sqrt{a^2-x^2-y^2}}dx\,dy \;,\; z \ge 0$$

$$\underbrace{\iint_S \vec{F}.\vec{n}.ds}_{LHS\ 1} = \underbrace{\iiint_D (\nabla.\vec{F}).dV}_{RHS\ 2} \quad \text{(Divergence Theorem)}$$

① LHS: $\displaystyle\iint_S \vec{F}.\vec{n}.ds = 2\iint_S \vec{F}.\vec{n}.ds \quad (z>0)$

$$= 2\iint_{R_{xy}} a.\frac{a}{\sqrt{a^2-x^2-y^2}}\,dxdy = 2a^2\int_{\theta=0}^{2\pi}\left(\int_{r=0}^{a}\frac{r}{\sqrt{a^2-r^2}}\,dr\right)d\theta = 2a^2\int_{\theta=0}^{2\pi} a\,d\theta = 4a^3\pi.$$

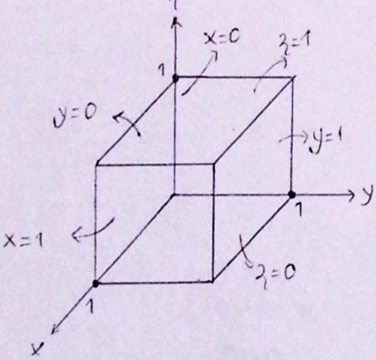
$$\begin{Bmatrix} a^2-r^2=u^2 \\ -2rdr=2udu \end{Bmatrix}$$

② RHS: $\nabla.\vec{F} = \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z} = 3$

$$\Rightarrow \iiint_D 3.dV = 3\underbrace{\iiint_D dV}_{\frac{4}{3}\pi a^3 \text{ (Volume of D)}} = 4a^3\pi.$$

① = ② Verified. ✓

2)



$$\underbrace{\iint_S \vec{F}\vec{n}.ds}_{①} = \underbrace{\iiint_D (\nabla.\vec{F})\,ds}_{②}$$

① $\displaystyle\iint_S \vec{F}\vec{n}\,ds = \iint_{S_1} \vec{F}\vec{n}_1 ds_1 + \dots + \iint_{S_6} \vec{F}\vec{n}_6 ds_6$

S_1: $x=0$, $\vec{n}_1 = -\vec{i}$, $\vec{F}.\vec{n}_1 = -xy = 0$, $ds_1 = \frac{dA}{|\vec{n}_1.\vec{p}|} = \frac{dA}{|\vec{n}_1.\vec{i}|} = dydz$

$\displaystyle\iint_{S_1} \vec{F}.\vec{n}_1.ds_1 = 0$ (Similarly S_2 and S_3 equal 0.)

$S_4: x=1,\ \vec{n}_4=\vec{i},\ \vec{F}.\vec{n}_4 = xy = y,\ ds_4 = \frac{dA}{|\vec{n}_4.\vec{i}|} = dydz$

$$\iint_{S_4} \vec{F}.\vec{n}_4.ds_4 = \iint_{R_{yz}} y.dydz = \int_{y=0}^{1}\left(\int_{z=0}^{1} ydz\right)dy = \frac{1}{2} \quad \text{(Similarly } S_5 \text{ and } S_6 \text{ equal } \tfrac{1}{2}\text{)}$$

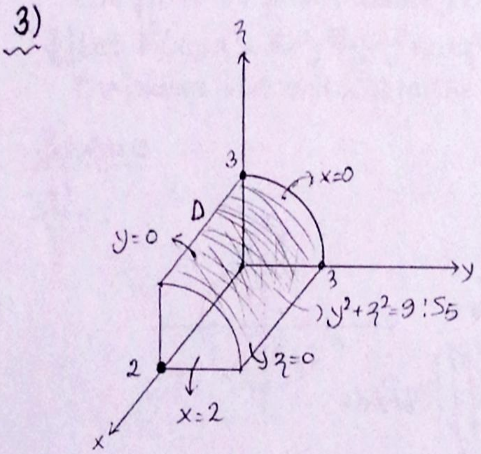
$$\Rightarrow \iint_S \vec{F}.\vec{n}.ds = 0+0+0+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}.$$

②: $(\nabla.\vec{F}) = y+z+x$

$$\Rightarrow \iiint_D (\nabla.\vec{F})dV = \iint_{R_{xy}}\left(\int_{z=0}^{1}(x+y+z)dz\right)dA = \frac{3}{2}.$$

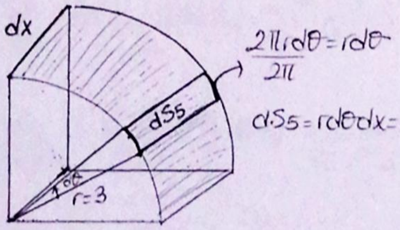
① = ② Verified. ✓

3)



$$\iiint_D (\nabla.\vec{F})dV = \iint_{R_{xy}}\left(\int_{z=0}^{\sqrt{9-y^2}}(\nabla.\vec{F})\,dz\right)dA = 180. ✓$$

*



$\frac{2\pi r d\theta}{2\pi} = rd\theta$

$dS_5 = rd\theta dx = 3dxd\theta$

$y^2+z^2=9$

$y = r\cos\theta$

$z = r\sin\theta$

$0 \le r \le 3$

$0 \le \theta \le \pi/2$

$$\Rightarrow \iint_{S_5} \vec{F}.\vec{n}_5.ds_5 = \iint_{S_5}\left(\frac{-y^3}{3}+\frac{4xz^3}{3}\right)dS_5 = \int_{x=0}^{2}\left(\int_{\theta=0}^{\pi/2}\left(\frac{-r^3\cos^3\theta}{3}+\frac{4xr^3\sin^3\theta}{3}\right)3\,d\theta\right)dx = 180. ✓$$

$r=3$