25 Şubat 2015
Çarşamba

Differentiation Under the Integral Sign

If the integrand is a function of one or more parameters in addition to the variable of integration, then the integral between the limits which may be constants or functions of the parameters is a function of these parameters.

Ex:

i) $$\int_0^{\pi/2} \sin\alpha x\,dx = -\frac{1}{\alpha}\cos\alpha x\Big|_0^{\pi/2} = \frac{1}{\alpha}-\frac{1}{\alpha}\cos\alpha\frac{\pi}{2} = F(\alpha)$$

ii) $$\int_1^2 (x+\alpha)^2 dx = \frac{(x+\alpha)^3}{3}\Big|_1^2 = \frac{(2+\alpha)^3-(1+\alpha)^3}{3} = G(\alpha)$$

Thus, in integral,

$$\int_a^b f(x,\alpha)dx = F(\alpha)$$

$$\int_a^b f(x,\alpha,\beta)dx = F(\alpha,\beta)$$

where a,b may be constants or functions of parameters.

Sometimes, $f(x,\alpha)$ is such that the evaluation of the integral is very complicated or impossible. However, the integral with integrand $\partial f/\partial\alpha$ may be easily evaluated.

Leibnitz's Rule for Differentiation Under the Integral Sign

Theorem: If $f(x,\alpha)$ and $\frac{\partial f(x,\alpha)}{\partial\alpha}$ are continuous functions of x and α for $a\leq x\leq b$, $c\leq\alpha\leq d$, a,b being independent of α, then,

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx = \int_a^b \frac{\partial f(x,\alpha)}{\partial\alpha}dx$$

Proof:

Let $F(\alpha) = \int_a^b f(x,\alpha)dx \quad (*)$

Let α changes $\alpha+\Delta\alpha$ ($[\alpha,\alpha+\Delta\alpha]$ in $[c,d]$)

$$F(\alpha+\Delta\alpha)=\int_a^b f(x,\alpha+\Delta\alpha)dx \quad (**)$$

$$(**)-(*): F(\alpha+\Delta\alpha)-F(\alpha)$$

$$=\int_a^b \big(f(x,\alpha+\Delta\alpha)-f(x,\alpha)\big)dx \quad (***)$$

By Lagrange's Mean Value Theorem (or the Mean Value Theorem):

$$f(x,\alpha+\Delta\alpha)-f(x,\alpha)=\Delta\alpha\,.\,\frac{\partial f}{\partial\alpha}(x,\alpha+\theta\Delta\alpha) \quad \text{where } 0<\theta<1$$

$$F(\alpha+\Delta\alpha)-F(\alpha)=\int_a^b \Delta\alpha\,.\,\frac{\partial f}{\partial\alpha}(x,\alpha+\theta\Delta\alpha)\,dx \quad , \; 0<\theta<1$$

$$\frac{F(\alpha+\Delta\alpha)-F(\alpha)}{\Delta\alpha}=\int_a^b \frac{\partial f}{\partial\alpha}(x,\alpha+\theta\Delta\alpha)\,dx$$

Taking limits as $\Delta\alpha\to 0$, we take,

$$\underbrace{\lim_{\Delta\alpha\to 0}\frac{F(\alpha+\Delta\alpha)-F(\alpha)}{\Delta\alpha}}_{F'(\alpha)=\frac{d(F(\alpha))}{d\alpha}}=\lim_{\Delta\alpha\to 0}\int_a^b \frac{\partial f}{\partial\alpha}(x,\alpha+\theta\Delta\alpha)\,dx$$

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx=\int_a^b \frac{\partial f}{\partial\alpha}(x,\alpha)\,dx$$

<u>Note 1</u>: Thus, Leibnitz's Rule states that,

$$F(\alpha)=\int_a^b f(x,\alpha)dx \Rightarrow \underbrace{F'(\alpha)}_{\frac{dF(\alpha)}{d\alpha}}=\int_a^b \frac{\partial f}{\partial\alpha}(x,\alpha)\,dx$$

Similarly;

$$F(\alpha,\beta)=\int_a^b f(x,\alpha,\beta)dx \Rightarrow \frac{\partial F(\alpha,\beta)}{\partial\alpha}=\int_a^b \frac{\partial f}{\partial\alpha}(x,\alpha,\beta)\,dx\,; \quad \frac{\partial(F(\alpha,\beta))}{\partial\beta}=\int_a^b \frac{\partial f}{\partial\beta}(x,\alpha,\beta)\,dx.$$

Note 2: From definite integrals, if a function "f" is,

i) Continuous on $[a,b]$ then $\exists$ a some number "c" in (a,b) such that,

$$\int_a^b f(x)dx = (b-a)f(c).$$

ii) Continuous on $[a,a+h]$ then $\exists$ a real number θ in $(0,1)$ such that,

$$\int_a^{a+h} f(x)dx = h\,f(a+\theta h).$$

Theorem: If $f(x,\alpha)$ and $\frac{\partial f}{\partial \alpha}(x,\alpha)$ are continuous functions of x and α ~~and~~ for $a \leq x \leq b$, $c \leq \alpha \leq d$ and a, b are themselves functions of parameter α, possessing continuous first order derivatives, then,

$$\boxed{\frac{d}{d\alpha}\int_a^b f(x,\alpha)dx = \int_a^b \frac{\partial f}{\partial \alpha}(x,\alpha)dx + \frac{db}{d\alpha}f(b,\alpha) - \frac{da}{d\alpha}f(a,\alpha) \quad \left\{\begin{array}{l} a=a(\alpha) \\ b=b(\alpha)\end{array}\right\}}$$

Proof:

Let $F(\alpha) = \int_a^b f(x,\alpha)dx \quad (*)$

$$F(\alpha+\Delta\alpha) = \int_{a+\Delta a}^{b+\Delta b} f(x,\alpha+\Delta\alpha)dx \quad (**) \qquad \left\{\begin{array}{l} \alpha \to \alpha+\Delta\alpha \\ a \to a+\Delta a \\ b \to b+\Delta b \end{array}\right\}$$

$(**)-(*)$: $F(\alpha+\Delta\alpha) - f(\alpha)$

$$= \int_{a+\Delta a}^{b+\Delta b} f(x,\alpha+\Delta\alpha)dx - \int_a^b f(x,\alpha)dx + \int_{a+\Delta a}^{b+\Delta b} f(x,\alpha)dx - \int_{a+\Delta a}^{b+\Delta b} f(x,\alpha)dx$$

$$= \int_{a+\Delta a}^{b+\Delta b} \left(f(x,\alpha+\Delta\alpha) - f(x,\alpha)\right)dx + \int_{a+\Delta a}^{b+\Delta b} f(x,\alpha)dx - \int_a^b f(x,\alpha)dx$$

$$\left\{ \int_{a+\Delta a}^{b+\Delta b} f(x,\alpha)dx = \int_{a+\Delta a}^{b} f(x,\alpha)dx + \int_{b}^{b+\Delta b} f(x,\alpha)dx = -\int_{b}^{a+\Delta a} f(x,\alpha)dx + \int_{b}^{b+\Delta b} f(x,\alpha)dx \right\}$$

$$= \int_{a+\Delta a}^{b+\Delta b} \Delta\alpha \cdot \frac{\partial f}{\partial \alpha}(x,\alpha+\theta\Delta\alpha)dx + \int_{b}^{b+\Delta b} f(x,\alpha)dx - \underbrace{\left(\int_{a}^{b} f(x,\alpha)dx + \int_{b}^{a+\Delta a} f(x,\alpha)dx \right)}_{\int_{a}^{a+\Delta a} f(x,\alpha)dx}$$

$$F(\alpha+\Delta\alpha)-F(\alpha) = \int_{a+\Delta a}^{b+\Delta b} \Delta\alpha \cdot \frac{\partial f}{\partial \alpha}(x,\alpha+\theta\Delta\alpha)dx + \Delta b \cdot f(b+\theta_1\Delta b,\alpha) - \Delta a \cdot f(a+\theta_2\Delta a,\alpha)$$

$$\frac{F(\alpha+\Delta\alpha)-F(\alpha)}{\Delta\alpha} = \int_{a+\Delta a}^{b+\Delta b} \frac{\partial f}{\partial \alpha}(x,\alpha+\theta\Delta\alpha)dx + \frac{\Delta b}{\Delta\alpha} f(b+\theta_1\Delta b,\alpha) - \frac{\Delta a}{\Delta\alpha} f(a+\theta_2\Delta a,\alpha)$$

