Definition: Let $\vec{F}$ be a vector field with continuous components defined along a smooth curve C parametrized by $\vec{r}(t)$ $a\leq t\leq b$. Then the line integral of $\vec{F}$ along C is,

$$\int_C \vec{F}.\vec{T}.ds = \int_C \vec{F}.\left(\frac{d\vec{r}}{ds}\right)ds = \int_C \vec{F}.d\vec{r}$$

Evaluating the Line Integral

$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ along,

$C: \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a\leq t\leq b$.

1) Express the vector field $\vec{F}$ in terms of the parametrized curve C as $\vec{F}(\vec{r}(t))$.
2) Find the derivative (velocity) vector $d\vec{r}/dt$.
3) Evaluate the line integral wrt parameter t, $a\leq t\leq b$ to obtain,

$$\int_C \vec{F}.d\vec{r} = \int_{t=a}^{t=b} \vec{F}(\vec{r}(t))\frac{d\vec{r}}{dt}dt.$$

Ex:

Evaluate $\int_C \vec{F}.d\vec{r}$ where,

$\vec{F}(x,y,z) = z\vec{i} + xy\vec{j} - y^2\vec{k}$ along,

$C: \vec{r}(t) = t^2\vec{i} + t\vec{j} + \sqrt{t}\vec{k}$, $0\leq t\leq 1$.

$$\int_C \vec{F}.d\vec{r} = \int_{t=a}^{b} \vec{F}(\vec{r}(t)).\frac{d\vec{r}}{dt}dt \Rightarrow \vec{r}(t) = \underbrace{t^2}_{x(t)}\vec{i} + \underbrace{t}_{y(t)}\vec{j} + \underbrace{\sqrt{t}}_{z(t)}\vec{k} \Rightarrow \frac{d\vec{r}}{dt} = 2t\vec{i} + \vec{j} + \frac{1}{2\sqrt{t}}\vec{k}$$

$$\Rightarrow \vec{F}(\vec{r}(t)) = \sqrt{t}\vec{i} + t^3\vec{j} - t^2\vec{k}$$

$$\Rightarrow \vec{F}(\vec{r}(t)).\frac{d\vec{r}}{dt} = 2t\sqrt{t} + t^3 - \frac{t^2}{2\sqrt{t}}$$

$$\Rightarrow \int_{t=a}^{b} \vec{F}(\vec{r}(t))\frac{d\vec{r}}{dt}dt = \int_{t=0}^{1}\left(2t\sqrt{t} + t^3 - \frac{t^2}{2\sqrt{t}}\right)dt = \int_0^1 \left(2t^{3/2} + t^3 - 1/2\, t^{3/2}\right)dt$$

$$= \left(\frac{3}{2}\cdot\frac{t^{5/2}}{5/2} + \frac{t^4}{4}\right)\Big|_0^1 = \frac{17}{20}.$$

Line Integrals with respect to the xyz-coordinates

Its sometimes useful to write a line integral of a scalar function with respect to one of the coordinates, such as,

$$\int_C P\,dx \quad \text{if} \quad \vec{F}(x,y,z) = P(x,y,z)\vec{i}$$

over the curve c parametrized by,

$\vec{r}(t) = x(t)\vec{i}$, $a \le t \le b$.

✓ $\vec{F}\cdot\underbrace{\frac{d\vec{r}}{dt}}\,dt = \vec{F}\left(\frac{dx}{dt}\vec{i}\right)dt = P.\underbrace{x'dt}_{dx} \Rightarrow \int_C \vec{F}d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)).\frac{d\vec{r}}{dt}dt = \int_C P\,dx$

$\left\{ \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} \right\}$

✓ $\vec{F}(x,y,z) = Q(x,y,z)\vec{j}$, $c: \vec{r}(t) = y(t)\vec{j} \Rightarrow \int_C \vec{F}d\vec{r} = \int_C Q\,dy$

✓ $\vec{F}(x,y,z) = R(x,y,z)\vec{k}$, $c: \vec{r}(t) = z(t)\vec{k} \Rightarrow \int_C \vec{F}d\vec{r} = \int_C R\,dz$

$$\int_C \vec{F}d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz \,.$$

Ex: Evaluate the line integral,

$$\int_C -y\,dx + z\,dy + 2x\,dz \text{ , where } c \text{ is the helix } \vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}, \; 0 \le t \le 2\pi.$$

$$\left.\begin{array}{l} x = \cos t \Rightarrow dx = -\sin t\,dt \\ y = \sin t \Rightarrow dy = \cos t\,dt \\ z = t \Rightarrow dz = dt \end{array}\right\} 0 \le t \le 2\pi. \qquad \left\{ \text{Helix}, \; x = \cos t,\; y = \sin t,\; z = t \right\}$$

$$\int_C -y\,dx + z\,dy + 2x\,dz = \int_{t=0}^{2\pi} (-\sin t)(-\sin t\,dt) + t(\cos t\,dt) + 2(\cos t)(dt) = \int_{t=0}^{2\pi} (\sin^2 t + t\cos t + 2\cos t)\,dt = \pi \;✓$$

$$\left\{ \begin{array}{l} \vec{F} = P\vec{i} + Q\vec{j} + R\vec{k} \Rightarrow \vec{F} = -y\vec{i} + z\vec{j} + 2x\vec{k} \\ \vec{r}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k} \end{array} \right\} \rightarrow \int_C \vec{F}.d\vec{r} = \int_C -y\,dx + z\,dy + 2x\,dz \;✓ \Big\}$$

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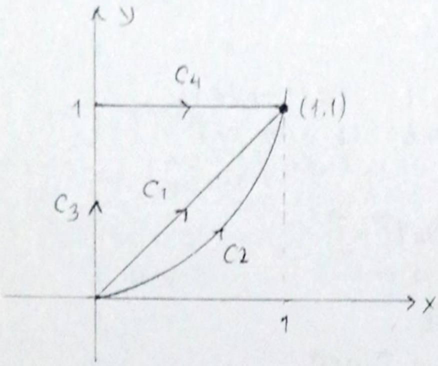
Çarşamba

Ex: Let $F(x,y)=y^2\vec{i}+2xy\vec{j}$ Evaluate the line integral $\int_C \vec{F}d\vec{r}$ from $(0,0)$ to $(1,1)$.

i) the straight line $y=x$,
ii) the curve $y=x^2$ and
iii) the piece smooth consisting of the straight line segments from $(0,0)$ to $(0,1)$ and from $(0,1)$ to $(1,1)$.



i) $C_1: \begin{cases} y=x & (0,0)\longrightarrow(1,1) \\ x=t, & t=0 \rightarrow t=1 \\ y=t \end{cases} \Rightarrow \begin{array}{l} \vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k} \\ \displaystyle\int_C \vec{F}.d\vec{r}=\int_{C_1}\vec{F}d\vec{r}=\int_{t=0}^{1}\left(\vec{F}(\vec{r}(t))\frac{d\vec{r}}{dt}\right)dt \end{array}$

$\left.\begin{array}{l} \dfrac{d\vec{r}}{dt}=\vec{i}+\vec{j} \\ \vec{F}(\vec{r}(t))=t^2\vec{i}+2t^2\vec{j} \end{array}\right\} \Rightarrow \vec{F}(\vec{r}(t)).\dfrac{d\vec{r}}{dt}=3t^2 \Rightarrow \displaystyle\int_0^1 3t^2dt=t^3\Big|_0^1=1.$

2^{nd} method:

$C_1: \{y=x \Rightarrow dy=dx$

$\int_C \vec{F}d\vec{r}=\int_C y^2dx+2xydy$

$\left.\begin{array}{l} \Rightarrow \vec{r}(x,y)=x\vec{i}+y\vec{j}=x\vec{i}+x\vec{j} \Rightarrow \dfrac{d\vec{r}}{dt}=\vec{i}+\vec{j} \\ \Rightarrow \vec{F}(\vec{r}(x))=x^2\vec{i}+2x^2\vec{j} \end{array}\right\} \Rightarrow \displaystyle\int_{x=0}^{1}3x^2dx=x^3\Big|_0^1=1.$

ii)

$$C_2: \begin{cases} y=x^2 & (0,0)\to(1,1) \\ x=t & , \quad t=0 \to t=1 \\ y=t^2 \end{cases}$$

$$\left.\begin{aligned} &\vec{r}(t)=t\vec{i}+t^2\vec{j} \\ &\Rightarrow \frac{d\vec{r}}{dt}=\vec{i}+2t\vec{j} \\ &\vec{F}(\vec{r}(t))=t^4\vec{i}+2t^3\vec{j} \end{aligned}\right\} \vec{F}(\vec{r}(t))\frac{d\vec{r}}{dt}=t^4+4t^4=5t^4 \Rightarrow \int_{t=0}^{1} 5t^4\,dt = 1.$$

iii)

$$C_3: \begin{cases} x=0 & (0,0)\to(0,1) \\ x=0 & , \quad t=0 \to t=1 \\ y=t \end{cases} \qquad C_4: \begin{cases} y=1 & , \quad (0,1)\to(1,1) \\ x=t & , \quad t=0\to t=1 \end{cases}$$

