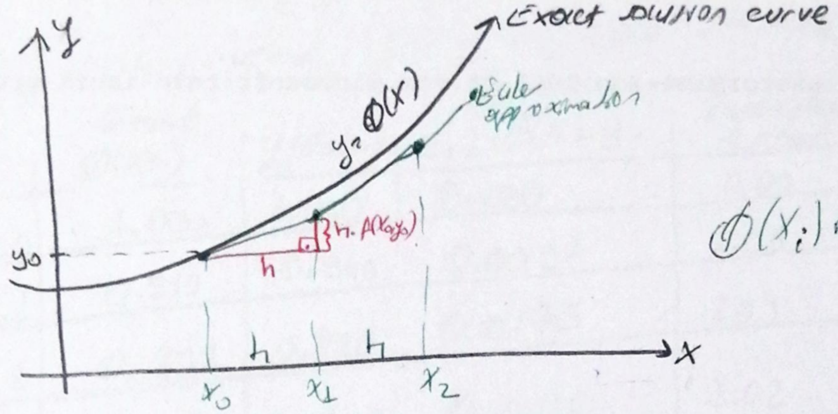


$\phi(x_i) \approx y_i, \ i=0,1,\ldots$

$\boxed{x_{n+1} = x_n + h}$ $\quad \boxed{y_{n+1} = y_n + h\, f(x_n, y_n)}$ $\quad n = 0, 1, \ldots$

EX Use Euler's method to find approximate values for the solution of the initial-value problem

$$y' = x - y, \quad y(0) = 1$$

on the interval $[0,1]$ using

a) 5 steps of size $h = 0.2$

b) 10 steps of size $h = 0.1$

Compare the approximations with the values of the exact solution $y = \phi(x) = x - 1 + 2e^{-x}$

Sol We have $f(x,y) = x - y$, $x_0 = 0$, and $y_0 = 1$

a) $x_{n+1} = x_n + h \longrightarrow y_{n+1} = y_n + h\, f(x_n, y_n) = y_n + h\,(x_n - y_n)$

$x_1 = x_0 + h = 0 + 0.2 = 0.2 \Rightarrow \phi(x_1) = \phi(0.2) \approx y_1 = y_0 + h\, f(x_0, y_0) = \overset{y_0}{1} + \overset{h}{0.2}\,(\overset{x_0 - y_0}{0 - 1}) = \boxed{0.8}_{y_1}$

$x_2 = 0.4 \rightarrow \phi(x_2) = \phi(0.4) \approx y_2 = y_1 + h\, f(x_1, y_1) = \overset{y_1}{0.8} + \overset{h}{0.2}\,(\overset{x_1 - y_1}{0.2 - 0.8}) = 0.68$

Table: Euler's method for $y' = x - y$, $y(0) = 1$ with $h = 0.2$ $\quad \dfrac{|\text{Exact v.} - \text{app. v.}|}{|\text{Exact v.}|} \times 100\%$

n (i)	x_n (x_i)	Exact solution	Euler approximation	Actual Error $e_n = \phi(x_n) - y_n$	Percentage Error	Error Bound
0	$x_0 = 0.0$	1.00	$y_0 = 1.00$	0.000	0.0	0.000
1	$x_1 = 0.2$	0.837	$y_1 = 0.800$	0.037	4.5	0.044
2	$x_2 = 0.4$	0.741	$y_2 = 0.680$	0.061	8.2	0.098
3	$x_3 = 0.6$	0.698	$y_3 = 0.624$	0.074	10.6	0.164
4	$x_4 = 0.8$	0.699	$y_4 = 0.619$	0.079 / 80	11.4	0.245
5	$x_5 = 1.0$	0.736	$y_5 = 0.655$	$\boxed{0.080 / 1} \rightarrow e_5$	10.9 / 11.0	0.344