where $0 < \theta, \theta_1, \theta_2 < 1$,

Taking limit as $\Delta\alpha, \Delta a, \Delta b \to 0$,

$$\underbrace{\lim_{\Delta\alpha\to 0} \frac{F(\alpha+\Delta\alpha)-F(\alpha)}{\Delta\alpha}}_{\downarrow\ \underbrace{\frac{d(F(\alpha))}{d\alpha}}_{F'(\alpha)}} = \lim_{\Delta\alpha\to 0} \int_{a+\Delta a}^{b+\Delta b} \frac{\partial f}{\partial \alpha}(x,\alpha+\theta\Delta\alpha)dx + \lim_{\Delta\alpha\to 0} \frac{\Delta b}{\Delta\alpha} \cdot \underbrace{\lim_{\Delta b\to 0} f(b+\theta_1\Delta b,\alpha)}_{\downarrow\ f(b,\alpha)}$$

$$- \lim_{\Delta\alpha\to 0} \frac{\Delta a}{\Delta\alpha} \cdot \underbrace{\lim_{\Delta a\to 0} f(a+\theta_2\Delta a,\alpha)}_{\downarrow\ f(a,\alpha)}$$

$$\Rightarrow \frac{d}{d\alpha} \int_{a}^{b} f(x,\alpha)dx = \int_{a}^{b} \frac{\partial f}{\partial \alpha}(x,\alpha)dx + \frac{db}{d\alpha} f(b,\alpha) - \frac{da}{d\alpha} f(a,\alpha)$$ ✓

Ex: 1) $\frac{d}{dx}\int_{\pi/4}^{x} \sin t . dt = ?$

2) $I = \int_{x^2}^{\sin^{-1}x} \frac{\sin t}{t} dt$ then $\frac{dI}{dx} = ?$

3) Find, $\int_{0}^{\infty} t^n . e^{-Kt^2} dt \quad (K>0)$ for odd n.

4) If, $\phi(\alpha) = \int_{\alpha}^{\alpha^2} \frac{\sin \alpha x}{x} dx$ find $\phi'(\alpha)$ where $\alpha \neq 0$.

5) Find, $I(x) = \int_{0}^{1} \ln(x^2+y^2) dy$.

Solution:

1) $\frac{d}{dx}\int_{\pi/4}^{x} \sin t . dt = \frac{d}{dx}\left(-\cos t \Big|_{\pi/4}^{x}\right) = \frac{d}{dx}\left(-\cos x + \cos \pi/4\right) = \sin x$.

2^{nd} method by using Leibnitz's Rule;

$$\begin{Bmatrix} \alpha = x \\ b = x \\ a = \pi/4 \end{Bmatrix}$$

$$\frac{d}{dx}\int_{\pi/4}^{x} \sin t dt = \underbrace{\int_{\pi/4}^{x} \frac{\partial}{\partial x} \sin t dt}_{0} + \underbrace{\frac{d(x)}{dx}}_{1} \sin x - \underbrace{\frac{d(\pi/4)}{dx}}_{0} . \sin \pi/4$$

$$= \sin x .$$

2) $$\frac{dI}{dx} = \frac{d}{dx}\int_{x^2}^{\sin^{-1}x} \frac{\sin t}{t}\,dt = \underbrace{\int_{x^2}^{\sin^{-1}x} \frac{\partial}{\partial x}\left(\frac{\sin t}{t}\right)dt}_{0} + \frac{\partial(\sin^{-1}x)}{\partial x}\cdot\frac{\sin(\sin^{-1}x)}{\sin^{-1}x} - \frac{\partial(x^2)}{\partial x}\cdot\frac{\sin(x^2)}{x^2}$$

$$= \frac{1}{\sqrt{1-x^2}}\cdot\frac{x}{\sin^{-1}x} - 2x\cdot\frac{\sin(x^2)}{x^2}.$$

27 Şubat 2015
Cuma

3) $$\int_0^\infty t^n e^{-Kt^2}\,dt = F(K)$$ {if dt then parameter is K} integration variable

$$n=1 \Rightarrow \underbrace{\int_0^\infty t\,e^{-Kt^2}dt}_{} = \int_0^\infty e^u\cdot\frac{du}{-2K} = -\frac{1}{2K}e^u\Big|_{t=0}^{\infty}$$

$$\left\{\begin{array}{l} -Kt^2 = u \\ -2Kt\,dt = du \end{array}\right\}$$

$$\Rightarrow I = -\frac{1}{2K}e^{-Kt^2}\Big|_0^\infty = \frac{1}{2K}\ \checkmark$$

$$\int_0^\infty t\,e^{-Kt^2}dt = \frac{1}{2K}$$ {Differentiate w.r.t K both sides}

$$\Rightarrow \frac{d}{dK}\left(\int_0^\infty t\,e^{-Kt^2}dt\right) = \frac{d}{dK}\left(\frac{1}{2K}\right) \Rightarrow \int_0^\infty \frac{\partial}{\partial K}\left(t e^{-Kt^2}\right)dt = \frac{d}{dK}\left(\frac{1}{2K}\right)$$

$$\Rightarrow -\int_0^\infty t^3 e^{-Kt^2}dt = -\frac{1}{2K^2}$$

$$\Rightarrow \int_0^\infty t^3 e^{-Kt^2}dt = \frac{1}{2K^2}\ \checkmark$$

$$\int_0^\infty t^3 e^{-Kt^2} dt = \frac{1}{2K^2} \quad \left\{ \text{Diff. wrt } K \right\}$$

$$\Rightarrow \frac{d}{dK}\left(\int_0^\infty t^3 e^{-Kt^2} dt \right) = \frac{d}{dK}\left(\frac{1}{2K^2} \right) \Rightarrow -\int_0^\infty t^5 e^{-Kt^2} dt = -\frac{1}{K^3}$$

$$\Rightarrow \int_0^\infty t^5 e^{-Kt^2} dt = \frac{1}{K^3} \checkmark$$

$$\int_0^\infty t^5 e^{-Kt^2} dt = \frac{1}{K^3} \quad \left\{ \text{Diff wrt } K \right\}$$

$$\Rightarrow \frac{d}{dK}\left(\int_0^\infty t^5 e^{-Kt^2} dt \right) = \frac{d}{dK}\left(\frac{1}{K^3} \right) \Rightarrow -\int_0^\infty t^7 e^{-Kt^2} dt = -\frac{3}{K^4}$$

$$\Rightarrow \int_0^\infty t^7 e^{-Kt^2} dt = \frac{3}{K^4} \checkmark$$

$\vdots$

$$\int_0^\infty t^{2n+1} \cdot e^{-Kt^2} \cdot dt = \frac{n!}{2 \cdot K^{n+1}} \quad \text{for } n = 0, 1, 2, \ldots$$

5) $$\int_0^1 \ln(x^2+y^2) dy = I(x)$$

$$\frac{dI}{dx} = \frac{d}{dx} \int_0^1 \ln(x^2+y^2) dy = \int_0^1 \frac{d}{dx}\left(\ln(x^2+y^2) dy \right) = \int_0^1 \frac{2x}{x^2+y^2} \cdot dy$$

$$\Rightarrow \frac{2x}{x^2} \int_0^1 \frac{dy}{1+(y/x)^2} \quad \left\{ \frac{y}{x} = u \Rightarrow \begin{array}{l} y = xu \\ dy = x\,du \end{array} \right\}$$

$$= \frac{2x}{x^2}\int_0^1 \frac{dy}{1+(y/x)^2} = \frac{2x}{x^2}\int \frac{x\,du}{1+u^2} = 2\int \frac{du}{1+u^2}$$

$$= 2\,\mathrm{Arctan}\,u \checkmark$$

$$\Rightarrow 2\,\mathrm{Arctan}(y/x)\Big|_{y=0}^{1} = 2\,\mathrm{Arctan}(1/x) - 2\,\mathrm{Arctan}(0)$$

$$= 2\,\mathrm{Arctan}(1/x) \checkmark$$

$$\Rightarrow \frac{dI}{dx} = 2\,\mathrm{Arctan}\left(\frac{1}{x}\right) \quad \left\{ \text{Integrate wrt } x \right\}$$

$$I(x) = 2\int \mathrm{Arctan}\left(\frac{1}{x}\right)dx + C \qquad \left\{ \begin{array}{l} u = \mathrm{Arctan}(1/x) \\ du = \dfrac{-1/x^2}{1+1/x^2}\,dx = \dfrac{-dx}{1+x^2} \\ dx = dv \\ x = v \end{array} \right\}$$

$$\Rightarrow I(x) = 2\left(x.\mathrm{Arctan}\left(\frac{1}{x}\right) + \int \frac{x\,dx}{1+x^2}\right) + C$$

$$= 2x\,\mathrm{Arctan}(1/x) + \ln(1+x^2) + C \checkmark$$

for $x=0$,

$$\Rightarrow I(0) = \int_0^1 \ln(y^2)\,dy = 2\int_0^1 \ln y\,dy$$

$$= 2\lim_{a\to 0}\int_a^1 \ln y\,dy = 2\lim_{a\to 0}(-a\ln a - 1)$$

$$= -2\underbrace{\lim_{a\to 0} a\ln a}_{0} - 2 = -2 \checkmark$$

$$I(0) = \underbrace{2.0.\mathrm{Arctan}(1/0)}_{0} + \underbrace{\ln(1)}_{0} + C = C$$

$$C = -2$$

$$\Rightarrow I(x) = 2x\,\mathrm{Arctan}\left(\frac{1}{x}\right) + \ln(1+x^2) - 2 .$$