$$\vec{r}(t)=0\vec{i}+t\vec{j}=t\vec{j} \qquad \vec{r}(t)=t\vec{i}+\vec{j}$$

$$\Rightarrow \frac{d\vec{r}}{dt}=\vec{j} \qquad \Rightarrow \frac{d\vec{r}}{dt}=\vec{i}$$

$$\vec{F}(\vec{r}(t))=t^2\vec{i} \qquad \vec{F}(\vec{r}(t))=\vec{i}+2t\vec{j}$$

$$\vec{F}(\vec{r}(t))\frac{d\vec{r}}{dt}=0 \qquad \vec{F}(\vec{r}(t))\frac{d\vec{r}}{dt}=1$$

$$\Rightarrow \int_{C_3}\vec{F}.d\vec{r}=\int_{t=0}^{1}0\,dt=0 \qquad \Rightarrow \int_{C_4}\vec{F}d\vec{r}=\int_{t=0}^{1}1\,dt=1$$

$$\int_C \vec{F}d\vec{r}=\int_{C_3}\vec{F}d\vec{r}+\int_{C_4}\vec{F}d\vec{r} \qquad \{C=C_3\cup C_4\}$$

$$=0+1=1.$$

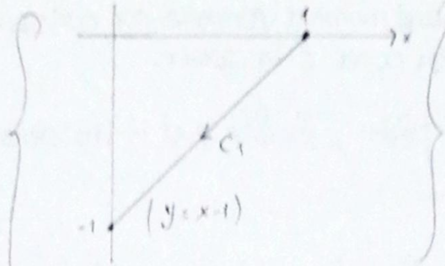
Ex: Let $\vec{F}=y\vec{i}-x\vec{j}$. Find $\int_C \vec{F}d\vec{r}$ from $(1,0)$ to $(0,-1)$.

a) the straight line segment joining the points

b) three quarters of the circle of the unit radius centered at the origin and traversed counterclockwise.

a)

$$C_1: \begin{cases} y = x-1 \\ x = t \\ y = t-1 \end{cases}, \quad (1,0) \to (0,-1), \quad t=1 \to t=0$$



$$\vec{r}(t) = t\vec{i} + (t-1)\vec{j}$$

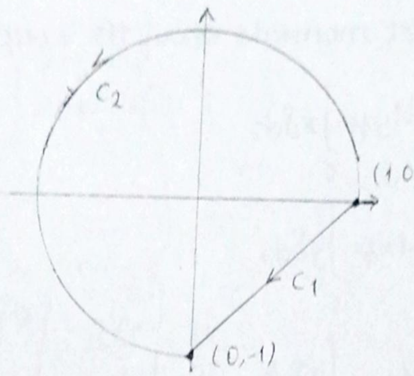
$$\Rightarrow \frac{d\vec{r}}{dt} = \vec{i} + \vec{j}$$

$$\vec{F}(\vec{r}(t)) = (t-1)\vec{j} - t\vec{i} = t\vec{j} - t\vec{i} - \vec{j}$$

$$\vec{F}(\vec{r}(t))\frac{d\vec{r}}{dt} = -t + t - 1 = -1 \Rightarrow \int_{C_1} \vec{F}\,d\vec{r} = \int_{t=1}^{0} -1\,dt = 1$$

b)

$$C_2: \begin{cases} x^2+y^2=1 \\ \left\{ \begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix} \right\} \to \theta = t,\ r=1 \\ x = \cos t \\ y = \sin t \end{cases}, \quad 0 \le t \le \frac{3\pi}{2}$$



$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\vec{F}(\vec{r}(t)) = \sin t\,\vec{i} - \cos t\,\vec{j}$$

$$\vec{F}(\vec{r}(t))\frac{d\vec{r}}{dt} = -\sin^2 t - \cos^2 t = -1 \Rightarrow \int_{C_2} \vec{F}\,d\vec{r} = \int_{0}^{3\pi/2} -1\,dt = \frac{-3\pi}{2}.$$

Remark:

$$\int_C f(x,y,z)\,ds = \int_{-C} f(x,y,z)\,ds \ ;$$

$$\int_C \vec{F}\,d\vec{r} = -\int_C \vec{F}\,d\vec{r} \quad \{ c = -c,\ \text{counter} \}$$

Mass and Moment Calculations

We treat coil springs and wires as masses distributed along smooth curves in space. The distribution is described by a continuous density function $\delta(x,y,z)$ represently mass per unit length. The curve,

$$C: \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, \quad a \le t \le b$$

$$ds = \left|\frac{d\vec{r}}{dt}\right| dt.$$

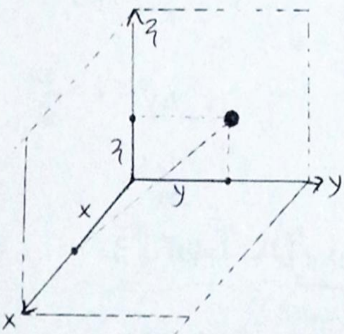
Table: Mass and moment formulas for coil springs, wires and thin rods lying along smooth curve C in space.

Mass: $M = \int_C \delta \, ds$, $\delta = \delta(x,y,z)$ is the density at (x,y,z).

$$\left\{ \begin{array}{ccccc} \underline{3D} & & \underline{2D} & & \underline{1D} \\ \delta = \dfrac{m}{V} & \rightarrow & \delta = \dfrac{m}{A} & \rightarrow & \delta = \dfrac{m}{\text{length}(ds)} \end{array} \right\}$$

* First moments about the coordinate planes;

$$\left. \begin{array}{l} M_{yz} = \int_C x \delta \, ds \\ M_{xz} = \int_C y \delta \, ds \\ M_{xy} = \int_C z \delta \, ds \end{array} \right\} \rightarrow$$



{ x is the perpendicular length to yz-plane. }

* Coordinates of the center of mass;

$$\bar{x} = \frac{M_{yz}}{M} , \quad \bar{y} = \frac{M_{xz}}{M} , \quad \bar{z} = \frac{M_{xy}}{M} .$$

* Moments of inertia about axes;

$$I_x = \int_C (y^2+z^2) \delta \, ds , \quad I_y = \int_C (x^2+z^2) \delta \, ds , \quad I_z = \int_C (x^2+y^2) \delta \, ds .$$

Ex 1: Evaluate $\int_C y \, ds$, where c is the parabola path $y^2 = 2x$ from $(0,0)$ to $(4, \sqrt{8})$.

Ex 2: Evaluate $\int_C (x+yz) ds$, where c is the polygonal path from $(0,0,2)$ to $(2,0,2)$ to $(1,1,1)$.

Ex 3: Evaluate $\int_C x^2 ds$, where c is the line of intersection of the planes $x-y+z=0$ and $x+y+3z=0$ from $(0,0,0)$ to $(3,1,-2)$.

Ex 4: Evaluate $\int_C z \, ds$, where c is the curve of intersection of the sphere $x^2+y^2+z^2=1$ and the plane $x+y=1$. $(z \geq 0)$.

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Sol 1:

$$C: \begin{cases} x=t \\ y=\sqrt{2t} \end{cases}, \quad \begin{matrix} (0,0) \to (4,\sqrt{8}) \\ t=0 \to t=4 \end{matrix}$$

$$ds = \left|\frac{d\vec{r}}{dt}\right| dt$$

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} = t\vec{i} + \sqrt{2t}\,\vec{j}$$

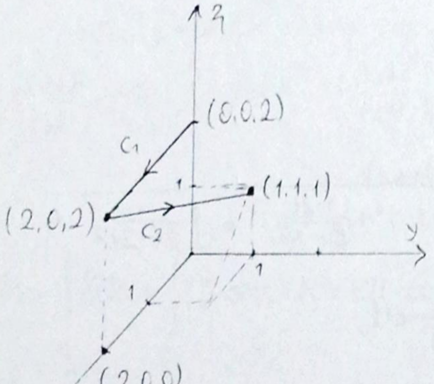
$$\frac{d\vec{r}}{dt} = \vec{i} + \frac{2}{2\sqrt{2t}}\vec{j} \Rightarrow \left|\frac{d\vec{r}}{dt}\right| = \sqrt{1^2 + \left(\frac{1}{\sqrt{2t}}\right)^2} = \frac{\sqrt{2t+1}}{\sqrt{2t}}.$$

$$I = \int_C y\,ds = \int_{t=0}^{4} \sqrt{2t}\cdot\frac{\sqrt{2t+1}}{\sqrt{2t}}\cdot dt = \int_0^4 \sqrt{2t+1}\,dt \qquad \left\{\begin{matrix} 2t+1 = u^2 \\ 2dt = 2u\,du \end{matrix}\right\}$$

$$= \int u(u\,du) = \frac{u^3}{3}$$

$$I = \left.\frac{(2t+1)^{3/2}}{3}\right|_0^4 = \frac{27-1}{3} = \frac{26}{3}.$$

Sol 2: $(0,0,2) \xrightarrow[C_1]{} (2,0,2) \xrightarrow[C_2]{} (1,1,1)$



$$C_1: \begin{cases} \vec{r} - \vec{r_0} = t\vec{v} \\ (x-0, y-0, z-2) = t(2-0, 0-0, 2-2) \\ (x, y, z-2) = (2t, 0, 0) \\ x = 2t \\ y = 0 \qquad , \quad t=0 \to t=1 \\ z = 2 \end{cases}$$

$$\vec{r}(t) = 2t\vec{i} + 0\vec{j} + 2\vec{k}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = 2\vec{i} \Rightarrow \left|\frac{d\vec{r}}{dt}\right| = 2 \Rightarrow ds = \left|\frac{d\vec{r}}{dt}\right|\cdot dt = 2dt$$

$$\Rightarrow \int_C (x+yz)\,ds = \int_{C_1} (x+yz)\,ds + \int_{C_2} \cdots = \int_0^1 (2t+0.2)2dt = 2t^2\Big|_0^1 = 2 + \int_{C_2} \cdots .$$

