$\vec{u} = -b\vec{i} + b\vec{j} = b(-\vec{i}+\vec{j})$

$\vec{u} = \frac{-1}{\sqrt{2}}\vec{i} + \frac{1}{2}\vec{j}$ or $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$

Ex: Find a vector tangent to the curve of intersection of the two surfaces $z = x^2 - y^2$ and $xyz + 30 = 0$ at the point $(-3, 2, 5)$

$\underbrace{x^2+y^2-z=0}_{F(x,y,z)}$ $\underbrace{xyz+30=0}_{G(x,y,z)}$

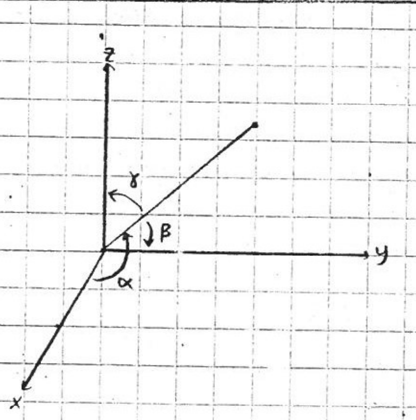
$\vec{n}_3 = \underbrace{\nabla F\Big|_{(3,2,5)}}_{\vec{n}_1} \times \underbrace{\nabla G\Big|_{(-3,2,5)}}_{\vec{n}_2} \Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix} = 9\vec{i} - 46\vec{j} + 130\vec{k}$

Remark: The directional derivative can be described as follows:

Let $\vec{u} = u_1 \cdot \vec{i} + u_2 \cdot \vec{j} + u_3 \cdot \vec{k}$ be a unit vector. Thus this vectors components are directional cosines.

That is,

$\boxed{\vec{u} = \cos\alpha\ \vec{i} + \cos\beta\ \vec{j} + \cos\gamma\ \vec{k}}$; $\|\vec{u}\| = 1$; $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$



Ex: In what directions at the point $(1,2)$ does the function $f(x,y) = 2xy$ have rate of change 4?
HW $\left(\vec{x} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j},\ \vec{x} = \vec{i}\right)$

Multiple Integration

21.11.14

Double Integrals:

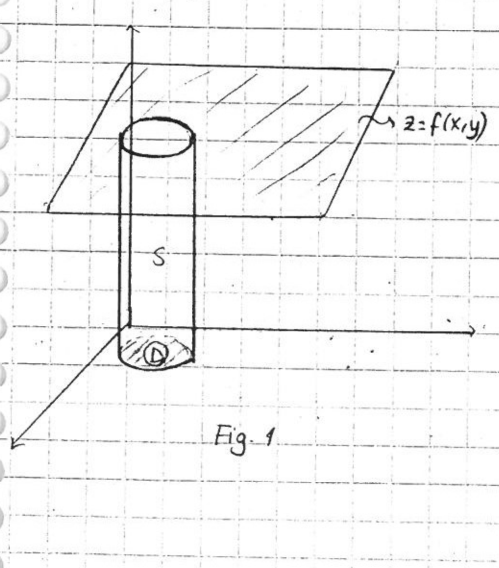
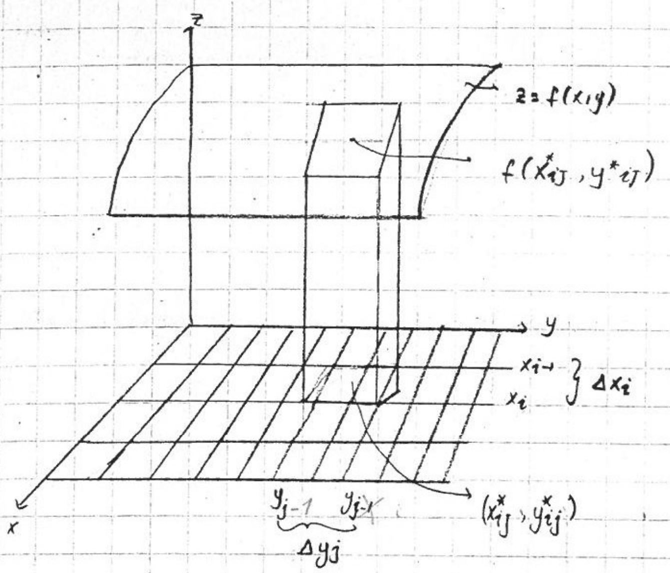


Fig. 1

Ⓢ: three dimensional region
Ⓓ: domain of integration



R_{ij} : Rectangular box

$$\Delta V = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\,\Delta x_i \cdot \Delta y_j$$

Fig. 2 : The Riemann Sum is a sum of values of such boxes.

Let us start with the case where D is a closed rectangle with sides parallel to the coordinate axes in the xy-plane, and f is is bounded function on D. If D consists of the points (x,y) into small rectangles by partitioning each of the intervals $[a,b]$ and $[c,d]$;

Say by points

$a = x_0 < x_1 < x_2 < \dots < x_{m-1} < x_m = b$

$c = y_0 < y_1 < y_2 < \dots < y_{n-1} < y_n = d$

R_{ij} $(1 \le i \le m \;,\; 1 \le j \le n)$

$x_{i-1} \le x \le x_i \;,\; y_{j-1} \le y \le y_j$ (See Fig. 2)

The rectangle R_{ij} has area

$\Delta R_{ij} = \Delta x_i \cdot \Delta y_j = (x_i - x_{i-1})(y_j - y_{j-1})$ and

Diameter (diagonal length) : $\operatorname{diam}(R_{ij}) = \sqrt{\Delta x_i^2 + \Delta y_j^2} = \sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}$

The norm of the partition P is the largest subrectangle diameters:

$\|P\| = \max \operatorname{diam}(R_{ij}) \qquad 1 \le i \le m \;;\; 1 \le j \le n$

Now we pick an arbitrary point (x^*_{ij}, y^*_{ij}) in each of the rectangles R_{ij} and form the Riemann sum

$$R(f,P) = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \cdot \Delta A_{ij} \quad \left(\|P\| \to 0 \quad \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \cdot \underbrace{\Delta A_{ij}}_{\Delta x_i \cdot \Delta y_j} \right)$$

The term corresponding to rectangle R_{ij} is, if $f(x^*_{ij}, y^*_{ij}) \ge 0$, the volume of the rectangular box whose base is R_{ij} and whose height is the value of f at (x^*_{ij}, y^*_{ij}).

Therefore, for positive functions f, the Riemann sum $R(f,P)$ approximates the volume above D and under the graph of f.

$$V = \iint f(x,y)\,dA \qquad \left(\lim_{\|P\|\to 0} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\cdot \Delta A_{ij} = \iint_D f(x,y)\,dA = V \right)$$

Remark: $dA = dx\,dy$
$\qquad\quad\;\; dy\,dx$ (integrasyon sınırı aynıysa dydx ya da dxdy fark etmez ; ama farklıysa değişken sınırı içte yazılır , sonuç fonksiyon olmasın diye.)