HW: Evaluate the integral,

$$I = \int_0^\infty t^n \cdot e^{-at^2} dt \text{ for } n=2,4,\dots,2m \text{ if } \int_0^\infty e^{-at^2} dt = \frac{1}{2}\sqrt{\pi/a}.$$

Ex:

Show that $y = \int_0^x f(u)\cdot\sin(x-u)du$ satisfies $y''+y=f(x)$.

$$y' = \frac{dy}{dx} = \frac{d}{dx}\int_0^x f(u)\sin(x-u)du = \int_0^x \frac{\partial}{\partial x}(f(u)\cdot\sin(x-u)du) + \underbrace{\frac{d(x)}{dx} f(x)\cdot\sin 0}_{0} - \underbrace{\frac{d(0)}{dx} f(0)\cdot\sin x}_{0}$$

$$= \int_0^x \frac{\partial}{\partial x}(f(u)\cdot\sin(x-u)du)$$

$$= \int_0^x f(u)\cdot\cos(x-u)du \checkmark$$

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\int_0^x f(u)\cdot\cos(x-u)du\right) = \int_0^x \frac{\partial}{\partial x}(f(u)\cos(x-u)du) + \underbrace{\frac{d(x)}{dx} f(x)\cos 0}_{f(x)} - \underbrace{\frac{d(0)}{dx} f(0)\cos x}_{0}$$

$$= \int_0^x \frac{\partial}{\partial x}(f(u)\cos(x-u)du) + f(x)$$

$$= -\int_0^x f(u)\cdot\sin(x-u)du + f(x) \checkmark$$

$$\Rightarrow y'' = -\int_0^x f(u)\cdot\sin(x-u)du + f(x)$$

$$y = \int_0^x f(u)\cdot\sin(x-u)du$$

$+$

$$y''+y=f(x).$$

Gamma Function

Definition: If $n>0$ then the integral,

$$\int_0^\infty x^{n-1}.e^{-x}.dx$$

which is obviously a function of n is called a Gamma Function and is denoted by $\Gamma(n)$. Thus,

$$\Gamma(n)=\int_0^\infty x^{n-1}.e^{-x}.dx \quad , \forall n>0.$$

Gamma function is also called the second Eulerian Integral. For example,

i) $\int_0^\infty x^3.e^{-x}.dx=\Gamma(4) \quad \{n-1=3\}$

ii) $\int_0^\infty x^{2/3}.e^{-x}.dx=\Gamma(5/3) \quad \{n-1=\frac{2}{3}\}$

Convergence of Gamma Function

Theorem: $\underbrace{\int_0^\infty x^{n-1}.e^{-x}.dx}_{\text{Converges}}$ iff $n>0$.

Proof: If $n\geq 1$, then integrand $x^{n-1}.e^{-x}$ is continuous at $x=0$.

If $n<1$, then integrand $\frac{e^{-x}}{x^{1-n}}$ has infinite discontinuity at $x=0$.

Thus, we have to examine the convergence at $x=0$ and $x=\infty$ both consider any positive number say 1, and examine the convergence of,

$$\left.\int_0^\infty x^{n-1}.e^{-x}.dx=\underbrace{\int_0^1 x^{n-1}.e^{-x}.dx}_{\text{Converges at } x=0}+\underbrace{\int_1^\infty x^{n-1}.e^{-x}.dx}_{\text{Converges at } x=\infty}\right\} \text{ iff } n>0.$$

Convergence at $x=0$ when $n<1$

Let $f(x)=\frac{e^{-x}}{x^{1-n}}$, take $g(x)=\frac{1}{x^{1-n}}$, then,

$$\lim_{x\to 0^+}\frac{f(x)}{g(x)}=\lim_{x\to 0^+}e^{-x}=1.$$

which is nonzero and finite. Also,

$$\int_0^1 g(x)\,dx=\int_0^1\frac{dx}{x^{1-n}} \text{ is convergent } \left\{1-n<1,\ n>0\right\}$$

$$\left\{ * \int_a^b \frac{dx}{(x-a)^n} \text{ is convergent iff } n<1 \right\}$$

By comparison test,

$$\int_0^1 f(x)\,dx=\int_0^1 x^{n-1}\cdot e^{-x}\cdot dx \text{ is convergent iff } 1-n<1, \text{ i.e., } n>0 \text{ at } x=0.$$

Convergence at $x=\infty$

Let $g(x)=\frac{1}{x^2}$, so that,

$$\left\{ \int_a^\infty \frac{dx}{x^p} \text{ is convergent iff } p>1 \ (a>0) \right\}$$

$$f(x)=x^{n-1}\cdot e^{-x}$$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x^{n+1}}{e^x}=0 \text{ for all } n \quad \left\{ \text{L'Hospital, } \lim_{x\to\infty}\frac{(n+1)n(n-1)\ldots 1}{e^x}=0 \right\}$$

As, $\int_1^\infty g(x)\,dx=\int_1^\infty \frac{1}{x^2}dx$ converges, here,

$$\int_1^\infty f(x)\cdot dx=\int_1^\infty x^{n-1}\cdot e^{-x}\cdot dx \text{ also converges for all } n.$$

Properties of Gamma Function

1) $\Gamma(n+1)=n\Gamma(n)$, $n>0$.

Proof: $\Gamma(n)=\int_0^\infty x^{n-1}.e^{-x}.dx$

$$\Gamma(n+1)=\int_0^\infty x^n.e^{-x}.dx=\lim_{m\to\infty}\int_0^m x^n.e^{-x}.dx \quad \left\{\begin{array}{l} u=x^n \Rightarrow nx^{n-1}dx=du \\ e^{-x}dx=dv \Rightarrow v=-e^{-x}\end{array}\right\}$$

$$=\lim_{m\to\infty}\left(-x^n.e^{-x}\Big|_0^m+n\int_0^m x^{n-1}.e^{-x}.dx\right)$$

$$=\underbrace{\lim_{m\to\infty}\left(-x^n.e^{-x}\Big|_0^m\right)}_{-m^n.e^{-m}}+n\underbrace{\int_0^\infty x^{n-1}.e^{-x}.dx}_{\Gamma(n)}$$

$$=\underbrace{\lim_{m\to\infty}(-m^n.e^{-m})}_{0}+n\Gamma(n)$$

$$\Gamma(n+1)=n\Gamma(n) \checkmark$$

2) $\Gamma(n+1)=n!$, $n=1,2,3,..$

Proof: $\Gamma(n+1)=n\Gamma(n)=n\Gamma(n-1+1)$

$$=n(n-1)\Gamma(n-1)=n(n-1)\Gamma(n-2+1)$$

$$=n(n-1)(n-2)\Gamma(n-2)$$

$$\vdots$$

$$=n(n-1)(n-2)\dots\Gamma(1)$$

$$\left\{\Gamma(1)=\int_0^\infty e^{-x}dx=-e^{-x}\Big|_0^\infty=1\right\}$$

$$=n(n-1)(n-2)\dots 1$$

$$\Gamma(n+1)=n! \checkmark$$

Ex: i) $\dfrac{\Gamma(6)}{2\Gamma(3)} = \dfrac{5!}{2.2!} = 30$

ii) $\Gamma(10) = 9!$

4 Mart 2015
Çarşamba

3) $\Gamma(1/2) = \sqrt{\pi}$

4) $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$, $n < 0$.