Sol3: $\left.\begin{aligned} x-y+z=0 \\ x+y+2z=0 \end{aligned}\right\}$ intersection curve

$$\left.\begin{aligned} 2x+3z=0 \\ 2y+z=0 \end{aligned}\right\} \Rightarrow \begin{aligned} x&=3t \\ y&=t \\ z&=-2t \end{aligned} \quad , \quad \begin{aligned} (0,0,0) &\rightarrow (3,1,-2) \\ t=0 &\rightarrow t=1 \end{aligned}$$

$$\vec{r}(t)=3t\vec{i}+t\vec{j}+(-2t)\vec{k}$$

$$\Rightarrow \frac{d\vec{r}}{dt}=3\vec{i}+\vec{j}-2\vec{k} \Rightarrow \left|\frac{d\vec{r}}{dt}\right|=\sqrt{3^2+1^2+(-2)^2}=\sqrt{14}$$

$$\Rightarrow ds=\sqrt{14}\,dt$$

$$\Rightarrow \int_C x^2\,ds=\int_0^1 (3t)^2\cdot\sqrt{14}\,dt=9\sqrt{14}\,\frac{t^3}{3}\Big|_0^1=3\sqrt{14}.$$

Sol4: $\left.\begin{aligned} x^2+y^2+z^2=1 \\ x+y=1 \end{aligned}\right\} \Rightarrow x^2+(1-x)^2+z^2=1$

$$x^2+1-2x+x^2+z^2=1 \Rightarrow z^2=2x-2x^2$$

$$z=\mp\sqrt{2x-2x^2}$$

$$z\geq 0 \Rightarrow z=\sqrt{2x-2x^2}$$

$$\Rightarrow C: \begin{cases} x=t \\ y=1-t \\ z=\sqrt{2t-2t^2} \end{cases} , \quad 0\leq t\leq 1$$

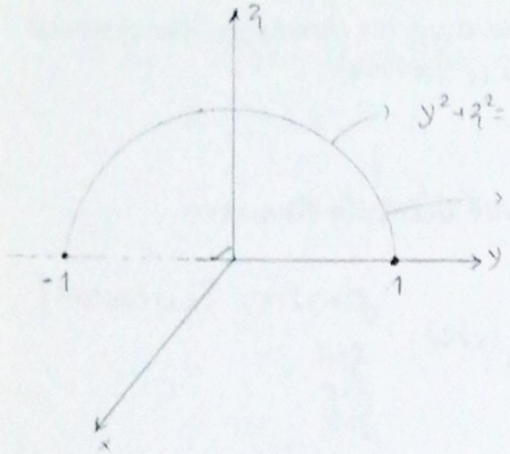
$$\vec{r}(t)=t\vec{i}+(1-t)\vec{j}+\sqrt{2t-2t^2}\,\vec{k}$$

$$\Rightarrow \frac{d\vec{r}}{dt}=\vec{i}-\vec{j}+\frac{2-4t}{2\sqrt{2t-2t^2}}\vec{k} \Rightarrow \left|\frac{d\vec{r}}{dt}\right|=\sqrt{1^2+(-1)^2+\left(\frac{1-2t}{\sqrt{2t-2t^2}}\right)^2}=\sqrt{\frac{1}{2t-2t^2}}$$

$$\Rightarrow ds=\frac{1}{\sqrt{2t-2t^2}}\,dt$$

$$\Rightarrow \int_C z\,ds=\int_0^1 \frac{\sqrt{2t-2t^2}}{\sqrt{2t-2t^2}}\,dt=1.$$

Ex: A slender metal arch, denser at the bottom than top, lies along the semicircle $y^2+z^2=1$, $z\geq 0$ in the yz-plane. Find the center of the arch's mass if the density at the (x,y,z) on the arch is $\delta(x,y,z)=2-z$.



$y^2+z^2=1 \quad (z \geq 0)$

slender metal arch $(\delta = 2-z)$

$\bar{x}=0$

$\bar{y}=0$

$\bar{z}=?$

$$M=\int_C \delta\, ds \quad , \quad M_{xy}=\int_C z\,\delta\, ds \quad , \quad \bar{z}=\frac{M_{xy}}{M}.$$

$$C:\begin{cases} y^2+z^2=1 \\ \left\{\begin{matrix} x=r\cos t \\ y=r\sin t \end{matrix}\right. \text{ polar cc} \left.\vphantom{\begin{matrix} x \\ y\end{matrix}}\right\} \to \theta=t,\ r=1 \\ y=\cos t \\ z=\sin t \end{cases} \quad , \quad 0 \leq t \leq \pi$$

$$\vec{r}(t)=y(t)\vec{j}+z(t)\vec{k}$$
$$=\cos t\,\vec{j}+\sin t\,\vec{k}$$

$$\Rightarrow \frac{d\vec{r}}{dt}=-\sin t\,\vec{j}+\cos t\,\vec{k} \Rightarrow \left|\frac{d\vec{r}}{dt}\right|=1$$

$$\Rightarrow ds=dt$$

$$M=\int_C \delta\, ds=\int_0^{\pi}(2-\sin t)\,dt=(2t+\cos t)\Big|_0^{\pi}=(2\pi-1)-(0+\cos 0)=2\pi-2.$$

$$M_{xy}=\int_C z\,\delta\, ds=\int_0^{\pi}\sin t\,(2-\sin t)\,dt=\int_0^{\pi}(2\sin t-\sin^2 t)\,dt \quad \left\{\begin{matrix} \cos 2t=1-2\sin^2 t \\ \Rightarrow \sin^2 t=\dfrac{1-\cos 2t}{2} \end{matrix}\right\}$$

$$=\left(-2\cos t-\frac{1}{2}\left(t-\frac{1}{2}\sin t\right)\right)\Big|_0^{\pi}=\left(2-\frac{1}{2}(\pi-0)\right)-\left(-2\cdot 1-\frac{1}{2}(0-0)\right)=\frac{8-\pi}{2}.$$

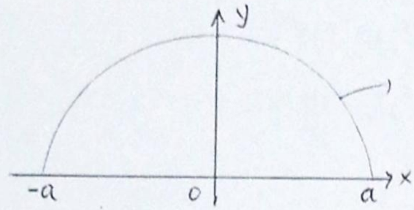
$$\Rightarrow \bar{z}=\frac{\frac{8-\pi}{2}}{2\pi-2}=\frac{8-\pi}{4\pi-4}\cong 0.57.$$

Ex: The mass density of semicircular wire of radius a varries directly as the distance from the diameter that joins the two endpoints of the wire.

a) Find the mass of the wire.

b) Locate the center of mass.

c) Determine the moment of inertia of the wire about the diameter.



$c: x^2+y^2=a^2 \quad (y \geq 0)$

$\delta(x,y)=ky$ {k is constant}

$\bar{z}=0$

$\bar{x}=0$

$\bar{y}=?$

a) $M=\int_C \delta ds$, $M_{xz}=\int_C y\delta ds$

$$c: \begin{cases} x=a\cos t \\ y=a\sin t \end{cases}, \quad 0 \leq t \leq \pi$$

$\vec{r}(t)=a\cos t\,\vec{i}+a\sin t\,\vec{j}$

$\Rightarrow \dfrac{d\vec{r}}{dt}=-a\sin t\,\vec{i}+a\cos t\,\vec{j} \Rightarrow \left|\dfrac{d\vec{r}}{dt}\right|=a$