Ex: If $0\le x\le 1$, $0\le y\le 1$ then evaluate $\iint_D (x^2+y)\,dA$

$$\int_{x=0}^{1}\int_{y=0}^{1}(x^2+y)\,dy\,dx = \int_{x=0}^{1}\left(x^2y+\frac{y^2}{2}\right)\Bigg|_{y=0}^{1}dx = \int_{x=0}^{1}\left(x^2+\frac{1}{2}\right)dx = \left(\frac{x^3}{3}+\frac{x}{2}\right)\Bigg|_{x=0}^{1} = \frac{5}{6}$$

OR $\qquad =$

$$\int_{y=0}^{1}\int_{x=0}^{1}(x^2+y)\,dx\,dy = \int_{y=0}^{1}\left(\frac{x^3}{3}+yx\right)\Bigg|_{x=0}^{1} = \int_{y=0}^{1}\left(\frac{1}{3}+y\right)dy = \left(\frac{y}{3}+\frac{y^2}{2}\right)\Bigg|_{y=0}^{1} = \frac{5}{6}$$

Double Integrals over more General Domains

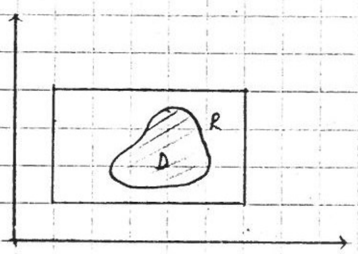


Fig: Bounded domain D is a subset of rectangle R.

Definition: If $f(x,y)$ is defined and bounded on domain D, let $\hat{f}$ be the extension of f that D zero everywhere outside D:

$$\hat{f}(x,y) = \begin{cases} f(x,y) & \text{if } f(x,y) \text{ belong to } D \\ 0 & \text{if } f(x,y) \text{ does not belong to } D \end{cases}$$

If $\hat{f}$ is integrable over R, we say that f is integrable over D and define the double integral of f over D to be

$$\boxed{\iint_R \hat{f}(x,y)\,dA = \iint_D f(x,y)\,dA + \iint_{(x,y)\in R,\ (x,y)\notin D} 0\,.\,dA = \iint_D f(x,y)\,dA}$$

Iteration of Double Integrals in Cartesian Coordinates:

The existence of the double integral $\iint_D f(x,y)\,dA$ depends on f and the domain D. As we shall see evaluation of double integrals easiest when the domain of integration is of simple type:

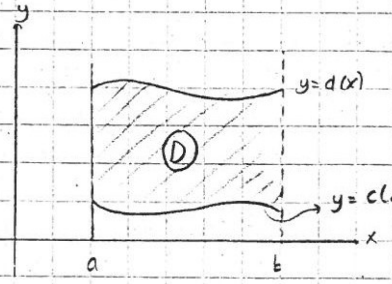


Fig 1.: A y-simple domain

D: $a\le x\le b$; $c(x)\le y\le d(x)$, $f(x,y)$ continuous in this domain

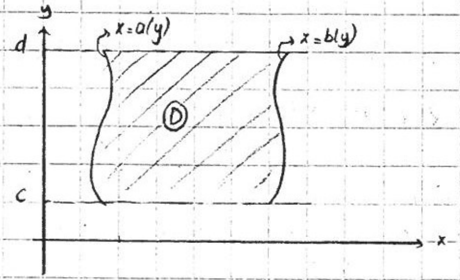


Fig 2.: A x-simple domain

D: $c\le y\le d$; $a(y)\le x\le b(y)$, $x=f(y)$ continuous in this domain

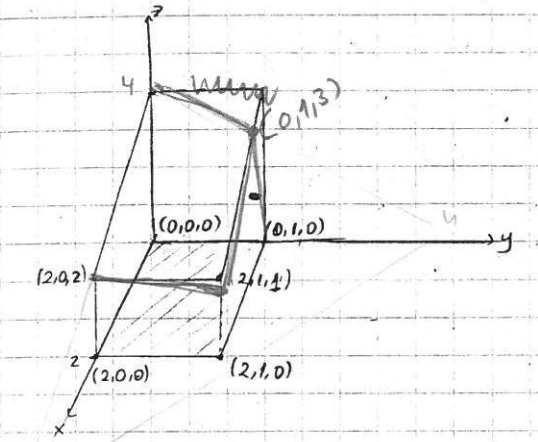
$\Rightarrow$ Fig 1 : $\displaystyle\iint_D f(x,y)\,dA = \int_{x=a}^{b}\left(\int_{y=c(x)}^{d(x)} f(x,y)\,dy\right)dx$

$\Rightarrow$ Fig 2 : $\displaystyle\iint_D f(x,y)\,dA = \int_{y=c}^{d}\left(\int_{x=a(y)}^{b(y)} f(x,y)\,dx\right)dy$

Ex: Calculate the volume under the plane $z = 4-x-y$ over the rectangular region R : $0 \le x \le 2$, $0 \le y \le 1$ in the xy-plane

$x+y+z=4$

$\frac{x}{4}+\frac{y}{4}+\frac{z}{4}=1$



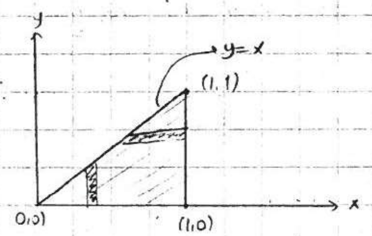
$$\iint_D f(x,y)\,\underbrace{dA}_{\text{base area}} = \iint_D (4-x-y)\,dA = \int_{x=0}^{2}\int_{y=0}^{1}(4-x-y)\,dy\,dx = \int_{y=0}^{1}\int_{x=0}^{2}(4-x-y)\,dx\,dy$$

$$= \int_{x=0}^{2}\left(4y-xy-\frac{y^2}{2}\right)\Big|_{y=0}^{1} dx$$

$$= \int_{x=0}^{2}\left(4-x-\frac{1}{2}\right)dx = \left(\frac{7x}{2}-\frac{x^2}{2}\right)\Big|_{x=0}^{2}$$

$= 7-2 = 5$ cubic units

Ex: Evaluate $\iint_T xy\,dA$ over the triangle T with vertices $(0,0)$, $(1,0)$ and $(1,1)$



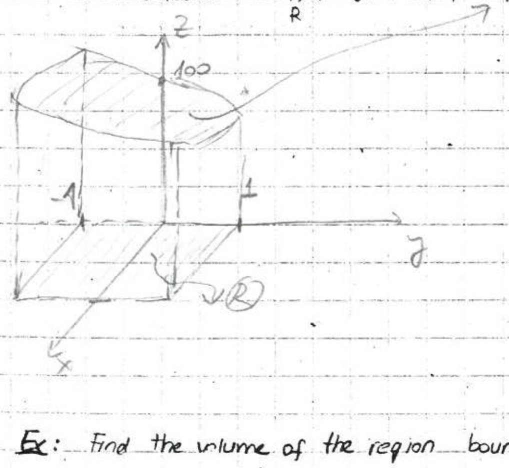
$I_1 = \iint_D xy\,dA = \int_{x=0}^{1}\left(\int_{y=0}^{x} xy\,dy\right)dx \Rightarrow \int_0^1\left(\frac{xy^2}{2}\Big|_{y=0}^{x}\right)dx = \int_0^1 \frac{x^3}{2}\,dx = \frac{x^4}{8}\Big|_0^1 = \frac{1}{8}$

$I_1 = I_2 = \cancel{\frac{1}{4}}$

$I_2 = \iint_D xy\,dA = \int_{y=0}^{1}\left(\int_{x=y}^{1} xy\,dx\right)dy \rightarrow \int_{y=0}^{1}\left(\frac{x^2y}{2}\right)\Big|_{x=y}^{1} dy = \int_0^1\left(\frac{y}{2}-\frac{y^3}{2}\right)dy = \frac{y^2}{4}-\frac{y^4}{8} = \frac{1}{4}-\frac{1}{8}$

24.11.14

Ex: Calculate $\iint_R f(x,y)\,dA$ for $f(x,y) = 100-6x^2y$ and $R = \{(x,y)\ 0 \le x \le 2 \text{ and } -1 \le y \le 1\}$.