Table: Euler's method for $y' = x - y$, $y(0) = 1$ with $h = 0.1$

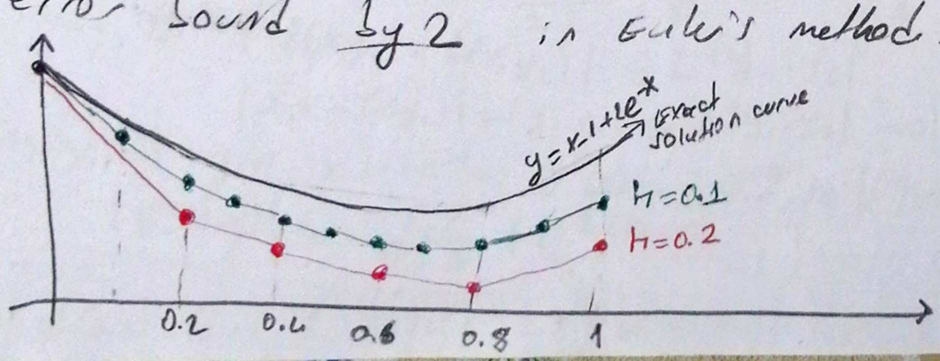
b)

n	x_n	Exact $\phi(x_n)$	y_n (Euler)	(Error) $e_n = \phi(x_n) - y_n$	Percentage error
0	0.0	1.000	1.000	0.000	0.00
1	0.1	0.910	0.900	0.0097	1.06
2	0.2	0.837	0.820	0.0175	2.09
3	0.3	0.782	0.758	0.0236	3.02
4	0.4	0.741	0.712	0.0284	3.84
5	0.5	0.713	0.681	0.0321	4.50
6	0.6	0.698	0.663	0.0347	4.98
7	0.7	0.693	0.657	0.0366	5.28
8	0.8	0.699	0.661	0.0377	5.40
9	0.9	0.713	0.675	0.0383	5.37
10	1.0	0.736	0.697	$\boxed{0.0384} = e_{10}$	5.22

For comparison, using $n=5$ in Euler's method yields a value of $\phi(x_5) \approx y_5 = \underline{0.655}$ as approximation for the solution at $x = 1.0$, whereas using $n = 10$ gives $\phi(x_{10}) \approx y_{10} = \underline{0.697}$. Thus, larger values for n (or smaller values for h) give more accurate results in general because tangent-line approximations are more accurate for smaller increments of x.

* Note that doubling the value of n divides the error bound by 2 in Euler's method. $\left(\underset{0.0384}{e_{10}} \approx \frac{\overset{0.080}{e_5}}{2} \right)$

error at $x = 1.0$



Error Bounds for Euler's Method:

Suppose $y' = f(x,y)$ is continuous and satisfies a Lipschitz condition with constant L on

$$R = \{(x,y) : x_0 \le x \le b \text{ and } -\infty < y < \infty\}$$

and that a constant M exists with

$$|y''(x)| \le M, \text{ for all } x \in [x_0, b],$$

where $y(x) = \phi(x)$ denotes the unique solution to the IVP

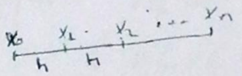
$$y' = f(x,y) \text{ with } y(x_0) = y_0, \quad x_0 \le x \le b.$$

Let $y_0, y_1, \ldots, y_n$ be the approximate values generated by Euler's method for some positive integer n.

Then, for each $i = 0, 1, 2, \ldots, n$ we have that

$$\boxed{|\phi(x_i) - y_i| \le \frac{h.M}{2.L}\left(e^{L(x_i - x_0)} - 1\right)}$$

where $h = \frac{b - x_0}{n}$ and $x_i = x_0 + i.h$



Ex Consider the IVP $y' = x - y$, $y(0) = 1$, $0 \le x \le 1$, with exact solution $y(x) = \phi(x) = x - 1 + 2e^{-x}$.
For the approximate values obtained by Euler's method with $h = 0.2$, compare the actual error at each step to the error bound.