$\Rightarrow ds=a\,dt$

$$M_{xz}=\int_C y\delta ds=\int_0^\pi (a\sin t)(ka\sin t)a\,dt=ka^3\int_0^\pi \sin^2 t\,dt=\dots$$

$$M=\int_C \delta ds=\int_0^\pi (ka\sin t)a\,dt=2ka^2.$$

b) $\bar{y}=\dfrac{M_{xz}}{M}=\dfrac{\pi a}{4}.$

c) $I_x=\int_C (y^2+\underbrace{z^2}_{z=0})\delta ds = I_x=\int_C y^2\delta ds$

$$\Rightarrow I_x = \int_C y^2 \delta\, ds = \int_0^{\pi} (a\sin t)^2 (ka\sin t)\, a\, dt$$

$$= ka^4 \int_0^{\pi} \sin^3 t\, dt = ka^4 \int_0^{\pi} (1-\cos^2 t)\sin t\, dt \qquad \left\{ \begin{array}{l} \cos t = u \\ -\sin t\, dt = du \end{array} \right\}$$

$$= ka^4 \int (1-u^2)(-du)$$

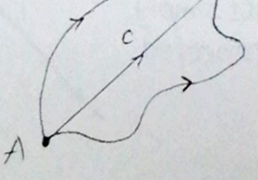
$$= ka^4 \left(\frac{u^3}{3} - u\right) = ka^4 \left(\frac{\cos^3 t}{3} - \cos t\right)\Big|_0^{\pi}$$

$$= ka^4 \left(\left(-\frac{1}{3}+1\right) - \left(\frac{1}{3}-1\right)\right) = \frac{4ka^4}{3}.$$

Path Independence, Conservative Fields and Potential Functions

Path Independence

Definition: Let $\vec{F}$ be a vector field defined on an open region D in space and suppose that for any two points A and B in the line integral $\int_C \vec{F} d\vec{r}$ along a path c from A to B in D is the same over all paths from A to B. Then the integral $\int_C \vec{F} d\vec{r}$ is path independent in D and the field $\vec{F}$ is conservative on D. When a line integral is independent of the path c from point A to point B, we sometimes represent the integral by the symbol $\int_A^B$ rather than the usual line integral symbol $\int_C$.



$\left\{ \text{If path independence, then } \operatorname{Curl} \vec{F} = 0 \right\}$

$\text{Curl}\,\vec{F}\,(\text{Rot}\,\vec{F}) = 0 \Rightarrow \vec{F} = \nabla \phi \quad \left\{ \phi \Big|_A^B = \dots \right\}$

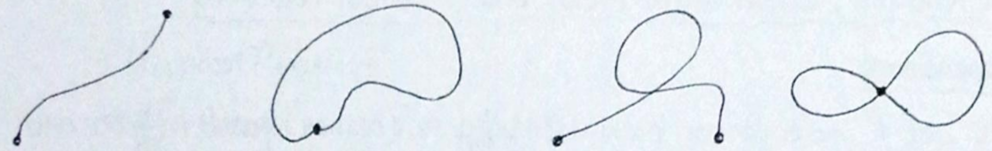
$\text{Curl}\,(\text{grad}) = 0$

$\nabla \times (\underbrace{\nabla \phi}_{\vec{F}}) = 0$

Definition: If $\vec{F}$ is a vector field on D and $\vec{F} = \nabla \phi$ for some scalar function ϕ on D, then ϕ is called a potential function for $\vec{F}$.

Assumptions on Curves, Vector Fields and Domains

A curve in the xy-plane is sample if it does not cross itself. When a curve starts and ends at the same point, it is closed curve or loop (Fig. 1).



simple not closed | simple closed | not simple not closed | not simple closed

Fig 1 - Distinguishing Curves

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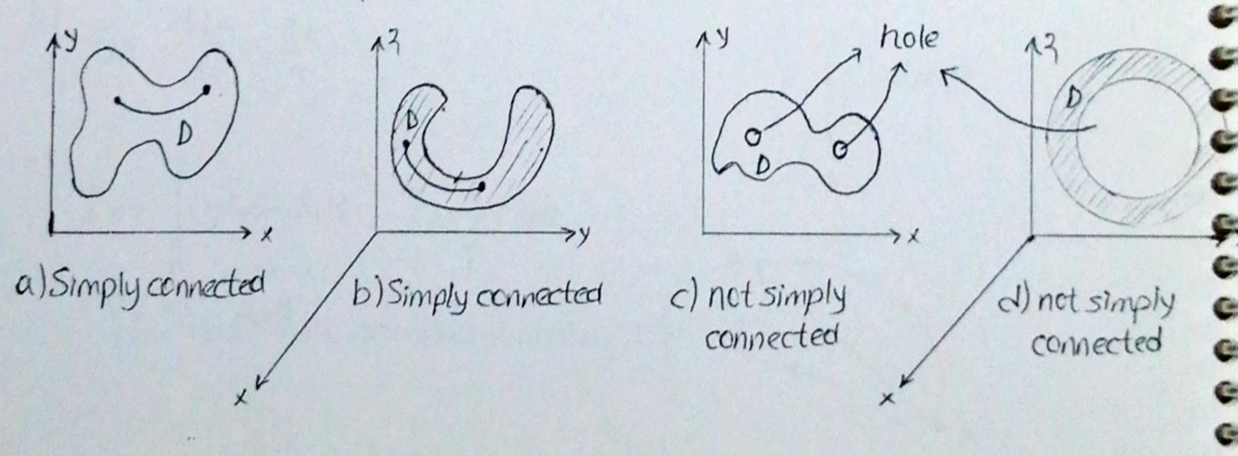
The curves we consider here are piecewise smooth such curves are made up of finitely many smooth pieces connected end to end. The domains D we consider are open regions in space, so every point in D is the center of an open ball that lies entirely in D. We also assume D to be connected. Finally, we assume D is simply connected, which means that every loop in D can be contracted to a point in D without ever leaving D.



Line Integrals in Vector Field

Theorem 1: Fundamental Theorem of Line Integrals

Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\vec{r}(t)$. Let ϕ be a differentiable function with a continuous gradient vector $\vec{F}=\nabla\phi$ on a domain D containing C, Then,

$$\int_C \vec{F}\,d\vec{r} = \phi(B)-\phi(A).$$

Proof of Theorem 1: Suppose that A and B are two points in region D.

$$c:\{\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}+z(t)\vec{k}\},\quad a\le t\le b$$

ϕ is a differentiable function of t.

$$\frac{d\phi}{dt}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$\Rightarrow \frac{\partial\phi}{\partial t}=\underbrace{\left(\frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial z}\vec{k}\right)}_{\vec{F}=\nabla\phi}\underbrace{\left(\frac{\partial x}{\partial t}\vec{i}+\frac{\partial y}{\partial t}\vec{j}+\frac{\partial z}{\partial t}\vec{k}\right)}_{\frac{d\vec{r}}{dt}}$$

$$\int_C \vec{F}\,d\vec{r}=\int_{t=a}^{b}\underbrace{\vec{F}(\vec{r}(t))\frac{d\vec{r}}{dt}}_{\frac{d\phi}{dt}}\cdot dt=\phi\Big|_a^b=\phi(x(b),y(b),z(b))-\phi(x(a),y(a),z(a))$$

$$=\phi(B)-\phi(A).$$

Ex: Suppose the force field $\vec{F}=\nabla\phi$ is the gradient of the function,

$$\phi(x,y,z)=-\frac{1}{x^2+y^2+z^2}.$$

Find the work done by $\vec{F}$ in moving an object along a smooth curve c joining the $(1,0,0)$ to $(0,0,2)$ that does not pass through the origin.

$$\text{Work}=W=\int_C \vec{F}\,d\vec{r}=\phi\Big|_{(1,0,0)}^{(0,0,2)}=\phi(0,0,2)-\phi(1,0,0)=\frac{3}{4}.$$

2^{nd} method:

$$\vec{F}=\frac{2x}{(x^2+y^2+z^2)^2}\vec{i}+\frac{2y}{(x^2+y^2+z^2)^2}\vec{j}+\frac{2z}{(x^2+y^2+z^2)^2}\vec{k}$$

$$C:\begin{cases} x=1-t \\ y=0 \\ z=2t \end{cases} \quad \begin{array}{l} ,\ \vec{r}=\vec{r_0}+t\vec{v} \\ (x-1,y-0,z-0)=t(0-1,0-0,2-0) \\ (x-1,y,z)=(-t,0,2t)\ ,\ 0\le t\le 1 \end{array}$$

$$\vec{r}(t)=(1-t)i+0\vec{j}+2t\vec{k}$$

$$\Rightarrow \frac{d\vec{r}}{dt}=-\vec{i}+2\vec{k}$$

$$\vec{F}(\vec{r}(t))=\frac{2(1-t)}{((1-t)^2+0^2+(2t)^2)^2}\vec{i}+0\vec{j}+\frac{2.2t}{((1-t)^2+0^2+(2t)^2)^2}\vec{k}$$

$$\vec{F}(\vec{r}(t))\frac{d\vec{r}}{dt}=\frac{-2(1-t)+2.2.(2t)}{((1-t)^2+(2t)^2)^2}=\frac{10t-2}{(5t^2-2t+1)^2}$$

$$\Rightarrow W=\int_C \vec{F}d\vec{r}=\int_{t=a}^{b}\vec{F}(\vec{r}(t))\frac{d\vec{r}}{dt}.dt$$

$$=\int_{t=0}^{1}\frac{10t-2}{(5t^2-2t+1)^2}dt \qquad \begin{cases} 5t^2-2t+1=u \\ (10t-2)dt=du \end{cases} \Rightarrow \begin{array}{l} t=0\Rightarrow u=1 \\ t=1\Rightarrow u=4 \end{array}\Bigg\}$$

$$=\int_{u=1}^{4}\frac{du}{u^2}=\dots=\frac{3}{4}.$$

Theorem 2: Conservative Fields are Gradient Fields

Let $\vec{F}=P\vec{i}+Q\vec{j}+R\vec{k}$ be a vector field whose components are continuous throughout an open connected region D in space, then $\vec{F}$ is conservative iff $\vec{F}$ is gradient field $\nabla\phi$, for a differentiable function ϕ. ($\vec{F}=\nabla\phi$)

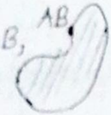
Theorem 3: Loop Property of Conservative Fields

The following statements are equivalent.