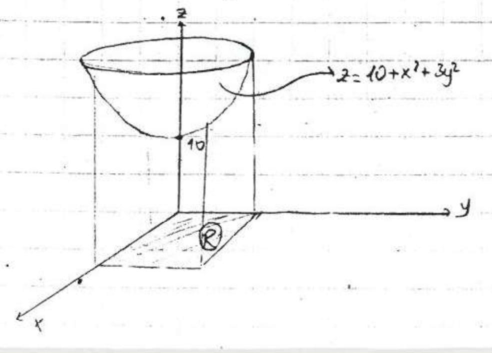


$$\int_{x=0}^{2}\left(\int_{y=-1}^{1}(100-6x^2y)\,dy\right)dx = \int_{x=0}^{2}(100y-3x^2y^2)\Big|_{y=-1}^{1} = \int_{y=0}^{2} 200 \cancel{-6x^2} = (200x \cancel{-2x^3})\Big|_{x=0}^{x=2}$$

$= 400$

Ex: Find the volume of the region bounded above the elliptical paraboloid $z = 10+x^2+3y^2$ and below by the rectangle

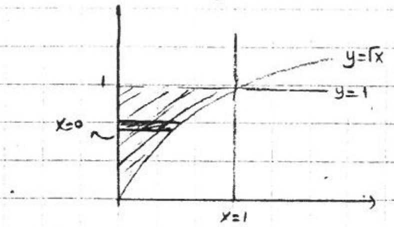
R: $0 \le x \le 1$, $0 \le y \le 2$



$$V = \iint_R f(x,y)\,dA = \int_{x=0}^{1}\left(\int_{y=0}^{2}(10+x^2+3y^2)\,dy\right)dx = \frac{86}{3}$$

Order Of Integration Reversed

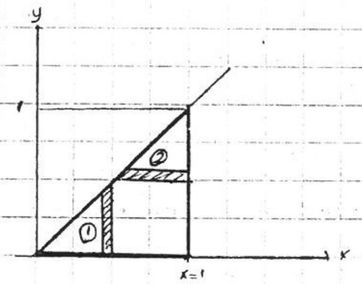
Ex: Evaluate the iterated integral $I = \int_0^1 dx \int_{\sqrt{x}}^1 e^{y^3}\,dy = \int_{x=0}^1 \left(\int_{y=\sqrt{x}}^1 e^{y^3}dy \right) dx = ?$



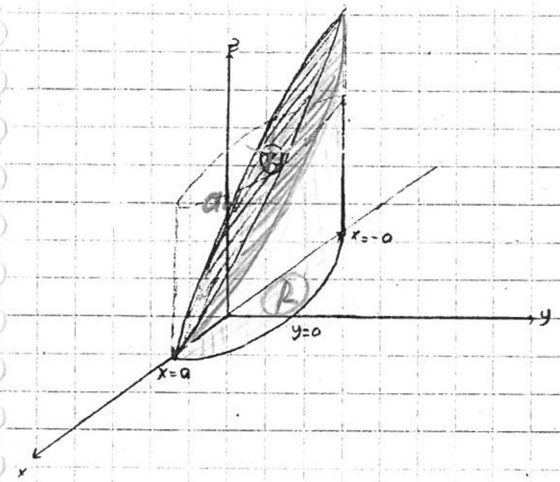
$$I = \int_{y=0}^1 \left(\int_{x=0}^{y^2} e^{y^3}\,dx \right) dy = \int_{y=0}^1 \left(x \cdot e^{y^3}\right)\Big|_{x=0}^{y^2} = \int_{y=0}^1 y^2 \cdot e^{y^3} = \frac{e^{y^3}}{3}\Big|_{y=0}^1 = \frac{e-1}{3}$$

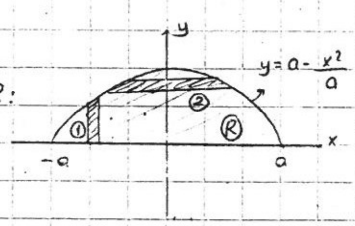
Ex: Calculate $\iint_R \frac{\sin x}{x}\,dA$ where R is the triangle in the xy-plane bounded by the x-axis, the line $y=x$ and the line $x=1$

① $$\int_{x=0}^1 \left(\int_{y=0}^x \frac{\sin x}{x}\,dy \right) dx = \int_{x=0}^1 \left(y \cdot \frac{\sin x}{x}\right)\Big|_{y=0}^{x} = \int_{x=0}^1 \sin x = -\cos x\Big|_{x=0}^1$$
$$= 1-\cos 1$$



Ex: Sketch and find the volume of the solid bounded by the planes $y=0$, $z=0$ and $z=a-x+y$ and the parabolic cylinder $y = a - \frac{x^2}{a}$, where a is a positive constant.





①: $$V = \iint_R \underbrace{f(x,y)}_{\text{height}}\,\underbrace{dA}_{\text{base area}} = \iint_R \overbrace{(a-x+y)}^{z}\,dA = \int_{x=-a}^{a} \left(\int_{y=0}^{a-\frac{x^2}{a}} (a-x+y)\,dy\,dx \right.$$

②: $$\int_{y=0}^{a} \int_{x=-\sqrt{a^2-ay}}^{\sqrt{a^2-ay}} (a-x+y)\,dA = \frac{28}{15}a^3.$$

Finding Integration Limits

Using Vertical Cross-Sections:

When faced with evaluating $\iint_R f(x,y)\,dA$, integrating with respect to y and then with respect to x, do the following three steps.

1) Sketch (Fig. a)

2) Finding the y-limits of integration (Fig. b)

3) Finding the x-limits of integration (Fig. c)

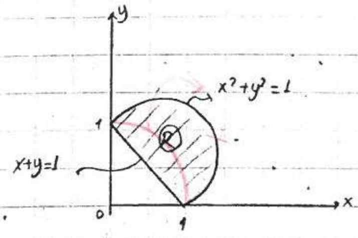


Fig. (a)

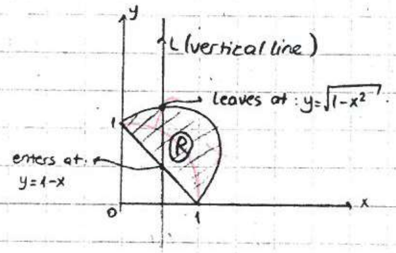


Fig. (b)

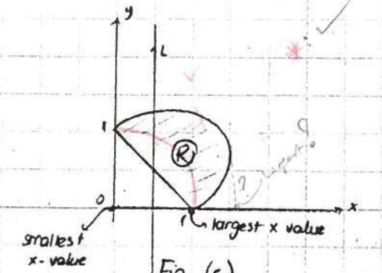
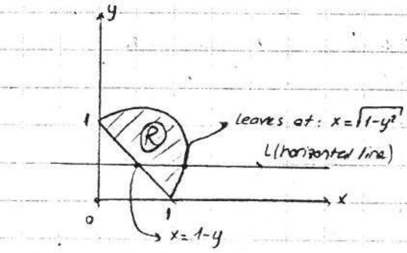


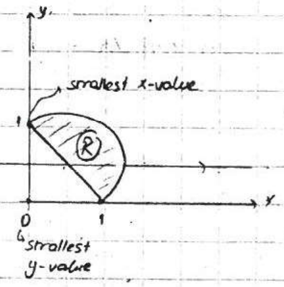
Fig. (c)

$$\iint_R f(x,y)\,dA = \int_{x=0}^{1}\left(\int_{y=1-x}^{\sqrt{1-x^2}} f(x,y)\,dy\right)dx$$