Ex: $\Gamma(-1/2) = \dfrac{\Gamma(-1/2+1)}{-1/2} = -2\Gamma(1/2) = -2\sqrt{\pi}$.

$$\Gamma(-5/2) = \frac{\Gamma(-5/2+1)}{-5/2} = -\frac{2}{5}\Gamma(-3/2) = -\frac{2}{5}\,\frac{\Gamma(-3/2+1)}{-3/2} = \frac{4}{15}\Gamma(-1/2) = -\frac{8}{15}\sqrt{\pi}$$

5) $\Gamma(n+1/2) = \dfrac{(2n)!\sqrt{\pi}}{4^n . n!}$

6) $\Gamma(n+k) = n(n+1)(n+2)\dots(n+k-1)\Gamma(n)$, k is a positive integer.

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma(n+2) = (n+1)\Gamma(n+1) = (n+1).n.\Gamma(n)$

$\vdots$

$$\begin{aligned}\Gamma(n+k-1+1) &= (n+k-1)\underbrace{\Gamma(n+k-1)}_{\Gamma(n+k-2+1) = (n+k-2).\Gamma(n+k-2)} \\ &= (n+k-1)(n+k-2).\underbrace{\Gamma(n+k-2)}_{\dots} \\ &= (n+k-1)(n+k-2)\dots(n+k-(k-1))\Gamma(n+k-(k-1)) \\ &= (n+k-1)(n+k-2)\dots(n+1)\underbrace{\Gamma(n+1)}_{n.\Gamma(n)}\end{aligned}$$

7) $\Gamma(p)\Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$, $0 < p < 1$

Ex: $p = \dfrac{3}{4}$, $1-p = \dfrac{1}{4}$ $\Rightarrow \Gamma(3/4)\Gamma(1/4) = \dfrac{\pi}{\sin\frac{3\pi}{4}} = \sqrt{2}\,\pi$.

8) Asymptotic formula for $\Gamma(n)$:

If n is large, the computational difficulties in a calculation of $\Gamma(n)$ are appeared. A useful result in such case is supplied by the relation,

$$\Gamma(n+1) = \sqrt{2\pi . n}\,.n^n.e^{-n}.e^{\frac{\theta}{12(n+1)}}, \quad 0 < \theta < 1.$$

For most practical purposes the last factor ($e^{\theta/12(n+1)}$) which is very close to 1, for large n can be omitted. If n is integer, we can write,

$$\Gamma(n+1) = n! \sim \sqrt{2\pi n}\, n^n e^{-n}$$

Stirling's factorial approximation of asymptotic formula for large n.

Ex: $5! \sim \sqrt{2\pi \cdot 5}\, 5^5 \cdot e^{-5} = 118.02$.

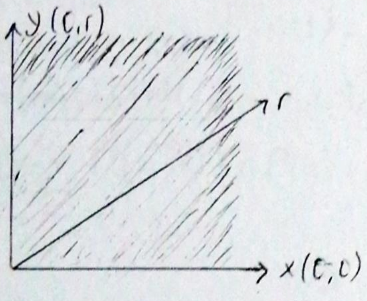
Ex:

Prove that $I = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

$$I = \int_0^\infty e^{-y^2} dy \Rightarrow I^2 = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dx\,dy$$

Use polar coordinates:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad dA = r\,dr\,d\theta \qquad \left\{ \begin{array}{c} r; \\ (x,y) \to (r,\theta) \\ J(r,\theta) = r \end{array} \right\}$$



$$\Rightarrow I^2 = \int_{\theta=0}^{\pi/2}\left(\int_{r=0}^{\infty} e^{-r^2} r\,dr\right) d\theta$$

$$\left\{ \begin{array}{l} r^2 = u \\ 2r\,dr = du \end{array} \right. \Rightarrow \int_0^\infty e^{-u} \cdot \frac{du}{2} = -\frac{1}{2} e^{-u}\Big|_0^\infty = \frac{1}{2} \Big\}$$

$$\Rightarrow I^2 = \int_{\theta=0}^{\pi/2} \frac{1}{2} d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}.$$

✓ 2nd method:

$$\Gamma(n) = \int_0^\infty x^{n-1} \cdot e^{-x} \cdot dx, \quad n > 0$$

$$I = \int_0^\infty e^{-x^2} dx \quad \left\{ \begin{array}{l} y = x^2 \Rightarrow dy = 2x dx \\ x = 0 \Rightarrow y = 0 \\ x = \infty \Rightarrow y = \infty \end{array} \right\}$$

$$= \int_0^\infty e^{-y} \frac{dy}{2y^{1/2}} = \frac{1}{2} \underbrace{\int_0^\infty y^{-1/2} e^{-y} dy}_{(n-1 = -1/2 \Rightarrow \Gamma(1/2))} = \frac{1}{2} \Gamma(1/2) = \frac{\sqrt{\pi}}{2}.$$

Ex: Show that $\Gamma(1/2) = \sqrt{\pi}$. (3rd property of $\Gamma(n)$)

$$\Gamma(n) = \int_0^\infty x^{n-1} \cdot e^{-x} \cdot dx \quad , n > 0$$

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} \cdot e^{-x} \cdot dx \quad \left\{ \begin{array}{l} x = u^2 \Rightarrow dx = 2u du \\ x = 0 \Rightarrow u = 0 \\ x = \infty \Rightarrow u = \infty \end{array} \right\}$$

$$= \int_0^\infty u^{-1} e^{-u^2} \cdot 2u du = 2 \int_0^\infty e^{-u^2} du = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

Ex: Evaluate each integral.

i) $\int_0^\infty \sqrt{y} \cdot e^{-y^3} dy$ ii) $\int_0^\infty 3^{-4x^2} \cdot dx$ iii) $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$

i) $\int_0^\infty y^{1/2} e^{\cancel{-y^3}} dy \quad \left\{ \begin{array}{l} y = x^{1/3} \Rightarrow y^3 = x \Rightarrow 3y^2 dy = dx \\ x = 0 \Rightarrow y = 0 \\ x = \infty \Rightarrow y = \infty \end{array} \right\}$

$$= \int_0^\infty (x^{1/3})^{1/2} e^{-x} \cdot \frac{dx}{3x^{2/3}} = \frac{1}{3} \int x^{-1/2} \cdot e^{-x} dx = \frac{1}{3} \Gamma(1/2) = \frac{\sqrt{\pi}}{3}.$$

ii) $3^{-4x^2} = e^{\ln 3^{-4x^2}} = e^{-(4x^2 \ln 3)}$

$$\left\{ \begin{array}{l} y = 4x^2 \ln 3 \Rightarrow dy = 8x \ln 3 dx \\ \qquad dx = \frac{dy}{8x \ln 3} \Rightarrow dx = \frac{dy}{(8\ln 3)(4\ln 3)^{-1/2} y^{1/2}} \\ x = 0 \Rightarrow y = 0 \\ x = \infty \Rightarrow y = \infty \end{array} \right\}$$

$$\Rightarrow \int_0^\infty 3^{-4x^2}\,dx = \int_0^\infty e^{\ln(3^{-4x^2})}\,dx = \int_0^\infty e^{-(4x^2\ln 3)}\,dx$$

$$= \int_0^\infty e^{-y}\cdot \frac{dy}{(8\ln 3)(4\ln 3)^{-1/2}}\cdot y^{-1/2}$$

$$= \int_0^\infty e^{-y}\cdot y^{-1/2}\cdot dy \cdot \frac{1}{(8\ln 3)(4\ln 3)^{-1/2}} = \Gamma(1/2)\cdot\frac{1}{(8\ln 3)(4\ln 3)^{-1/2}}$$

$$= \frac{\sqrt{\pi}}{4(\ln 3)^{1/2}}.$$

iii) $-\ln x = y$
$$\left\{\begin{array}{lll} x = e^{-y} & \Rightarrow & dx = -e^{-y}dy \\ x = 0 & \Rightarrow & y = \infty \\ x = 1 & \Rightarrow & y = 0 \end{array}\right\}$$

$$\Rightarrow I = \int_\infty^0 \frac{1}{\sqrt{y}}(-e^{-y})\,dy = \int_0^\infty y^{-1/2}\cdot e^{-y}\,dy = \Gamma(1/2) = \sqrt{\pi}.$$

Beta Function

Definition: If $m,n>0$ then the integral,

$$\int_0^1 x^{m-1}\cdot(1-x)^{n-1}\cdot dx$$

which is obviously a function of m and n, is called a Beta Function, and is denoted by; $\beta(m,n)$. Thus,

$$\beta(m,n) = \int_0^1 x^{m-1}\cdot(1-x)^{n-1}\cdot dx\ , \quad m,n>0.$$

Beta Function is called the first Eulerian integral. For example;

i) $\int_0^1 x^3(1-x)^5\,dx = \beta(4,6)$ ii) $\int_0^1 \sqrt{x}\cdot(1-x)^3\,dx = \beta(3/2,4)$

iii) $\int_0^1 x^{-2/3}\cdot(1-x)^{-1/2}\,dx = \beta(1/3,1/2)$ iv) $\int_0^1 x^{-3}\cdot(1-x)^5\,dx = \beta(-2,6)$ is not β function.