1) $\oint_C \vec{F}d\vec{r}=0$ around every loop.

2) The field $\vec{F}$ is conservative on D

$$\vec{F}=\nabla\phi \to \phi\Big|_A^B=0 \quad \left\{ A=B, \right.$$



$$\text{Curl}\,\vec{F}=\nabla\times\vec{F}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{2x}{(x^2+y^2+z^2)^2} & \frac{2y}{(x^2+y^2+z^2)^2} & \frac{2z}{(x^2+y^2+z^2)^2} \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}\left(\frac{2z}{(x^2+y^2+z^2)^2}\right)-\frac{\partial}{\partial z}\left(\frac{2y}{(x^2+y^2+z^2)^2}\right)\right]\vec{i}+\dots$$

$$\frac{-2(x^2+y^2+z^2)2y.2z}{\cancel{(x^2+y^2+z^2)^4}}\;\frac{-2(x^2+y^2+z^2)2z.2y}{\cancel{(x^2+y^2+z^2)^4}}+\dots$$

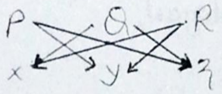
$=0$ so $\vec{F}=\nabla\phi$ (Conservative, path independence)

Finding Potentials for Conservative Fields

Component Test for Conservative Fields

Let $\vec{F}=P(x,y,z)\vec{i}+Q(x,y,z)\vec{j}+R(x,y,z)\vec{k}$ be a field on a connected and simply connected domain whose components functions have continuous first partial derivatives. Then $\vec{F}$ is conservative (irrotational) iff,

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x},\quad \frac{\partial P}{\partial z}=\frac{\partial R}{\partial x},\quad \frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}$$



Ex: Show that, $\vec{F}(x,y,z)=(e^x\cos y+yz)\vec{i}+(xz-e^x\sin y)\vec{j}+(xy+z)\vec{k}$ is conservative over its natural domain and a find a potential function for it.

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\Rightarrow -e^x\sin y+z=z-e^x\sin y \ \checkmark$$

$$\frac{\partial P}{\partial z}=\frac{\partial R}{\partial x}\Rightarrow y=y \ \checkmark$$

$$\frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}\Rightarrow x=x \ \checkmark$$

$\vec{F}$ is conservative.

$\Rightarrow \vec{F}=\nabla\phi.$

$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k} = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$

$\frac{\partial \phi}{\partial x} = e^x \cos y + yz$ integrate x $\phi(x,y,z) = e^x \cos y + xyz + h(y,z)$

$\frac{\partial \phi}{\partial y} = xz - e^x \sin y$

$\frac{\partial \phi}{\partial z} = xy + z$

$\frac{\partial \phi(x,y,z)}{\partial y} = -e^x \sin y + xz + \frac{\partial h(y,z)}{\partial y} = xz - e^x \sin y$

$\Rightarrow \frac{\partial h(y,z)}{\partial y} = 0 \Rightarrow h(y,z) = g(z)$. ✓

$\frac{\partial \phi(x,y,z)}{\partial z} = xy + \frac{\partial g(z)}{\partial z} = xy + z$

$\Rightarrow \frac{\partial g(z)}{\partial z} = z \Rightarrow g(z) = \frac{z^2}{2} + C$ ✓

$\Rightarrow \phi(x,y,z) = e^x \cos y + xyz + \frac{z^2}{2} + C.$

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EX1: Show that $\vec{F} = (2x-3)\vec{i} + (-z)\vec{j} + (\cos z)\vec{k}$ is not conservative. (Curl $\vec{F} \neq 0$ or Comp. Test)

EX2: Show that the vector field,

$$\vec{F} = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} + 0\vec{k}$$

satisfies the component test, but is not conservative over its natural domain. Explain why is this possible.

Ex3: Evaluate the integral,

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz$$

over any path from $(2,3,-1)$ to $(1,1,1)$.

HW: Integrate the vector field $\vec{u}(x,y) = y^2\vec{i} + (2xy - e^{2y})\vec{j}$ over the circular path.

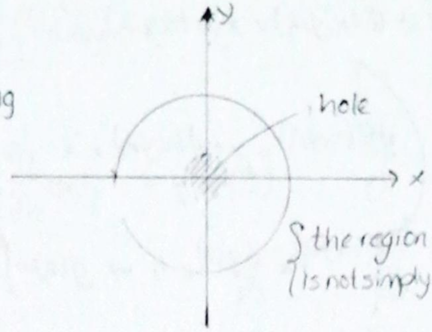
$$c: \begin{cases} \vec{r}(u) = \cos u\,\vec{i} + \sin u\,\vec{j} \\ 0 \leq u \leq 2\pi \end{cases} \quad \left(\text{Result: } \frac{1}{2}(1-e^2)\right)$$

Sol 2: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x} \Rightarrow \dfrac{y^2-x^2}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ ✓

$\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y} \Rightarrow 0 = 0$ ✓

$\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x} \Rightarrow 0 = 0$ ✓

if excluding the origin



$\left\{ \text{the region (or domain) is not simply connected} \right\}$

Sol 3: $\vec{F} = y\vec{i} + x\vec{j} + 4\vec{k} \Rightarrow \text{Curl}\,\vec{F} = \nabla \times \vec{F} \overset{?}{=} 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x & 4 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$$ (so the line integral is path independent)

$$C: \begin{cases} x = 1+t & , \ \vec{r} - \vec{r_0} = t\vec{v} \\ y = 1+2t & (x-1, y-1, z-1) = t(2-1, 3-1, -1-1) \\ z = 1-2t & (x-1, y-1, z-1) = (t, 2t, -2t), \quad 0 \le t \le 1 \end{cases}$$

$\vec{r}(t) = (1+t)\vec{i} + (1+2t)\vec{j} + (1-2t)\vec{k}$

$\Rightarrow \dfrac{d\vec{r}}{dt} = \vec{i} + 2\vec{j} - 2\vec{k}$

$\vec{F}(\vec{r}(t)) = (1+2t)\vec{i} + (1+t)\vec{j} + 4\vec{k}$

$\vec{F}(\vec{r}(t)) \dfrac{d\vec{r}}{dt} = 1 + 2t + 2 + 2t - 8 = 4t - 5.$

$\Rightarrow \displaystyle\int_0^1 (4t-5)\,dt = (2t^2 - 5t)\Big|_0^1 = -3.$ ✓

2^{nd} method:

$\nabla \times \vec{F} = 0 \Rightarrow \vec{F} = \nabla \phi \Rightarrow y\vec{i} + x\vec{j} + 4\vec{k} = \dfrac{\partial \phi}{\partial x}\vec{i} + \dfrac{\partial \phi}{\partial y}\vec{j} + \dfrac{\partial \phi}{\partial z}\vec{k}$

$\dfrac{\partial \phi}{\partial x} = y \Rightarrow \phi(x,y,z) = xy + h(y,z)$

$\dfrac{\partial \phi}{\partial y} = x \Rightarrow \phi(x,y,z) = yz + h(x,z)$

$\dfrac{\partial \phi}{\partial z} = 4 \Rightarrow \phi(x,y,z) = 4z + h(x,y)$

$$\frac{\partial \phi}{\partial x} = y \Rightarrow \phi(x,y,z) = xy + h(y,z)$$

$$\frac{\partial \phi(x,y,z)}{\partial y} = x + \frac{\partial h(y,z)}{\partial y} = x \Rightarrow h(y,z) = g(z)$$

$$\frac{\partial \phi(x,y,z)}{\partial z} = \frac{\partial g(z)}{\partial z} = 4 \Rightarrow g(z) = \int 4\,dz + C = 4z + C$$

$$\Rightarrow \phi(x,y,z) = xy + 4z + C \quad \text{(Potential Function)}$$

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz = \phi(x,y,z)\Big|_{(1,1,1)}^{(2,3,-1)} = (xy+4z+C)\Big|_{(1,1,1)}^{(2,3,-1)} = (6-4+C)-(1+4+C) = -3. \checkmark$$

Green's Theorem in the Plane

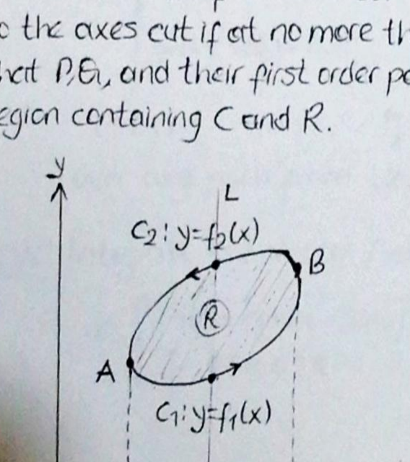
Theorem: Let C be a piecewise smooth, simple closed curve enclosing a region R in the xy-plane. Let $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ be a vector field with P and Q having continuous first partial derivatives in an open region containing R, then the counterclockwise circulation of $\vec{F}$ around C equals the double-integral over R.

$$\underbrace{\oint_C \vec{F}.\vec{T}\,ds = \oint_C \vec{F}\,d\vec{r} = \oint_C P\,dx + Q\,dy}_{①} = \underbrace{\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy}_{②} \Bigg\} \text{Green's Theorem in the plane}$$

① LHS (Left Hand Side) ② RHS (Right Hand Side)

Proof of Green's Theorem for Special Regions

Let C be a simple closed curve in the xy-plane with the property that lines parallel to the axes cut it at no more than two points. Let R be the region enclosed by C and suppose that P, Q, and their first order partial derivatives are continuous every point of some open region containing C and R.