ⓑ Using Horizontal cross-sections

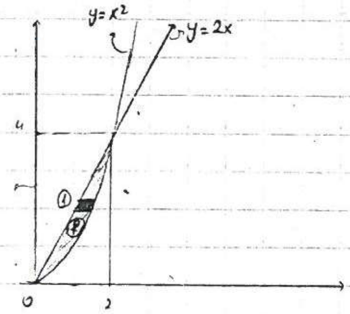
To evaluate the same double integral as an iterated integral with the order integration reversed, use horizontal lines instead of vertical lines in step ① and ③.





$$\iint_R f(x,y)\,dA = \int_{y=0}^{1}\left(\int_{x=1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\right)dy$$

Ex: Sketch the region of integration for the integral $\int_0^2\int_{x^2}^{2x}(4x+2)\,dA$ and write an equivalent integral with the order of integration reversed.



$$\int_{y=0}^{4}\left(\int_{x=y/2}^{\sqrt{y}}(4x+2)\,dx\right)dy = \int_{y=0}^{4}(2x^2+2x)\Big|_{x=y/2}^{\sqrt{y}}$$

$$= \int_{y=0}^{4}\left(2y+2\sqrt{y}-\frac{y^2}{2}-y\right)dy = \int_{y=0}^{4}\left(y+2\sqrt{y}-\frac{y^2}{2}\right)dy$$

$$= \left(\frac{y^2}{2}+\frac{4}{3}y^{3/2}-\frac{y^3}{6}\right)\Big|_{y=0}^{4}$$

$$= 8+\frac{32}{3}-\frac{32}{3} = 8$$

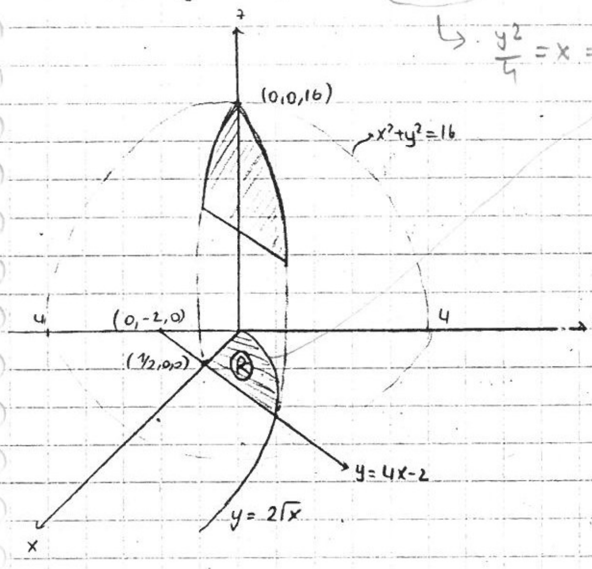
Area by Double Integration

$\iint_R f(x,y)\,dA \rightarrow f(x,y)=1$

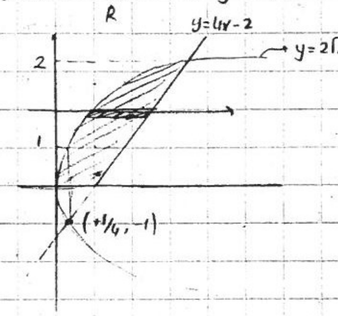
Definition: The area of a closed, bounded plane region R is $\boxed{A=\iint_R dA}$ $\rightarrow dA = dx\,dy = dy\,dx$

Ex: Find the volume of the solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region Ⓡ bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$ and x axis.

$\hookrightarrow \frac{y^2}{4} = x \Rightarrow y = \pm 2\sqrt{x} \Rightarrow y = 2\sqrt{x} > 0 \checkmark$

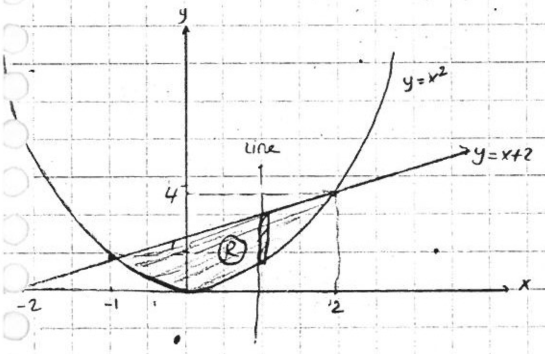


$$V = \iint_R f(x,y)\, dA = \iint_R (16 - x^2 - y^2)\, dA$$



$$\int_{y=0}^{2} \int_{x=y^2/4}^{\frac{y+2}{4}} (16 - x^2 - y^2)\, dx\, dy \cong 12.4$$

Ex: Find the area of the region Ⓡ enclosed by the parabola $y = x^2$ and the line $y = x + 2$



$$\iint_R f(x,y)\, dA = \int_{x=-1}^{2} \int_{y=x^2}^{x+2} dy\, dx = \frac{9}{2} \text{ cubic units}$$

Symmetry in Double-Integration

1) Suppose that Ⓡ is symmetric about y-axis.

a) If ⓕ is odd in ⓧ [if $f(-x,y) = -f(x,y)$], then $\iint_R f(x,y)\, dx\, dy = 0$

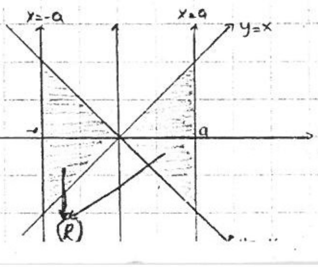
b) If ⓕ is even in ⓧ [if $f(-x,y) = f(x,y)$], then $\iint_R f(x,y)\, dx\, dy = 2 \cdot \iint_{\text{right half of } R} f(x,y)\, dx\, dy$

2) Suppose that Ⓡ is symmetric about x-axis

a) If ⓕ is odd in ⓨ [if $f(x,-y) = -f(x,y)$], then $\iint_R f(x,y)\, dx\, dy = 0$

b) If ⓕ is even in ⓨ [if $f(x,-y) = f(x,y)$], then $\iint_R f(x,y)\, dx\, dy = 2 \iint_{\text{upper half of } R} f(x,y)\, dx\, dy$

Ex: Take the region bounded by $x+y=0$, $x-y=0$, $x=a$, $x=-a$. Suppose that we want to calculate $\iint_R (2x - \sin x^2 y)\, dx\, dy$.



$$\underbrace{\iint_R 2x\, dx\, dy}_{I_1} - \underbrace{\iint_R \sin x^2 y\, dx\, dy}_{I_2} = ?$$

R is Symmetric about y-axis

$I_1 = 0$; $I_2 = -2 \cdot \iint_{\text{right half of } R} \sin x^2 y \, dx dy$

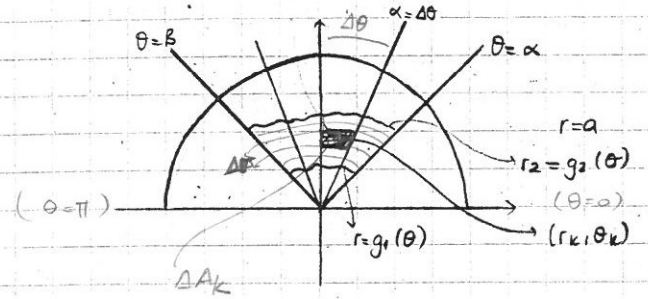
R is Symmetric about x-axis ... right half of

$I_2 = 0 \Rightarrow$ ① and ② $\underline{I = 0}$

Double Integrals in Polar Forms

Integrals in Polar Coordinates

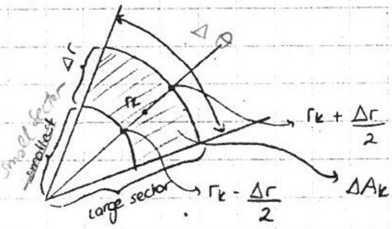
Suppose that a function $f(r,\theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $g_2(\theta)$ $(0 \le g_1(\theta) \le g_2(\theta) \le a)$