$m=-2<0$

Convergence of Beta Function

Theorem:

$$\int_0^1 x^{m-1}\cdot(1-x)^{n-1}\,dx \quad \text{exists iff } m \text{ and } n \text{ are both positive}$$

Proof: If $m \geq 1$ and $n \geq 1$, then the integral is proper.

If $m<1$ and $n<1$, then $f(x)=\dfrac{1}{x^{1-m}\cdot(1-x)^{1-n}}$.

For $m<1$ and $n<1$,

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \underbrace{\int_0^{1/2} x^{m-1}(1-x)^{n-1}\cdot dx}_{\substack{\text{infinite}\\ \text{discontinuity}\\ x=0}} + \underbrace{\int_{1/2}^1 x^{m-1}\cdot(1-x)^{n-1}\cdot dx}_{\substack{\text{infinite}\\ \text{discontinuity}\\ x=1}}$$

Convergence at $x=0$ when $m<1$:

Let $f(x)=\dfrac{(1-x)^{n-1}}{x^{1-m}}$, take $g(x)=\dfrac{1}{x^{1-m}}$

$$\lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+}(1-x)^{n-1} = 1$$, which is nonzero, ~~in~~finite

$$\int_0^{1/2} g(x)dx = \int_0^{1/2}\frac{dx}{x^{1-m}}$$ is convergent iff $1-m<1$, i.e, $m>0$.

By comparison test,

$$\int_0^{1/2} f(x)dx = \int_0^{1/2} x^{m-1}\cdot(1-x)^{n-1}\cdot dx$$ is also convergent at $x=0$ for $m>0$.

Convergence at $x=1$, when $n<1$:

Let $f(x)=\dfrac{x^{m-1}}{(1-x)^{1-n}}$. take $g(x)=\dfrac{1}{(1-x)^{1-n}}$

$\lim\limits_{x\to 1^-}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to 1^-}x^{m-1}=1$ which is nonzero, ~~in~~finite.

$\displaystyle\int_{1/2}^{1} g(x)\,dx=\int_{1/2}^{1}\frac{dx}{(1-x)^{1-n}}$ is convergent $1-n<1$, i.e, $n>0$.

By comparison test,

$\displaystyle\int_{1/2}^{1} f(x)\,dx=\int_{1/2}^{1} x^{m-1}\cdot(1-x)^{n-1}\cdot dx$ is convergent at $x=1$ for $n>0$

$$\int_0^1 x^{m-1}\cdot(1-x)^{n-1}\cdot dx=\underbrace{\int_0^{1/2} x^{m-1}\cdot(1-x)^{n-1}\cdot dx}_{m<1}+\underbrace{\int_{1/2}^{1} x^{m-1}\cdot(1-x)^{n-1}}_{n<1}\quad , m,n>0$$

Properties of Beta Function

1)
$$\beta(m,n)=\int_0^1 x^{m-1}\cdot(1-x)^{n-1}\cdot dx\quad , m>0\, , n>0.$$

$$\beta(m,n)=\beta(n,m)$$

Proof: Use transformation $x=1-y$.

$$\left.\begin{array}{l} x=1-y \Rightarrow dx=-dy \\ x=0 \Rightarrow y=1 \\ x=1 \Rightarrow y=0 \end{array}\right\} \Rightarrow \beta(m,n)=\int_1^0 (1-y)^{m-1}\cdot(1-(1-y))^{n-1}\cdot(-dy)$$

$$=\underbrace{\int_0^1 y^{n-1}\cdot(1-y)^{m-1}dy}_{\begin{pmatrix} m-1=n-1 \Rightarrow m=n \\ n-1=m-1 \Rightarrow n=m \end{pmatrix}}=\beta(n,m)\ \checkmark$$

2)

$$\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta.\cos^{2n-1}\theta.d\theta$$

Proof: Use transformation $x=\sin^2\theta$.

$$\left.\begin{array}{l} x=\sin^2\theta \Rightarrow dx=2\sin\theta\cos\theta d\theta \\ x=0 \Rightarrow \theta=0 \\ x=1 \Rightarrow \theta=\pi/2 \end{array}\right\} \Rightarrow \beta(m,n)=\int_0^{\pi/2}\sin^{2m-2}\theta.\underbrace{(1-\sin^2\theta)}_{\cos^2\theta}{}^{n-1}.2\sin\theta.\cos\theta.d\theta$$

$$=2\int_0^{\pi/2}\sin^{2m-1}\theta.\cos^{2n-1}\theta d\theta. \checkmark$$

3)

$$\beta(m,n)=\frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$$

Proof: Use transformation $z=x^2$.

$$\left\{\Gamma(m)=\int_0^\infty z^{m-1}.e^{-z}.dz \quad , m>0\right\}$$

$$\left.\begin{array}{l} z=x^2 \Rightarrow dz=2xdx \\ 0\le z\le\infty \end{array}\right\} \Rightarrow \Gamma(m)=\int_0^\infty x^{2m-2}.e^{-x^2}2xdx=2\int_0^\infty x^{2m-1}.e^{-x^2}.dx$$

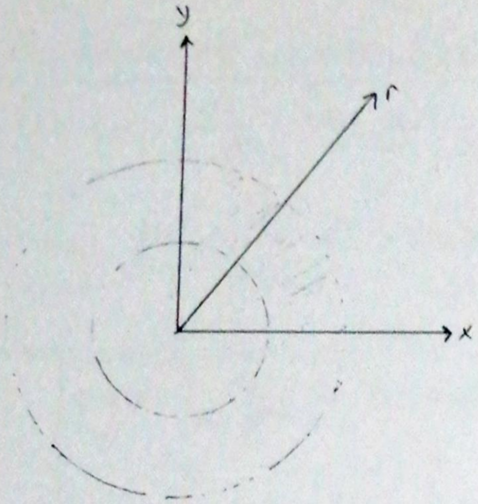
$$\Gamma(n)=2\int_0^\infty y^{2n-1}.e^{-y^2}.dy$$

$$\Gamma(m).\Gamma(n)=\left(2\int_0^\infty x^{2m-1}.e^{-x^2}.dx\right)\left(2\int_0^\infty y^{2n-1}.e^{-y^2}.dy\right)$$

$$=4\int_0^\infty\int_0^\infty x^{2m-1}.y^{2n-1}.e^{-(x^2+y^2)}dxdy.$$

Use polar coordinates:

$$\left.\begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \\ dA=|J(r,\theta)|drd\theta=rdrd\theta \end{array}\right\} \quad \begin{array}{l} 0\le\theta\le\pi/2 \\ 0\le r\le\infty \end{array}$$



$$\Big\} \Rightarrow \begin{matrix} 0 \leq \theta \leq \pi/2 \\ 0 \leq r \leq \infty \end{matrix}$$

$$\Rightarrow 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} r^{2m-1} . \cos^{2m-1}\theta . r^{2n-1} . \sin^{2n-1}\theta . e^{-r^2} . r\, dr\, d\theta$$

$$= 4 \underbrace{\left(\frac{1}{2} . 2 \int_0^{\pi/2} \sin^{2n-1}\theta . \cos^{2m-1}\theta\, d\theta \right)}_{\frac{1}{2}\beta(m,n)} \underbrace{\left(\int_0^{\infty} r^{2(m+n)-1} . e^{-r^2} . dr \right)}$$

$$\left.\begin{matrix} r^2 = u \Rightarrow 2r dr = du \\ r = 0 \Rightarrow u = 0 \\ r = \infty \Rightarrow u = \infty \end{matrix}\right\} \Rightarrow \int_0^{\infty} u^{m+n-1/2} . e^{-u} . \frac{du}{2u^{1/2}}$$

$$= \underbrace{\frac{1}{2} \int_0^{\infty} u^{m+n-1} . e^{-u} . du}_{\Gamma(m+n)}$$

$$\Rightarrow \Gamma(m) . \Gamma(n) = 4 . \left(\frac{1}{2} \beta(m,n) \right) \left(\frac{1}{2} \Gamma(m+n) \right) ;$$

$$\beta(m,n) = \frac{\Gamma(m) . \Gamma(n)}{\Gamma(m+n)} \checkmark$$

4)

$$\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} . dx \quad , \; m,n > 0 .$$