$C = C_1 \cup C_2$

R

$C_1: y = f_1(x), \quad a \leq x \leq b$

$C_2: y = f_2(x), \quad b \geq x \geq a$.

$$\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dx\,dy = -\iint_R \frac{\partial P}{\partial y}\,dx\,dy + \iint_R \frac{\partial Q}{\partial x}\,dx\,dy$$

$$\oint_C P\,dx + \oint_C Q\,dy = -\iint_R \frac{\partial P}{\partial y}\,dx\,dy + \iint_R \frac{\partial Q}{\partial x}\,dx\,dy \quad \checkmark \quad (\square)$$

$$\oint_C P\,dx = \oint_C P(x,y)\,dx = \int_{C_1} P(x,y)\,dx + \int_{C_2} P(x,y)\,dx = \int_{C_1} P(x,f_1(x))\,dx + \int_{C_2} P(x,f_2(x))\,dx$$

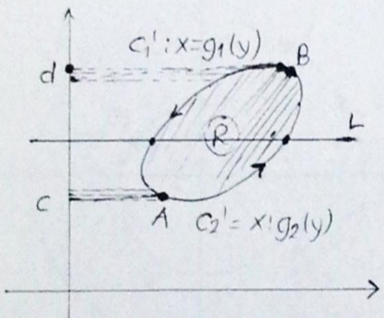
$$= \int_{x=a}^{b} P(x,f_1(x))\,dx + \int_{x=b}^{a} P(x,f_2(x))\,dx$$

$$= \int_{x=a}^{b} \big(P(x,f_1(x)) - P(x,f_2(x))\big)\,dx \quad (\star)$$

$$\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_{x=a}^{b}\left(\int_{y=f_1(x)}^{y=f_2(x)} \frac{\partial P}{\partial y}\,dy\right)dx = \int_{x=a}^{b} P(x,y)\Big|_{y=f_1(x)}^{y=f_2(x)}\,dx = \int_{x=a}^{b}\big(P(x,f_2(x)) - P(x,f_1(x))\big)\,dx$$

$$=$$

$$-\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_{x=a}^{b}\big(P(x,f_1(x)) - P(x,f_2(x))\big)\,dx \quad (\Delta)$$



$$\underbrace{C = C_1' \cup C_2'}_{R}$$

$C_1' : x = g_1(y)\,,\ d \geq y \geq c$

$C_2' : x = g_2(y)\,,\ c \leq y \leq d$

$$\oint_C Q\,dy = \oint_C Q(x,y)\,dy = \int_{C_1'} Q(x,y)\,dy + \int_{C_2'} Q(x,y)\,dy = \int_{C_1'} Q(g_1(y),y)\,dy + \int_{C_2'} Q(g_2(y),y)\,dy$$

$$= \int_d^c Q(g_1(y),y)\,dy + \int_c^d Q(g_2(y),y)\,dy$$

$$= \int_c^d \left(Q(g_2(y),y) - Q(g_1(y),y)\right)dy . \quad (*)$$

$$\iint_R \frac{\partial Q}{\partial x}\,dx\,dy = \int_{y=c}^{d}\left(\int_{x=g_2(y)}^{x=g_2(y)} \frac{\partial Q}{\partial x}\,dx\right)dy = \int_c^d Q(x,y)\Big|_{x=g_1(y)}^{g_2(y)} dy = \int_c^d \left(Q(g_2(y),y) - Q(g_1(y),y)\right)dy . \quad (-)$$

$$\Rightarrow \left.\begin{aligned} \oint_C P\,dx &= -\iint_R \frac{\partial P}{\partial y}\,dx\,dy \\ \oint_C Q\,dy &= \iint_R \frac{\partial Q}{\partial x}\,dx\,dy \end{aligned}\right\} + \Rightarrow \oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dx\,dy . \checkmark$$

Ex! Verify Green's Theorem for the vector field $\vec{F}(x,y) = (x-y)\vec{i} + x\vec{j}$ and the region R bounded by the unit circle. $C: \{\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j},\ 0 \leq t \leq 2\pi .$

$$\underset{①\ LHS}{\oint_C P\,dx + Q\,dy} = \underset{②\ RHS}{\iint_R (Q_x - P_y)\,dx\,dy}$$

= LHS : 1 ≠

$$\oint_C (x-y)\,dx + x\,dy = \int_{t=0}^{2\pi} (\cos t - \sin t)(-\sin t\,dt) + (\cos t)(\cos t\,dt)$$

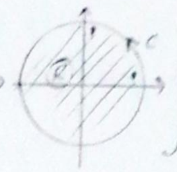
$$= \int_0^{2\pi} (-\underbrace{\sin t . \cos t}_{u} + 1)\,dt = \left(\frac{\cos^2 t}{2} + t\right)\Big|_0^{2\pi}$$

$$= \left(\frac{1}{2} + 2\pi\right) - \left(\frac{1}{2} + 0\right) = 2\pi .$$

RHS: 2

$$\left.\begin{aligned} Q_x &= \frac{\partial Q}{\partial x} = 1 \\ P_y &= \frac{\partial P}{\partial y} = -1 \end{aligned}\right\} Q_x - P_y = 1-(-1) = 2$$

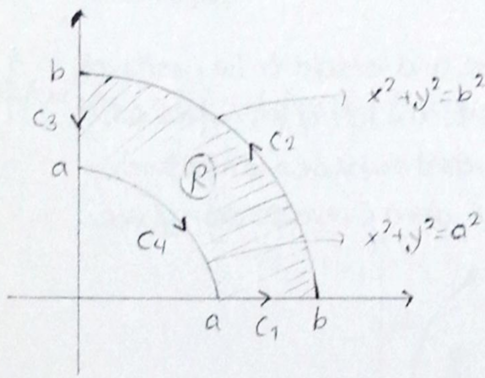
$\left\{ \text{Area; } \pi \cdot 1^2 = \pi \right.$ →



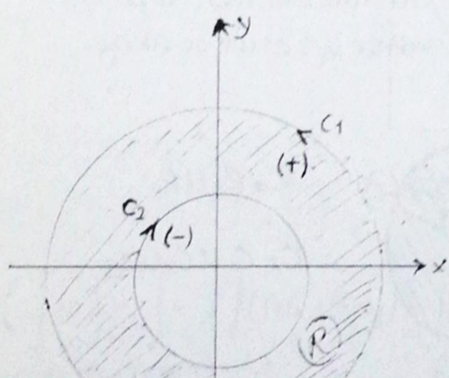
$$\Rightarrow \iint_R (Q_x - P_y)\,dA = \iint_R 2\,dA = 2\iint_R dA = 2\pi.$$

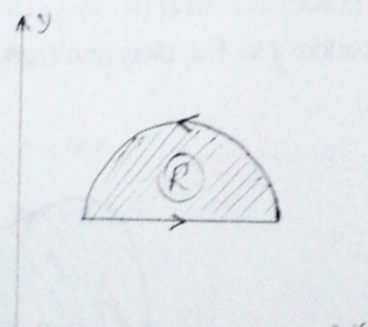
⇒ ① = ② ✓

Remark: Green's Theorem is valid for regions that are not simply connected.



$C: C_1 \cup C_2 \cup C_3 \cup C_4$





Calculating Area with Green's Theorem

If a simple closed curve C in the xy-plane and the region R it encloses satisfy the hypotheses of Green's Theorem, the area of R is given by,

$$\text{Area of } R = A = \frac{1}{2}\oint_C -y\,dx + x\,dy \,,$$

$$A = \underbrace{\oint_C -y\,dx}_{\iint_R dA = \oint_C P\,dx} = \underbrace{\oint_C x\,dy}_{\iint_R = \oint_C Q\,dy} \quad \checkmark \qquad \left\{ \begin{array}{l} R \text{ is bounded by the} \\ \text{piecewise smooth,} \\ \text{simple closed curve} \end{array} \right\}$$

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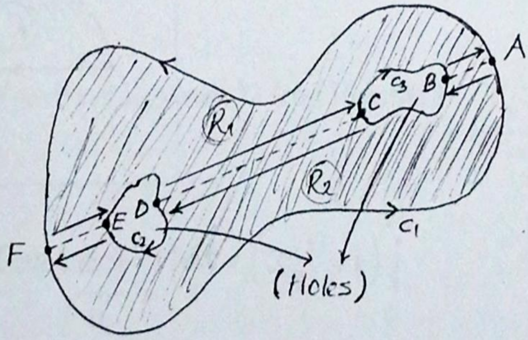
<u>Remark!</u> The curve is traversed counterclockwise and is said to be positively oriented if the region it encloses is always to the left of an object as it moves along the path. Otherwise, it is traversed clockwise and negatively oriented. The line integral of a vector field $\vec{F}$ along C reverses sign if we change the orientation.