We let (r_k, θ_k) be any point in the polar rectangle whose area is ΔA_k. We then form the sum;

$$\boxed{S_n = \sum_{k=1}^{n} f(r_k,\theta_k)\,\Delta A_k \Rightarrow \lim_{n\to\infty} S_n = \iint_R f(r,\theta)\,dA}$$

To evaluate this limit, we first have to write the sum S_n in a way that express ΔA_k in terms of Δr and $\Delta\theta$:



The area of a vedge shaped sector of a circle having radius r and angle θ is $A = \frac{1}{2}\theta r^2$ $\left(A = (\pi r^2)\cdot\frac{\theta}{2\pi}\right)$. So the areas of the circular sectors subtended by these areas at the origin are

$\Rightarrow$ inner radius: $\frac{1}{2} \cdot \Delta\theta \left(r_k - \frac{\Delta r}{2}\right)^2$

$\Rightarrow$ outer radius: $\frac{1}{2} \cdot \Delta\theta \left(r_k + \frac{\Delta r}{2}\right)^2$

Therefore, ΔA_k = area of large sector − area of small sector

$$= \frac{1}{2} \cdot \Delta\theta \cdot \left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right] = r_k \cdot \Delta r \cdot \Delta\theta$$

As $n \to \infty$ and values of Δr and $\Delta\theta$ approach zero, these sum converge to the double integral

$$\lim_{n\to\infty} S_n = \iint_R f(r,\theta) \cdot r \cdot dr \cdot d\theta$$

$$\iint_R f(r,\theta)\,dA = \iint_R f(r,\theta)\,r\,dr\,d\theta = \int_{\theta=\alpha}^{\beta} \int_{r=g_1(\theta)}^{g_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

Area in Polar Coordinate

$A = \iint_R \underbrace{r\,dr\,d\theta}_{dA}$ R: Closed and bounded region

Finding the Limits of Integration

To evaluate $\iint_R f(r,\theta)\,dA$ over a region Ⓡ in polar coordinates, integrating first with respect to ⓡ and then with respect to θ, take the following steps:

1) Sketch the graph (fig. a)

2) Find the ⓡ limits of integration (Fig. b)

3) Find the Ⓣ limits of integration (Fig. c)

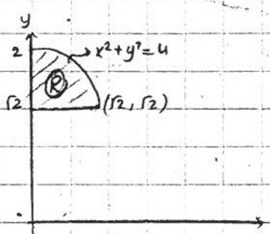


Fig ⓐ

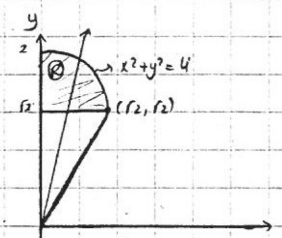


Fig ⓑ

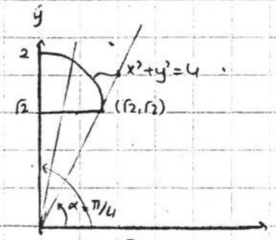


Fig ⓒ

$x = r.\cos\theta$
$y = r.\sin\theta$
$dA = r\,dr\,d\theta$
} polar coordinate transformation

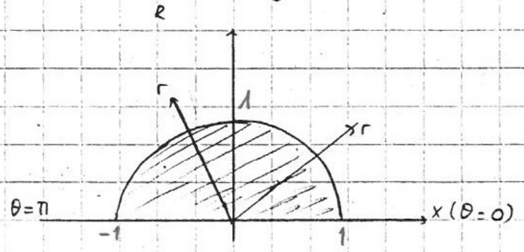
$y=\sqrt{2} \Rightarrow \sqrt{2} = r.\sin\theta$
$r = \sqrt{2}\csc\theta$

$\sqrt{2}\csc\theta \le r \le 2$

$x^2+y^2=4 \Rightarrow (r.\cos\theta)^2 + (r\sin\theta)^2 = 4$
$r^2 = 4$
$r = \mp 2 \Rightarrow r \neq -2$
$r=2$

$$\Rightarrow \boxed{\iint_R f(x,y)\,dA = \int_{\theta=\pi/4}^{\pi/2} \int_{r=\sqrt{2}\csc\theta}^{2} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

Ex: Evaluate $\iint_R e^{x^2+y^2}\,dy\,dx$, where R is the semi-circular region bounded by the x-axis and the curve $\underbrace{y=\sqrt{1-x^2}}_{x^2+y^2=1}$



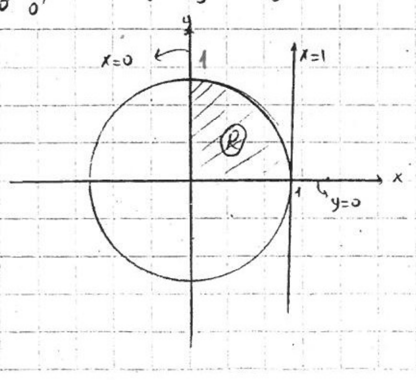
$x = r.\cos\theta$
$y = r\sin\theta$
$\underbrace{dA = r\,dr\,d\theta}_{\text{polar coor. trans.}}$

$0 \le \theta \le \pi$
$0 \le r \le 1$

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_{\theta=0}^{\pi}\left(\int_{r=0}^{1} e^{r^2}\,r.dr\right)d\theta = \int_{\theta=0}^{\pi} \frac{e-1}{2}\,d\theta = \frac{\pi}{2}.(e-1)$$

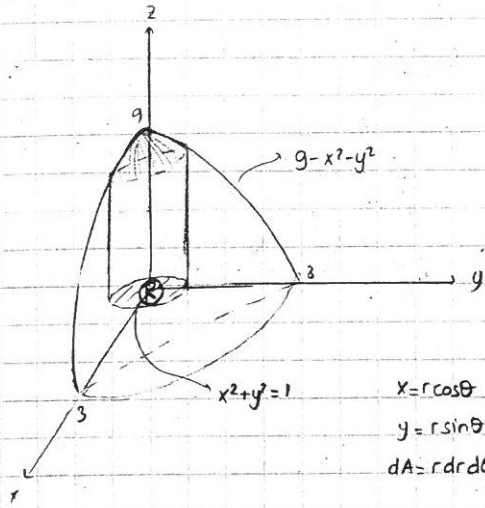
Ex: Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$ by Polar Coordinate.