Proof: Use transformation $z = \frac{x}{1+x}$ $\left(x = \frac{z}{1-z}\right)$

$$\left\{ \beta(m,n) = \int_0^1 z^{m-1}(1-z)^{n-1}\,dz \quad , \quad m,n>0 \right\}$$

$$\left.\begin{array}{l} z = \frac{x}{1+x} \Rightarrow dz = \frac{dx}{(1+x)^2} \\ z=0 \Rightarrow x=0 \\ z=1 \Rightarrow x=\infty \end{array}\right\} \Rightarrow \beta(m,n) = \int_{x=0}^{\infty} \left(\frac{x}{1+x}\right)^{m-1} \cdot \left(1-\frac{x}{1+x}\right)^{n-1} \cdot \frac{dx}{(1+x)^2}$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m-1}} \cdot \frac{1}{(1+x)^{n-1}} \cdot \frac{dx}{(1+x)^2}$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \checkmark$$

5)

$$\beta(p,1-p) = \frac{\Gamma(p)\Gamma(1-p)}{\Gamma(1)} = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}$$

$$= \int_0^\infty \frac{x^{p-1}}{(1+x)}\,dx \quad \left\{\begin{array}{l} 0<p<1 \\ m=p \\ n=1-p \end{array}\right\}$$

Ex: Evaluate each of the following integrals.

i) $\int_0^1 x^4(1-x)^2\,dx$ ii) $\int_0^2 \frac{x^2\,dx}{\sqrt{2-x}}$ iii) $\int_0^a y^4\sqrt{a^2-y^2}\,dy$

i) $\beta(5,3) = \frac{\Gamma(5).\Gamma(3)}{\Gamma(8)} = \frac{4!\,2!}{7!} = \frac{1}{105}$.

ii) $\int_0^2 x^2(2-x)^{-1/2}\,dx$ $\quad \left\{\begin{array}{l} x=2y \Rightarrow dx=2dy \\ x=0 \Rightarrow y=0 \\ x=2 \Rightarrow y=1 \end{array}\right\}$

$$\Rightarrow \int_0^1 (2y)^2 \cdot (2-2y)^{-1/2} dy \cdot 2 = \underbrace{\int_0^1 4\sqrt{2} \cdot y^2 (1-y)^{-1/2} \cdot dy}$$

$$\begin{pmatrix} m-1=2 \Rightarrow m=3 \\ n-1=-1/2 \Rightarrow n=1/2 \end{pmatrix} \Rightarrow 4\sqrt{2} \cdot \beta(3, 1/2) = 4\sqrt{2} \cdot \frac{\Gamma(3) \cdot \Gamma(1/2)}{\Gamma(7/2)}$$

$$= 4\sqrt{2} \cdot \frac{2! \cdot \sqrt{\pi}}{\underbrace{\Gamma(7/2)}}$$

$$\left\{ \begin{array}{l} \Gamma(n+1/2) = \frac{(2n)! \cdot \sqrt{\pi}}{4^n \cdot n!} \\ \Gamma(3+1/2) = \frac{6! \cdot \sqrt{\pi}}{4^3 \cdot 3!} = \frac{15\sqrt{\pi}}{8} \end{array} \right\}$$

$$= 4\sqrt{2} \cdot \frac{2! \sqrt{\pi}}{\frac{15\sqrt{\pi}}{8}} = \frac{64\sqrt{2}}{15}.$$

iii) Use transformation $y^2 = a^2 x$ or $y = a \sin t$.

$\left\{ \text{Result: } \frac{\pi a^6}{16} \right\}$

Ex: Evaluate each integrals.

i) $\int_0^{\pi/2} \sin^6 \theta \, d\theta$ ii) $\int_1^3 \frac{x^2 dx}{\sqrt{(x-1)(3-x)}}$ iii) $\int_0^{\pi} \cos^6 \theta \, d\theta$

i) $$\int_0^{\pi/2} \sin^6 \theta \, d\theta = \frac{1}{2} \cdot 2 \underbrace{\int_0^{\pi/2} \sin^6 \theta \cdot \cos^0 \theta \cdot d\theta}$$

$$\begin{pmatrix} 2m-1=6 \Rightarrow m=7/2 \\ 2n-1=0 \Rightarrow n=1/2 \end{pmatrix} \Rightarrow \frac{1}{2} \cdot \beta(7/2, 1/2) = \frac{1}{2} \cdot \frac{\Gamma(7/2)\Gamma(1/2)}{\Gamma(4)}$$

$$= \frac{1}{2} \frac{15\sqrt{\pi}/8 \cdot \sqrt{\pi}}{3!}$$

$$= \frac{5\pi}{32}.$$

11 Mart 2015
Çarşamba

ii) Use transformation $x-1=u \quad (u+1=x)$

$$\left.\begin{array}{l} x-1=u \Rightarrow dx=du \\ x=1 \Rightarrow u=0 \\ x=3 \Rightarrow u=2 \end{array}\right\} \Rightarrow \int_0^2 \frac{du}{\sqrt{u(2-u)}} = \underbrace{\int_0^2 u^{-1/2}\cdot(2-u)^{-1/2}\,du}$$

1. dönüşüm

$$\left\{\begin{array}{c} u=2x \\ du=2dx \\ u=0 \Rightarrow x=0 \\ u=2 \Rightarrow x=1 \end{array}\right\}$$

2. dönüşüm

$$= \int_0^1 (2x)^{-1/2}\cdot(2-2x)^{-1/2}\cdot 2dx$$

$$= \int_0^1 2^{-1/2}\cdot x^{-1/2}\cdot 2^{-1/2}\cdot(1-x)^{-1/2}\cdot 2dx$$

$$= \underbrace{\int_0^1 x^{-1/2}\cdot(1-x)^{-1/2}dx}$$

$$\left(\begin{array}{l} m-1=-1/2 \Rightarrow m=1/2 \\ n-1=-1/2 \Rightarrow n=1/2 \end{array}\right) \Rightarrow \beta(1/2, 1/2) = \frac{\Gamma(1/2)\cdot\Gamma(1/2)}{\Gamma(1)} = \pi$$

veya tek dönüşüm:

$x-1=2u \qquad x=1 \; u=0$
$dx=2du \qquad x=3 \; u=1$

$$I = \int_0^1 \frac{2du}{\sqrt{2u}\cdot\sqrt{2-2u}} = \int_0^1 u^{-1/2}(1-u)^{-1/2}\,du$$

$3-x = 3-(1+2u)$
$= 2-2u$

iii) $$\int_0^{\pi} \cos^6\theta\,d\theta = \int_0^{\pi/2} \cos^6\theta\,d\theta + \underbrace{\int_{\pi/2}^{\pi} \cos^6\theta\,d\theta}_{*}$$

$*$: Use transformation $\theta=\pi-x$.

$$\left.\begin{array}{l} \theta=\pi-x \Rightarrow d\theta=-dx \\ \theta=\pi/2 \Rightarrow x=\pi/2 \\ \theta=\pi \Rightarrow x=0 \end{array}\right\} \Rightarrow * = \int_{\pi/2}^{\pi} \cos^6\theta\,d\theta = \underbrace{\int_{\pi/2}^{0} \cos^6(\pi-x)(-dx)}$$