Regions with Many Holes

Green's Theorem holds for a region R with any finite number of holes as long as the bounding curves are smooth, simple and closed and we integrate over each component of the boundary in the direction that keeps R on our immediate left as we go along.



$R = R_1 \cup R_2$

$C = C_1 \cup C_2 \cup C_3$

Multiply Connected

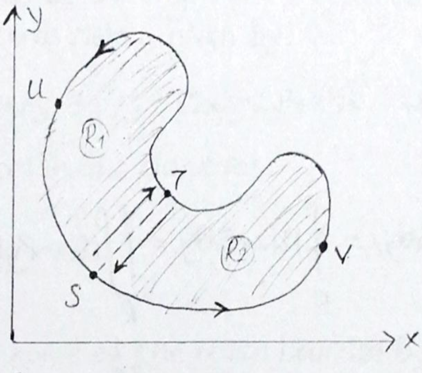
$$\left(\int_{FA}+\cancel{\int_{AB}}+\int_{BC}+\cancel{\int_{CD}}+\int_{DE}+\cancel{\int_{EF}}\right)(Pdx+Qdy)+\left(\int_{AF}+\cancel{\int_{FE}}+\int_{ED}+\cancel{\int_{DC}}+\int_{CB}+\cancel{\int_{BA}}\right)(Pdx+Qdy)=\iint_{R_1}(Q_x-P_y)dA+\iint_{R_2}(Q_x-P_y)dA$$

$$\left\{\int_{AB}=-\int_{BA},\ \int_{CD}=-\int_{DC},\ \int_{EF}=-\int_{FE}\right\}$$

$$\left(\int_{BC}+\int_{DE}+\int_{FA}\right)(Pdx+Qdy)+$$

$$\left(\int_{CB}+\int_{ED}+\int_{AF}\right)(Pdx+Qdy)=$$

$$\oint_{C_1}Pdx+Qdy+\oint_{C_2}Pdx+Qdy+\oint_{C_3}Pdx+Qdy=\iint_{R_1}(Q_x-P_y)dA+\iint_{R_2}(Q_x-P_y)dA \checkmark$$

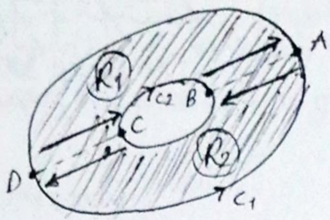


Simply Connected

$R=R_1\cup R_2$

$$\left(\int_{SV7}+\cancel{\int_{7S}}+\int_{7US}+\cancel{\int_{S7}}\right)(Pdx+Qdy)=\iint_R(Q_x-P_y)dA$$

$$\oint_C Pdx+Qdy=\iint_R(Q_x-P_y)dA \checkmark$$



$R = R_1 \cup R_2$

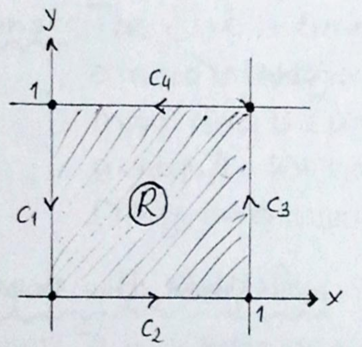
$C = C_1 \cup C_2$

$$\oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA \quad \checkmark$$

Ex: Evaluate the line integral,

$$\oint_C xy\,dy - y^2\,dx$$

where c is the square cut from the first quadrant by the lines $x=1$ and $y=1$.



$C = C_1 \cup C_2 \cup C_3 \cup C_4$

$$\left(\oint_{C_1} + \oint_{C_2} + \oint_{C_3} + \oint_{C_4} \right)(P\,dx + Q\,dy) = \oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$$

$$= \int_0^1\int_0^1 (y + 2y)\,dy\,dx = \frac{3}{2}.$$

2^{nd} method:

$C_1: \begin{cases} x=0; \\ x=0 \\ y=y \end{cases}$, $C_2: \begin{cases} y=0; \\ x=x \\ y=0 \end{cases}$, $C_3: \begin{cases} x=1; \\ x=1 \\ y=y \end{cases}$, $C_4: \begin{cases} y=1; \\ x=x \\ y=1 \end{cases}$

$$\int_{C_1} -y^2dx + xydy = \int_{y=1}^{0} -y^2.0 + 0.ydy = 0$$

$$\int_{C_2} -y^2dx + xydy = \int_{x=0}^{1} -0^2dx + x.0.0 = 0$$

$$\int_{C_3} -y^2dx + xydy = \int_{y=0}^{1} -y^2.0 + 1ydy = \frac{1}{2}$$

$$\int_{C_4} -y^2dx + xydy = \int_{x=1}^{0} -1^2dx + x.1.0 = 1$$

$$\oint_C Pdx + Qdy = 0 + 0 + \frac{1}{2} + 1 = \frac{3}{2}.$$

Examples

1) Find the area of the ellipse by using the line integral (Result: πab)

2) Find the work done in moving a particle once around an ellipse c in the xy-plane. If the ellipse has its center at the origin with semimajor and semiminor axes 4 and 3 respectively. The force field is given by,

$$\vec{F} = (3x-4y+2z)\vec{i} + (4x+2y-3z^2)\vec{j} + (2xz-4y^2+z^3)\vec{k}.$$ (Result: 96π)

3) Verify the Green's Theorem in the plane for,

$$\oint_C (2xy-x^2)dx + (x+y^2)dy$$

where c is the closed curve of the region bounded by $y=x^2$ and $y^2=x$. (Result: $1/30$)

4) Let,

$$\vec{F} = \frac{-y\vec{i} + x\vec{j}}{x^2+y^2},$$

a) Calculate $\nabla \times \vec{F}$.

b) Evaluate

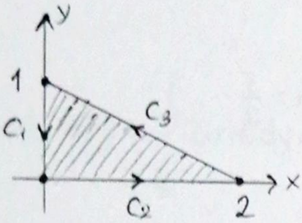
$$\oint_C \vec{F} d\vec{r}$$

around any closed path and explain results. (Result: $0 \to$ exc. origin, $2\pi \to$ inc. origin)

5) Evaluate,

$$\oint_C (2x-y^2)dx+(xy-1)dy.$$

If the curve is given by,



(Result: 1)

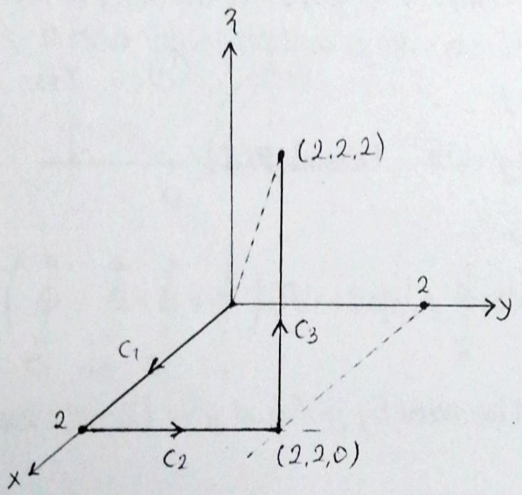
6) Let,

$$\vec{F}(x,y,z)=(y^2+2z^2-1)\vec{i}+(2xy)\vec{j}+(2x^2z)\vec{k}.$$

Evaluate,

$$\oint_C \vec{F}d\vec{r}$$

where c is the path from $(0,0,0)$ to $(2,2,2)$ given in figure,



(Result: 22)

7) Evaluate,

$$\oint_C (2x^2+xy-y^2)dx+(3x^2-xy+2y^2)dy$$

where, $c: \{ (x-a)^2+y^2=4,\ a>0.$ (Result: $20\pi a$)

8) If,

$$\oint_C (1+y^2)dx + ydy$$

where c is closed curve of the region bounded by $y=\sin x$, $y=2\sin x$, $0 \leq x \leq \pi$. (Result: $-3\pi/2$)

9) Let,

$\vec{F}(x,y) = 3y\vec{i} + x\vec{j}$,

$$c: \begin{cases} x^2+y^2=16 \text{ and} \\ x^2-2x+y^2=3 \end{cases}$$

i) Evaluate,

$$\oint_C \vec{F}d\vec{r}.$$

ii) Evaluate directly. (Result: -24π)

10) If,

$\vec{F}(x,y,z) = z^2\vec{i} + y^2\vec{j} + x\vec{k}$

and c is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counterclockwise rotation. Find,

$$\int_C \vec{F}d\vec{r}$$

along the curve c. (Result: $-1/6$)