$x=0$	$y=0$
$x=1$	$y=\sqrt{1-x^2}$

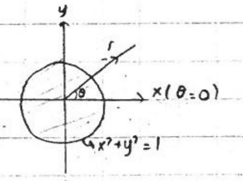


$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r^2.r\,dr\,d\theta = \frac{1}{4}.\frac{\pi}{2} = \frac{\pi}{8}$$

Ex: Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.



$$V = \iint_R \overbrace{f(x,y)}^{z}\,dA = \iint (9 - x^2 - y^2)\,dA$$



$0 \le \theta \le 2\pi$

$0 \le r \le 1$

$x = r\cos\theta$

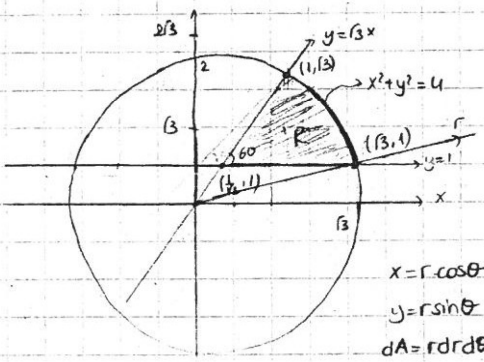
$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1} (9 - r^2)\,r\,dr\,d\theta = \frac{17}{4}\cdot\int_0^{2\pi} d\theta = \frac{17\pi}{2}$$

$$\frac{9}{2}r^2 - \frac{r^4}{4}\Big|_0^1 = \frac{17}{4}$$

Ex: Using the polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2+y^2=4$, above the line $y=1$ and below the line $y=\sqrt{3}x$.



$$A_{rea} = \iint_R dA = \iint_R r\,dr\,d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$y=1 \rightarrow r\sin\theta = 1$

$r = \csc\theta$

$x^2+y^2=4 \rightarrow r=2$

$\csc\theta \le r \le 2$

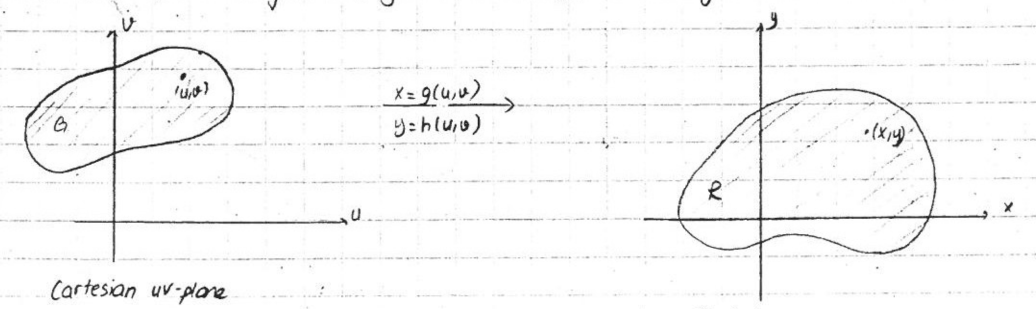
$\tan\theta = 1/\sqrt{3} \rightarrow \theta = \pi/6$

$\tan\theta = \sqrt{3} \rightarrow \theta = \pi/3$

$$= \int_{\theta=\pi/6}^{\pi/3}\left(\int_{r=\csc\theta}^{2} r\,dr\right)d\theta = \frac{1}{2}\cdot\int_{\theta=\pi/6}^{\pi/3}(4 - \csc^2\theta)\,d\theta = \frac{1}{2}\left(4\theta + \cot\theta\right)\Big|_{\pi/6}^{\pi/3} = \frac{\pi - \sqrt{3}}{3}$$

Substitutions in Double Integrals

Suppose that a region G in the uv-plane is transformed one-to-one into the region R in the xy-plane by equations of the form $x = g(u,v)$, $y = h(u,v)$ as suggested in Figure 1.



Cartesian uv-plane

Cartesian xy-plane

Figure 1

51

we call Ⓡ the image of Ⓖ under the transformation, and Ⓖ the preimage of Ⓡ. Any function $f(x,y)$ defined on Ⓡ can be throught of as a function of $f(g(u,v), h(u,v))$ defined on Ⓖ as well.

If ⓖ, ⓗ and ⓕ has continuous partial derivatives and $J(u,v)$ is zero only at isolated points $(J(u,v) \neq 0)$

$$\boxed{\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v), h(u,v))\,|J(u,v)|\,du\,dv} \qquad J(u,v): \text{Jacobian}$$

Definition: The Jacobian determinant or Jacobian of the coordinate transformation $x = g(u,v)$, $y = h(u,v)$ is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

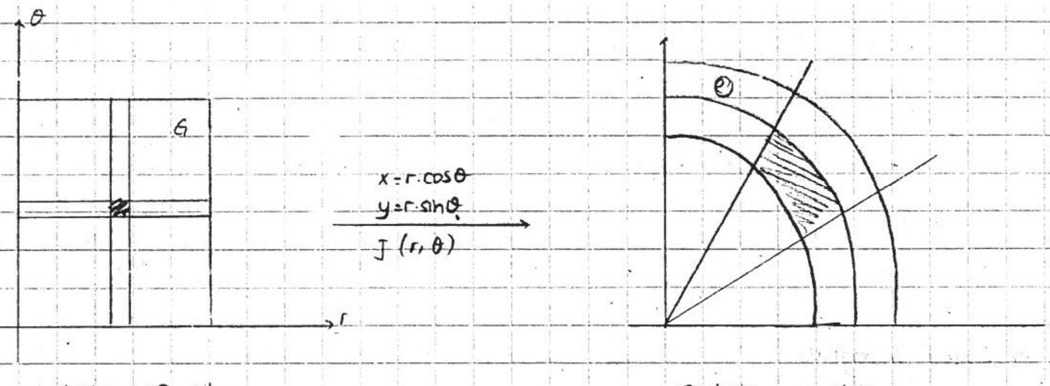
Find the Jacobian for the polar coordinate transformation $x = r.\cos\theta$, $y = r.\sin\theta$ and use equation above (uv-form) to write the cartesian integral $\iint_R f(x,y)\,dx\,dy$ as a polar integral.

$x = r.\cos\theta$
$y = r.\sin\theta$

$$(x,y) \xrightarrow[(J(u,v)\neq 0)]{} (r,\theta) \qquad J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}$$

$$J(r,\theta) = \begin{vmatrix} \cos\theta & -r.\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\Rightarrow \iint_R f(x,y)\,dx\,dy = \iint_G f(r.\cos\theta, r.\sin\theta)\,\underbrace{|J(\theta,r)|}_{=r}\,dr\,d\theta$$

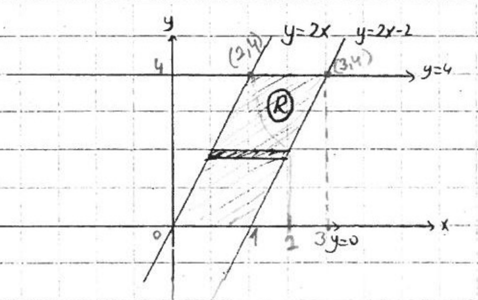


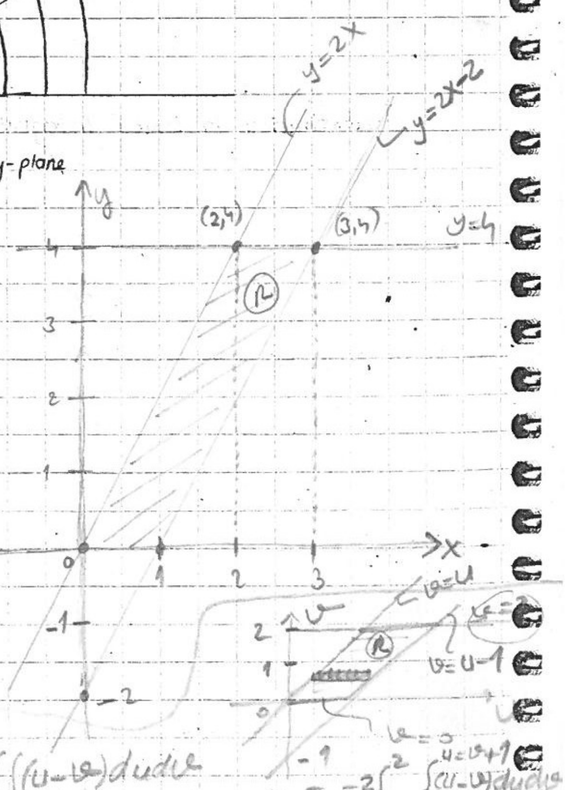
cartesian rθ-plane — cartesian xy-plane

Ex: Evaluate $\displaystyle\int_0^4 \int_{x=y/2}^{x=\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy$