$\{ \cos(\pi-x) = \cos\pi\cdot\cos x + \sin\pi\cdot\sin x = -\cos x \}$

$$= \int_0^{\pi/2} \cos^6 x\,dx \Rightarrow \int_0^{\pi/2} \cos^6\theta\,d\theta.$$

$$\Rightarrow \int_0^{\pi} \cos^6\theta d\theta = \int_0^{\pi/2} \cos^6\theta d\theta + \int_0^{\pi/2} \cos^6\theta d\theta = 2\int_0^{\pi/2} \cos^6\theta d\theta$$

$$= 2\underbrace{\int_0^{\pi/2} \sin^0\theta . \cos^6\theta . d\theta}$$

$$\begin{pmatrix} 2m-1=0 \Rightarrow m=1/2 \\ 2n-1=6 \Rightarrow n=7/2 \end{pmatrix} = \beta(1/2, 7/2) = \frac{\Gamma(1/2).\Gamma(7/2)}{\Gamma(4)}$$

$$= \frac{\sqrt{\pi}.\frac{15\sqrt{\pi}}{8}}{3!} = \frac{5\pi}{16}.$$

$$\left\{ \text{If } \int_0^{2\pi} \dots = \int_0^{\pi} \dots + \int_{\pi}^{2\pi} \dots \quad \text{then use transformation } \theta = 2\pi - x. \right\}$$

HW:

i) $\displaystyle\int_0^{\infty} \frac{dy}{1+y^4} = ?$ $\left\{ \text{Result} = \frac{\sqrt{2}.\pi}{4} \right\}$

ii) $\displaystyle\int_0^{2} x.\sqrt[3]{8-x^3}.dx = ?$ $\left\{ \text{Result} = \frac{16\pi}{9\sqrt{3}} \right\}$

VECTOR FIELDS, VECTOR CALCULUS

Vector and Scalar Fields

A function whose domain and range are subsets of R^3 is called a vector field.

$$\vec{F}(x,y,z) = \underbrace{f_1(x,y,z)}\vec{i} + \underbrace{f_2(x,y,z)}\vec{j} + \underbrace{f_3(x,y,z)}\vec{k}$$

Components of vector field
(Scalar values)

If $F_3(x,y,z)=0$ and F_1 and F_2 are independent of z, then $\vec{F}$ reduces to,

$$F(x,y)=F_1(x,y)\vec{i}+F_2(x,y)\vec{j}$$

plane vector field (or vector field in the xy-plane). The position of vector of (x,y,z),

$$\vec{r}(x,y,z)=x\vec{i}+y\vec{j}+z\vec{k}.$$

Gradient, Divergence and Curl (or Rotational (Rot))

✓ $\text{grad}\, f(x,y,z)=\nabla f(x,y,z)=\frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j}+\frac{\partial f}{\partial z}\vec{k}.$ $\left\{ \nabla=\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k} \right.$ (Vector-Differentiate Operator) $\left.\right\}$

↓
Nabla (or Del)

✓ $\text{div}\,\vec{F}=\nabla\cdot\vec{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$

$$=\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)\underbrace{\left(F_1\vec{i}+F_2\vec{j}+F_3\vec{k}\right)}_{\vec{F}}$$

✓ $\text{Curl}\,\vec{F}=\text{Rot}\,\vec{F}=\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}$

$$=\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\vec{i}+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\vec{j}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\vec{k}$$

✓ $\nabla^2=\nabla\cdot\nabla\Rightarrow$ Laplacian Operator $\left(\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$

ϕ : Scalar field

$\vec{F}$: Vector field

$$\nabla^2\cdot\phi=\nabla\cdot\nabla\phi=\text{div}(\text{grad}\,\phi)=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi$$

$$\nabla^2\cdot\vec{F}=\nabla^2\left(F_1\vec{i}+F_2\vec{j}+F_3\vec{k}\right)=\nabla^2F_1\vec{i}+\nabla^2F_2\vec{j}+\nabla^2F_3\vec{k}=\left(\frac{\partial^2F_1}{\partial x^2}+\frac{\partial^2F_1}{\partial y^2}+\frac{\partial^2F_1}{\partial z^2}\right)\vec{i}+\left(\frac{\partial^2F_2}{\partial x^2}+\frac{\partial^2F_2}{\partial y^2}+\frac{\partial^2F_2}{\partial z^2}\right)\vec{j}+\left(\frac{\partial^2F_3}{\partial x^2}+\frac{\partial^2F_3}{\partial y^2}+\frac{\partial^2F_3}{\partial z^2}\right)\vec{k}$$

$\Rightarrow \nabla\cdot(\nabla\times\vec{F})=0$ (div(curl) = 0)

$\Rightarrow \nabla\times(\nabla\phi)=0$ (Curl(grad) = 0)

Irrotational Vector Fields

A vector field $\vec{F}$ is called irrotational in a domain D if $\text{Curl}\,\vec{F}=0$ in D.

$$\vec{F}=\text{grad}\,\phi \iff \text{Curl}\,\vec{F}=0$$

✓ Every conservative vector field is irrotational. (ϕ: Potential Function)

Integration in Vector Fields

Line integrals are used to find the work done by a force in moving an object along a path and to find the mass of curved wire with variable density. Surface integrals are used to find the rate of flow of a fluid across a surface.

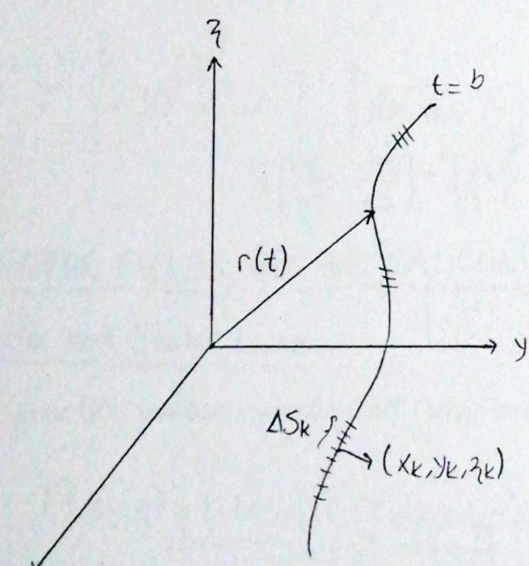
Line Integral

We need to integrate over a curve C rather than over an interval $[a,b]$. We make our definitions for space curves and the curves in the xy-plane.

Suppose that $f(x,y,z)$ is a real valued function. We wish to integrate over the curve C lying within the domain of f and parametrized by,

$$\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k} \quad , a\leqslant t\leqslant b.$$

The values of along the curve are given by $f(x(t),y(t),z(t))$. We are going to integrate this function with respect to arc length from $t=a$ to $t=b$. To begin, we first partition the curve C into a finite number n of subarcs. (Fig. 1)



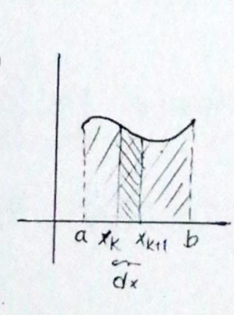
- Figure 1 -

The curve $f(t)$ partitioned into small arcs from $t=b$. The typical subarc is ΔS_k.

In each subarc we choose a point (x_k,y_k,z_k) and from the sum;

$$S_n=\sum_{k=1}^{n} f(x_k,y_k,z_k)\Delta S_k$$

which is similar to Riemann Sum.



$$ds^2=dx^2+dy^2+dz^2$$

If f is continuous and the functions $x(t), y(t), z(t)$ have continuous first order derivatives then these sums approach a limit as n increases and the length ΔS_k approaches to zero.

$n \to \infty$, $\Delta S_k \to 0$.