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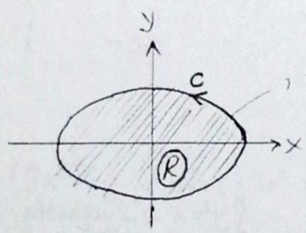
Solutions

1)



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\left.\begin{array}{l} x = ar\cos\theta \\ y = br\sin\theta \end{array}\right\} r=1 \Rightarrow \begin{array}{l} x = a\cos\theta \Rightarrow dx = -a\sin\theta d\theta \\ y = b\sin\theta \Rightarrow dy = b\cos\theta d\theta \end{array}$

$(0 \leq \theta \leq 2\pi)$

$$A = \frac{1}{2}\oint_C -ydx + xdy = \frac{1}{2}\int_0^{2\pi} (-b\sin\theta)(-a\sin\theta d\theta) + (a\cos\theta)(b\cos\theta d\theta) = \frac{ab}{2}\int_0^{2\pi} d\theta = \pi ab.$$

2)

$$C: \begin{cases} x = 4\cos\theta \\ y = 3\sin\theta \\ z = 0 \end{cases}$$

$$\vec{r}(\theta) = 4\cos\theta\,\vec{i} + 3\sin\theta\,\vec{j} + 0\vec{k}$$

$$\Rightarrow \frac{d\vec{r}}{d\theta} = -4\sin\theta\,\vec{i} + 3\cos\theta\,\vec{j}$$

$$\vec{F}(\vec{r}(\theta)) = \left(3(4\cos\theta) - 4(3\sin\theta) + 2.0\right)\vec{i} + \left(4(4\cos\theta) + 2(3\sin\theta) - 3.0^2\right)\vec{j} + \left(2(4\cos\theta)0 - 4(3\sin\theta)^2 + 0^3\right)\vec{k}$$

$$= \underbrace{(12\cos\theta - 12\sin\theta)\vec{i} + (16\cos\theta + 6\sin\theta)\vec{j}} + (-36\sin^2\theta)\vec{k}$$

$$W = \oint_C \vec{F}d\vec{r} = \oint_C \left(\vec{F}.\frac{d\vec{r}}{d\theta}\right).d\theta = \int_{\theta=0}^{2\pi} \left[(-4\sin\theta)(12\cos\theta - 12\sin\theta) + (3\cos\theta)(16\cos\theta + 6\sin\theta)\right]d\theta$$

$$= \int_0^{2\pi} (-48\sin\theta\cos\theta + 48\sin^2\theta + 48\cos^2\theta + 18\sin\theta\cos\theta)\,d\theta$$

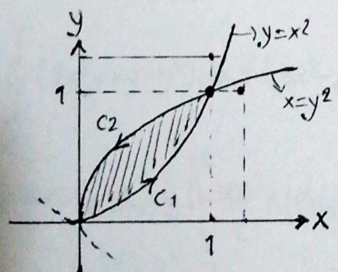
$$= \int_0^{2\pi} (-30\sin\theta\underbrace{\cos\theta}_{u} + 48)\,d\theta = \ldots = 96\pi.$$

2^{nd} method:

$$\oint_C \vec{F}d\vec{r} = \oint_C Pdx + Qdy = \iint_R (Q_x - P_y)\,dA \quad \begin{Bmatrix} Q_x = 4 \\ P_y = -4 \end{Bmatrix}$$

$$= 8\underbrace{\iint_R dA}_{\substack{\text{Area} \\ (\pi ab)}} = 96\pi.$$

3)



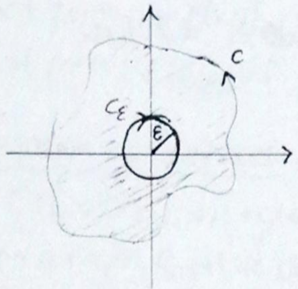
$$C_1: \begin{cases} y = x^2 \Rightarrow dy = 2x\,dx \\ x = x \Rightarrow dx = dx \\ x: 0 \to 1 \end{cases} \qquad C_2: \begin{cases} y^2 = x \Rightarrow 2y\,dy = dx \\ y = y \Rightarrow dy = dy \\ y: 1 \to 0 \end{cases}$$

4) $\vec{F} = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} + 0.\vec{k}$

a)

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]\vec{k} = \vec{0}. \left\{\begin{matrix}\text{If excluding} \\ \text{the origin} \\ (x,y) \neq (0,0)\end{matrix}\right\}$$

b)



$$C_\varepsilon : \begin{cases} x = \varepsilon\cos\theta \\ y = \varepsilon\sin\theta \\ 2\pi \geq \theta \geq 0 \end{cases}$$

$$\oint_C Pdx + Qdy + \oint_{C_\varepsilon} Pdx + Qdy = \iint_{R'} (Q_x - P_y)dA = 0$$

$$\Rightarrow \oint_C Pdx + Qdy = -\oint_{C_\varepsilon} Pdx + Qdy = -\int_{2\pi}^{0} \frac{(-\varepsilon\sin\theta)}{\varepsilon^2}(-\varepsilon\sin\theta d\theta) + \frac{(\varepsilon\cos\theta)}{\varepsilon^2}(\varepsilon\cos\theta d\theta) = -\int_{2\pi}^{0} d\theta = 2\pi.$$

$$\Rightarrow \oint_C \vec{F}d\vec{r} = \begin{cases} 0 \ , & \text{excluding origin} \\ 2\pi \ , & \text{including origin.} \end{cases}$$

7)

$$C: \begin{cases} (x-a)^2 + y^2 = 4 \\ a > 0 \end{cases}, \quad \begin{matrix} x - a = r\cos\theta \\ y = r\sin\theta \\ J = r \end{matrix}, \quad \begin{matrix} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \\ dA = rdrd\theta \end{matrix}$$

$$\oint_C \underbrace{(2x^2 + xy - y^2)}_{P_y = x - 2y} dx + \underbrace{(3x^2 - xy + 2y^2)}_{Q_x = 6x - y} dy$$

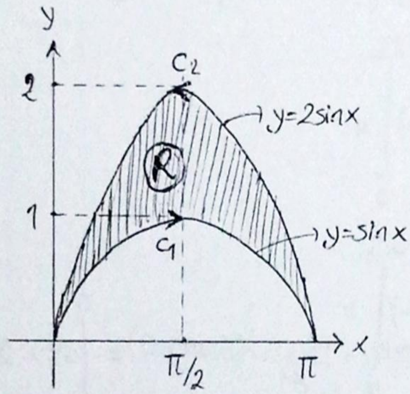
$$Q_x - P_y = 5x + y = 5(a + r\cos\theta) + r\sin\theta$$

$$\Rightarrow \oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$$

$$= \iint_R (5x+y)\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (5(a+r\cos\theta) + r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left(\int_{r=0}^{2} (5(a+r\cos\theta)+r\sin\theta)\,r\,dr\right)d\theta = \dots = 20\pi a.$$

8)

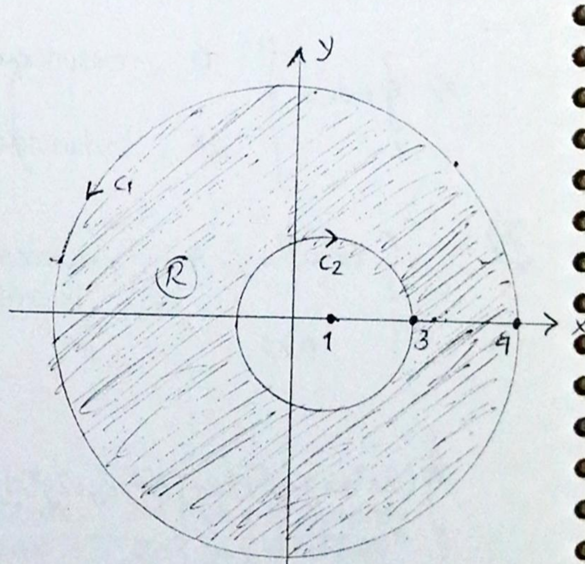


$C_1: \begin{cases} y=\sin x \\ x=x \\ x: 0\to\pi \end{cases}$ $C_2: \begin{cases} y=2\sin x \\ x=x \\ x=\pi\to 0 \end{cases}$

⋮

9)

$C: \begin{cases} x^2+y^2=16 \\ x^2-2x+y^2=3 \Rightarrow (x-1)^2+y^2=4 \end{cases} \Rightarrow$



i) for $\oint_C \vec{F}\,d\vec{r}$

$C_1: \begin{cases} x=4\cos\theta \\ y=4\sin\theta \\ 0\le\theta<2\pi \end{cases}$, $C_2: \begin{cases} x-1=2\cos\theta \\ y=2\sin\theta \\ 0\le\theta<2\pi \end{cases}$

⋮

ii) $$\iint_R (Q_x-P_y)\,dA = \iint_R (1-(+3))\,dA = -2\iint_R dA = -2(\pi 4^2-\pi 2^2) = -24\pi.$$