$x = \frac{y}{2}$	$y = 0$
$x = \frac{y}{2} + 1$	$y = 4$

$y = 2x$
$y = 2x - 2$





veya $\begin{cases} x = u \\ \frac{y}{2} = v \end{cases}$ $\quad J(u,v) = 2 \Rightarrow \iint_G f(u,v)\,|J(u,v)|\,du\,dv = 2\iint_G (u-v)\,du\,dv$

$\Rightarrow I = 2\int_{v=0}^{2} \int_{u=v}^{u=v+1} (u-v)\,du\,dv$

$$\left.\begin{array}{l} u = \dfrac{2x-y}{2} \\ \\ v = \dfrac{y}{2} \end{array}\right\} \quad \text{polar transform} \quad \begin{array}{l} x = u+v \\ y = 2v \end{array}$$

$$(x,y) \xrightarrow[J(u,v)]{} (u,v)$$

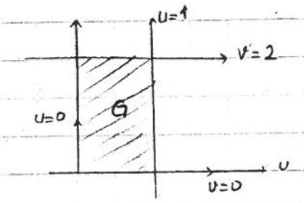
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 \Rightarrow |J(u,v)| = 2$$

$y=0 \Rightarrow v=0$

$y=4 \Rightarrow v=2$

$x=y/2 \Rightarrow u=0$

$x=y/2+1 \Rightarrow u+v=v+1 \Rightarrow u=1$



$$\int_0^4 \int_{x=y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy = \int_{u=0}^{1} \underbrace{\int_{v=0}^{2} (u)\cdot\frac{|J(u,v)|}{2}\,dv}_{2uv\big|_{v=0}^{2} = 4u}\,du = 4\frac{u^2}{2}\Big|_{u=0}^{1} = 2$$

Ex: Evaluate $\int_0^1\int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$

HW*: $\left.\begin{array}{l} x+y=v \\ y-2x=u \end{array}\right\} \Longrightarrow \begin{cases} x = \dfrac{-u+v}{3} \\ y = \dfrac{u+2v}{3} \end{cases}$

$x=0$	$y=0$
$x=1$	$y=1-x$

$x+y=u$

$y-2x=v$

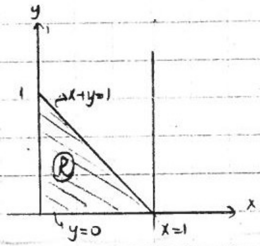
$y=\dfrac{2u+v}{3}$; $x=\dfrac{u-v}{3}$ $\Rightarrow J(u,v) = \begin{vmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{vmatrix} = 1/3$

$x=0 \rightarrow v=u$

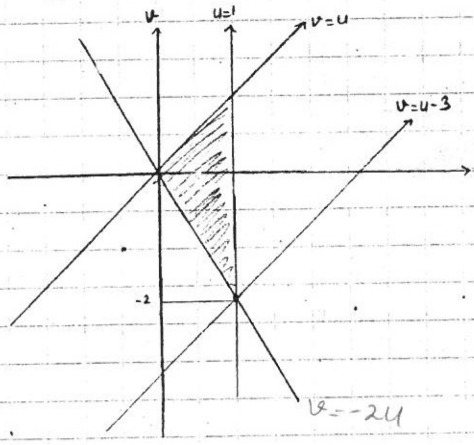
$x=1 \rightarrow u-v=3 \rightarrow v=u-3$

$y=0 \rightarrow v=-2u$

$y=1-x \rightarrow u=1$



$\xrightarrow[J(u,v)]{}$



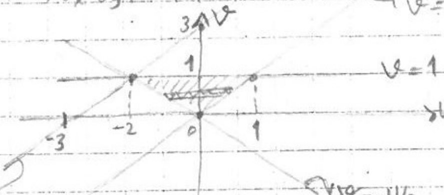
HW*: $J(u,v) = \dfrac{1}{J(x,y)} = \dfrac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \dfrac{1}{\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}} = -\dfrac{1}{3}$ $\quad |J(u,v)| = |-1/3| = 1/3$

$x=0 \Rightarrow v=u$

$x=1 \Rightarrow v=u+3$

$y=0 \Rightarrow v=-u/2$

$x+y=1 \Rightarrow v=1$



$$I = \int_{v=0}^{1}\int_{u=-2v}^{u=v} \sqrt{v}\cdot u^2\cdot\frac{1}{3}\,du\,dv = \int_0^1 v^3\sqrt{v}\,dv = \frac{2}{9}v^{9/2}\Big|_0^1 = 2/9$$

$$I = \int_{u=0}^{1}\int_{v=-2u}^{u} \underbrace{\sqrt{u}\cdot v^2\cdot \underbrace{|J(u,v)|}_{1/3}\,dv}_{\frac{1}{3}\cdot\frac{v^3}{3}\cdot\sqrt{u}\big|_{-2u}^{u}}\,du = 2/9$$

Ex: Evaluate the integral $\int_1^2\int_{1/y}^{y} \sqrt{y/x}\; e^{\sqrt{xy}}\,dx\,dy$ $\quad$ (Result: $2e(e-2) = 2e^2-4e$) $\quad$ ($u=y/x$, $xy=v$)

HW

Ex: If R is that part of the annulus (halka) $0 \le a^2 \le x^2+y^2 \le b^2$ lying in the first quadrant and below the line

HW

$y=x$, evaluate $I = \iint_R \dfrac{y^2}{x^2}\,dA$ $\quad$ (Result: $\dfrac{(b^2-a^2)}{2}\cdot\left(1-\dfrac{\pi}{4}\right)$)

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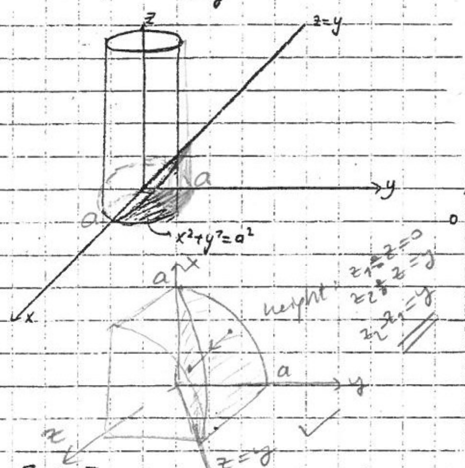
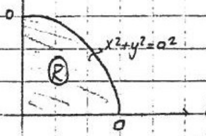
Ex: Find the volume of the solid lying in the first octant, inside the cylinder $x^2+y^2=a^2$ and under the plane $z=y$.



$$V=\iint_R \underbrace{f(x,y)}_{\text{height}}\,dA=\iint_R z\,dA=\iint_R y\,dA=\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y\,dy\,dx$$

$$=\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\sin\theta\, r\,dr\,d\theta$$

$$=\frac{a^3}{3}\int_{\theta=0}^{\pi/2}\sin\theta\,d\theta=a^3/3$$



$x=r\cos\theta$

$y=r\sin\theta$

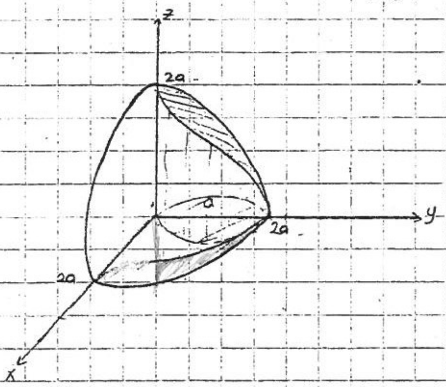
$dA=r\,dr\,d\theta$

$0\le\theta\le\pi/2$

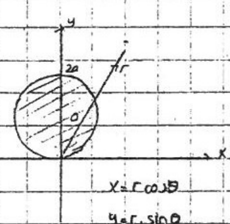
$0\le r\le a$

Ex: Find the volume of the solid lying inside both the sphere $x^2+y^2+z^2=4a^2$ and the cylinder $x^2+y^2=2ay$ where $a>0$

$$x^2+(y-a)^2=a^2$$



$$V=\iint_R f(x,y)\,dA \Rightarrow V=2\cdot\iint\sqrt{4a^2-x^2-y^2}\,dA$$



$$V=2\cdot2\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\sin\theta}\sqrt{4a^2-r^2}\,(r)\,dr\,d\theta$$