Solution

$$|\phi(x_i) - y_i| \le \frac{h.M}{2.L}\left(e^{L(x_i - x_0)} - 1\right)$$

• Finding L: $|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2|$

$$|x - y_1 - (x - y_2)| = |y_2 - y_1| \le 1|y_1 - y_2| \Rightarrow \boxed{L = 1}$$

• Finding M: $y(x) = x - 1 + 2e^{-x} \Rightarrow y''(x) = 2e^{-x}$, so $|f''(x)| = |2e^{-x}| \le \boxed{2} = M$, $0 \le x \le 1$

$$|\phi(x_i) - y_i| \le \frac{0.2.2}{2.1}\left(e^{1.(x_i - 0)} - 1\right) = \boxed{0.2\,(e^{x_i} - 1)}$$

• For $x_1 = 0.2$

$$|\phi(0.2) - y_1| \le 0.2.(e^{0.2} - 1) = 0.044$$

• For $x_2 = 0.4$ $\quad |\phi(x_2) - y_2| \le 0.2(e^{0.4} - 1) = 0.098$

x_i	0.2	0.4	0.6	0.8	1.0
Actual error	0.037	0.061	0.074	0.079	0.080
Error bound	0.044	0.098	0.164	0.245	0.344

EX Find a bound on the step size h to guarantee that the error in using Euler's method to solve

$$y' = e^{-2y}, \quad y(0) = 0$$

Over $0 \leq x_n \leq 1$ is less than $\varepsilon = 0.001$.

Sol Recall the upper bound for error is given by:

$$|\phi(x_i) - y_i| \leq \frac{h \cdot M}{2 \cdot L}\left(e^{L(x_i - x_0)} - 1\right)$$

For $\cancel{x_i =}$ $x_n = 1$ ($i = n$), $|\phi(x_n) - y_n| \leq \underbrace{\frac{h \cdot M}{2L}\left(e^{L(1-0)} - 1\right) \leq 0.001}_{h = ?}$

Finding L: $L = \max\limits_{y \geq 0} |f_y(x,y)| = \max\limits_{y \geq 0} |-2e^{-2y}| = 2.$

Finding M: Since $y'' = e^{-2y}(-2y') = -2e^{-4y}$, we have $|y''(x)| = |-2e^{-4y}| \leq 2 = M$ $(0 \leq x \leq 1)$

$$|\phi(x_n) - y_n| \leq \frac{h \cdot 2}{2 \cdot 2}\left(e^{2(1-0)} - 1\right) \leq 0.001$$

$$h \leq \frac{0.002}{e^2 - 1} \cong 0.0003130$$

$h = \frac{b - x_0}{n} \Rightarrow n = \frac{1-0}{h}$

$(n \geq 3195)$

$n \cong 3194{,}5$

$n \cong 3195$

EX Find a bound on the step size h guarantee that the error in using Euler's method to solve

$$y' = \sin(y), \quad y(0) = \frac{\pi}{2}$$

is less than $\varepsilon = 0.001$. on $0 \leq x_n \leq \ln 2$

$n \geq 17\cancel{7}^{4}$

$n = \frac{\ln 2 - 0}{0.004} \cong 173$

Sol $y'' = \cos(y) \cdot y' = \cos(y) \underbrace{\sin(y)}_{y'} = \frac{1}{2}\sin(2y) \Rightarrow |y''(x)| \leq \frac{1}{2} = M$

$f_y(x,y) = \cos(y) \Rightarrow |\cos y| \leq 1 = L$

$h = \frac{b - x_0}{n} \Rightarrow n = \frac{b - x_0}{h}$?

$n \geq 250$

$$|\phi(x_n) - y_n| \leq \frac{h \cdot (\frac{1}{2})}{2 \cdot 1}\left(e^{1(\ln 2 - 0)} - 1\right) = \frac{h \cdot \frac{1}{2}}{2 \cdot 1}(2 - 1) = \frac{h}{4} \leq 0.001$$

$h \cong 0.004$