13 Mart 2015
Cuma

Definition: If f is defined on a curve c given parametrically by,

$$f(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \quad , \quad a \leq t \leq b$$

then the integral of f over c is,

$$\int_c f(x,y,z)\,ds = \lim_{\substack{n \to \infty \\ (\Delta S_k \to 0)}} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta S_k$$

provided the limit exists.

$$\left.\begin{aligned} ds^2 = dx^2 + dy^2 + dz^2 \quad &\Rightarrow \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \\ \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \quad &\Rightarrow \quad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \end{aligned}\right\} \quad ds = \left|\frac{d\vec{r}}{dt}\right| dt$$

How to Evaluate a Line Integral

To integrate a continuous function $f(x,y,z)$ over a curve c,

1) Find a smooth parametrization of c

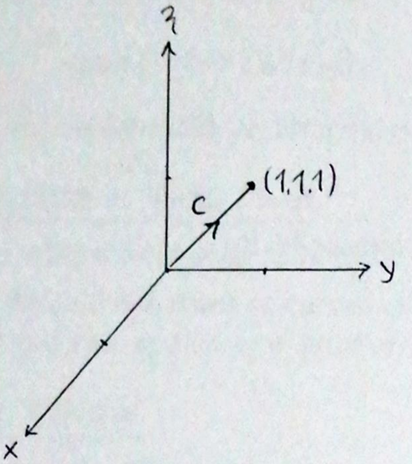
$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k} \quad , \quad a \leq t \leq b$$

2) Evaluate the integral as,

$$\int_c f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \left|\frac{d\vec{r}}{dt}\right| dt$$

* If f has the constant value 1, then the integral of f over c gives the length of the c from $t=a$ to $t=b$ (Fig 1)

Ex: Integrate $f(x,y,z)=x-3y^2+z$ over the line segment c joining the origin to the point $(1,1,1)$.



$$\int_C f(x,y,z)\,ds=\int_a^b f(x(t),y(t),z(t))\left|\frac{d\vec{r}}{dt}\right|dt=?$$

$$c:\begin{cases} x=t \\ y=t \\ z=t \end{cases} \quad \begin{matrix}(0,0,0)\to(1,1,1)\\ t=0 \to t=1\end{matrix} \quad \left\{ \frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}=\frac{z-z_0}{z_1-z_0}=t,\ -\infty<t<\infty \right\}$$

(Parametric form of the line equation)

$$\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$$

$$\vec{r}(t)=t\vec{i}+t\vec{j}+t\vec{k} \Rightarrow \frac{d\vec{r}}{dt}=\vec{i}+\vec{j}+\vec{k}$$

$$\Rightarrow ds=\sqrt{1^2+1^2+1^2}\,dt=\sqrt{3}\,dt \quad (0\leq t\leq 1)$$

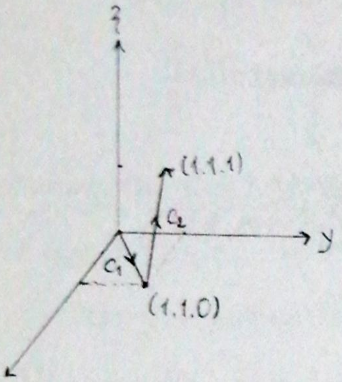
$$f(t,t,t)=t-3t^2+t=2t-3t^2 \Rightarrow \int_0^1 (2t-3t^2)\sqrt{3}\,dt=\sqrt{3}\,(t^2-t^3)\Big|_0^1=0.$$

Additivity

Line integrals have the useful property that if a piecewise smooth curve c is made by joining a finite number of smooth curves $c_1, c_2, \ldots, c_n$ end to end, then the integral of a function over c is the sum of the integrals over the curves that make it up:

$$\int_C f(x,y,z)\,ds=\int_{C_1} f(x,y,z)\,ds_1+\int_{C_2} f(x,y,z)\,ds_2+\ldots+\int_{C_n} f(x,y,z)\,ds_n.$$

Ex: Integrate $f(x,y,z)=x-3y^2+z$ over the line segments c_1 and c_2. $(c_1 \cup c_2)$



$$\int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds_1 + \int_{C_2} f(x,y,z)\,ds_2$$

$C_1: \begin{cases} x=t \\ y=t \\ z=0 \end{cases}$ $(0.0.0) \to (1.1.0)$, $t=0 \to t=1$, $(-\infty<t<\infty)$

$\vec{r_1}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$

$=t\vec{i}+t\vec{j}+0\vec{k}$

$\Rightarrow \dfrac{d\vec{r_1}}{dt}=\vec{i}+\vec{j} \Rightarrow \left|\dfrac{d\vec{r_1}}{dt}\right|=\sqrt{1^2+1^2}=\sqrt{2}$

$\Rightarrow ds_1=\sqrt{2}\,dt$

$C_2: \begin{cases} x=1 \\ y=1 \\ z=t \end{cases}$ $(1.1.0) \to (1.1.1)$, $t=0 \to t=1$, $(-\infty<t<\infty)$

$\vec{r_2}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$

$=\vec{i}+\vec{j}+t\vec{k}$

$\Rightarrow \dfrac{d\vec{r_2}}{dt}=\vec{k} \Rightarrow \left|\dfrac{d\vec{r_2}}{dt}\right|=\sqrt{1^2}=1$

$\Rightarrow ds_2=dt$

$$\int_{C_1} f(x(t),y(t),z(t))\,ds_1=\int_0^1 (t-3t^2)\sqrt{2}\,dt$$

$$=\sqrt{2}\left(\frac{t^2}{2}-t^3\right)\Big|_0^1=-\frac{\sqrt{2}}{2}$$

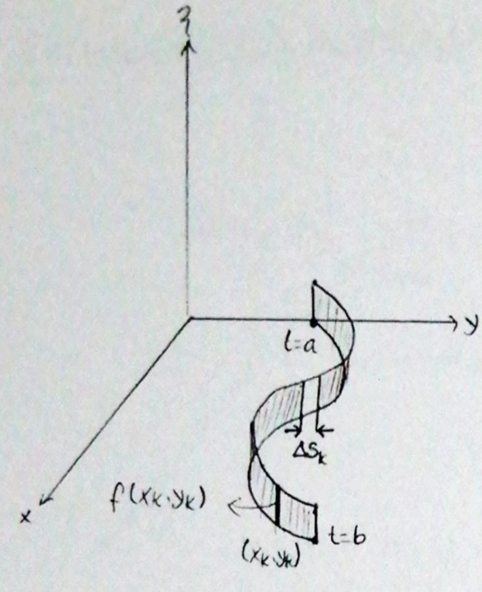
$$\int_{C_2} f(x(t),y(t),z(t))\,ds_2=\int_0^1 (-2+t)\,dt$$

$$=\left(-2t+\frac{t^2}{2}\right)\Big|_0^1=-\frac{3}{2}$$

$$\int_C f(x,y,z)\,ds=-\frac{\sqrt{2}}{2}+\left(-\frac{3}{2}\right)=\frac{-\sqrt{2}-3}{2}.$$

Line Integrals in the Plane

Let C be a smooth curve in the plane and parametrized by $\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}$, $a<t<b$. We generate a cylindrical surface by moving a straight line along c orthogonal to the plane. If $z=f(x,y)$ is nonnegative continuous function over a region in the plane containing the curve C then the graph of f is a surface that lies above the plane



$z = f(x,y) \geqslant 0$ from the definition.

$$\int_C f(x,y,z)\,ds = \lim_{\substack{n \to \infty \\ (\Delta S_k \to 0)}} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta S_k$$

The line integral,

$\int_C f(x,y,z)\,ds$ is the area of the "wall".

Vector Fields and Line Integrals

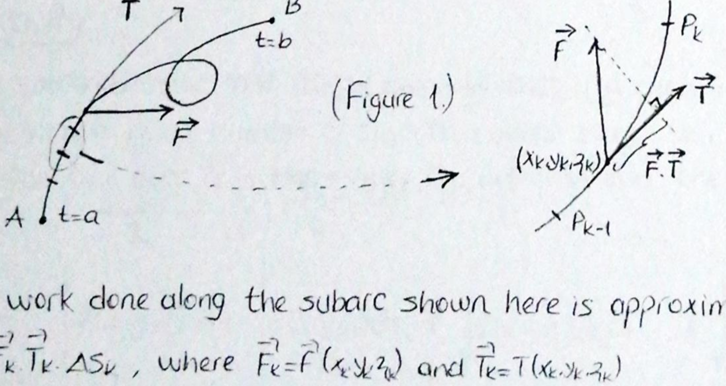
Now, we turn our attention to the idea of a line integral of a vector field $\vec{F}$ along the curve C. Assume that the vector field,

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

has continuous components, and that the curve C has a smooth parametrization,

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}\ ,\quad a \leqslant t \leqslant b.$$

The parametrization $\vec{r}(t)$ defines a forward direction along C. At each point along the path C, the tangent $\vec{T} = \frac{d\vec{r}}{ds}$ is a unit vector tangent to the path and pointing in this forward direction. The line integral of the vector field is the line integral of the scalar tangential component of $\vec{F}$ along C.



(Figure 1.)

The work done along the subarc shown here is approximately,

$\vec{F_k}.\vec{T_k}.\Delta S_k$, where $\vec{F_k} = \vec{F}(x_k, y_k, z_k)$ and $\vec{T_k} = T(x_k, y_k, z_k)$

The tangential component is given the dot product,

$$\vec{F}.\vec{T} = \vec{F}.\frac{d\vec{r}}{ds}$$