$u^2=4a^2-r^2$

$2u\,du=-2r\,dr$

$u\,du=-r\,dr$

$x=r\cos\theta$

$y=r\sin\theta$

$dA=r\,dr\,d\theta$

$0\le\theta\le\pi/2$

$x^2+y^2=2ay\Rightarrow r^2-2ar\sin\theta=0$

$r=0\quad,\quad r=2a\sin\theta$

$$I=4\int_0^{\pi/2}\left(-\frac{\left(\sqrt{4a^2-r^2}\right)^3}{3}\Bigg|_{r=0}^{2a\sin\theta}\right)d\theta=4\int_0^{\pi/2}\left[-\frac{(2a)^3\cos^3\theta}{3}+\frac{(2a)^3}{3}\right]d\theta=\frac{32a^3}{3}\int_0^{\pi/2}(1-\underbrace{\cos^3\theta}_{\cos\theta(1-\sin^2\theta)})\,d\theta$$

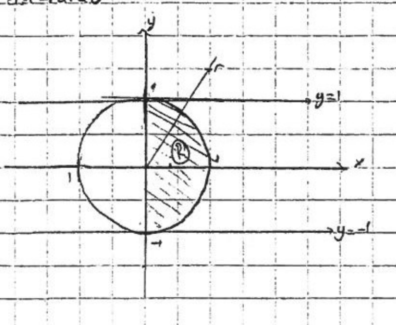
$$=\frac{16}{9}(3\pi-4)a^3 \text{ cubic units}$$

Ex: Evaluate the integral $\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\sqrt{x^2+y^2}\,dx\,dy$ by using polar coord.

$x=r\cos\theta$ $y=1$ $x=\sqrt{1-y^2}$

$y=r\sin\theta$ $y=-1$, $x=0$

$dA=r\,dr\,d\theta$



$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$0\le r\le1$

$$I=\int_{-\pi/2}^{\pi/2}\int_0^1 r\cdot r\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\frac{1}{3}\,d\theta=\frac{\pi}{3}$$

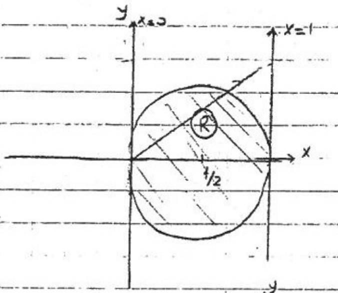
Ex: 1) Evaluate the integral $\int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} (x^2+y^2)\,dy\,dx$ by using Polar Coordinate

Ex: 2) Evaluate $I = \int_1^{\sqrt{2}} \int_0^{x} xy\,dy\,dx + \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-x^2}} xy\,dy\,dx$ by using Polar coordinate

Ex: 1) $x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$



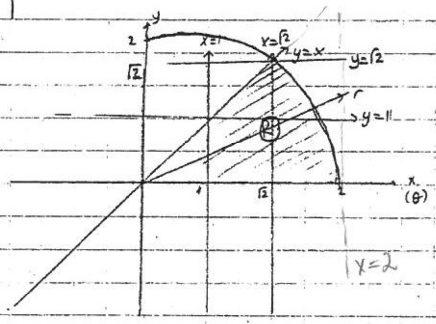
$\left(x-\frac{1}{2}\right) + y^2 = \left(\frac{1}{2}\right)^2$

$-\pi/2 \le \theta \le \pi/2$

$0 \le r \le \cos\theta$

$I = \int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{\cos^4\theta}{4}\,d\theta$

$= \frac{3\pi}{32}$



Ex: 2) $x = 1$ | $y = 0$

$x = \sqrt{2}$ | $y = x$

$x = \sqrt{2}$ | $y = 0$

$x = 2$ | $y = \sqrt{4-x^2}$

$I = \int_0^{\pi/4} \int_{1/\cos\theta}^{2} (r^2 \sin\theta \cos\theta)(r\,dr\,d\theta) = 7/8$

$y = r\cos\theta = 1$

$r = 1/\cos\theta$

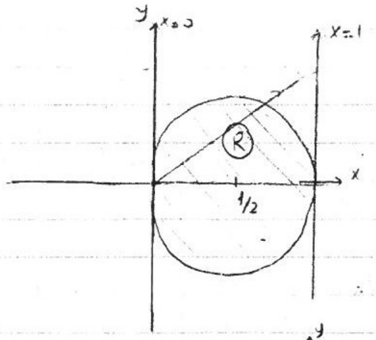
Ex: 1) Evaluate the integral $\int_0^1 \int_{-\sqrt{x-x^2}}^{\sqrt{x-x^2}} (x^2+y^2)\,dy\,dx$ by using Polar coordinate.

Ex: 2) Evaluate $I = \int_1^{\sqrt{2}} \int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} xy\,dy\,dx$ by using Polar coordinate.

Ex: 1) $x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$



$$\left(x-\frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$$

$-\pi/2 \le \theta \le \pi/2$

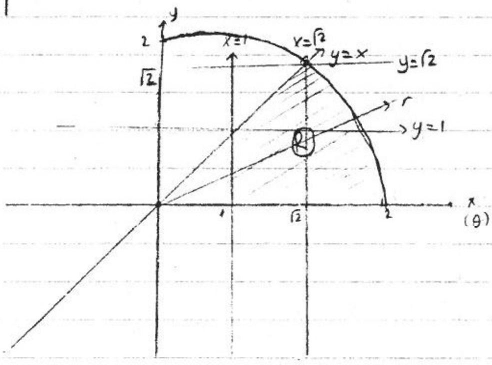
$0 \le r \le \cos\theta$

$$I = \int_{-\pi/2}^{\pi/2} \int_0^{\cos\theta} r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{\cos^4\theta}{4}\,d\theta = \frac{3\pi}{32}$$

Ex: 2)

$x = 1$	$y = 0$
$x = \sqrt{2}$	$y = x$

$x = \sqrt{2}$	$y = 0$
$x = 2$	$y = \sqrt{4-x^2}$



$$I = \int_0^{\pi/4} \int_{1/\cos\theta}^{2} (r^2\sin\theta\cos\theta)\,r\,dr\,d\theta = 7/8$$

$x = r\cdot\cos\theta = 1$

$r = 1/\cos\theta$

08.12.14

Triple Integrals in Cartesian Coordinates

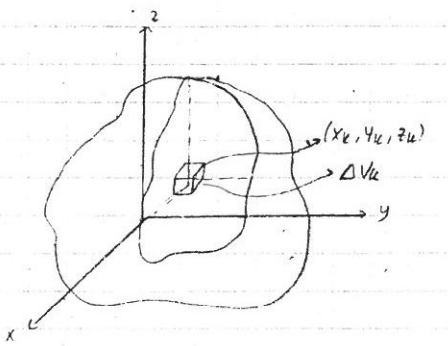
The integral of $F(x,y,z)$ over D may be defined in the following way:

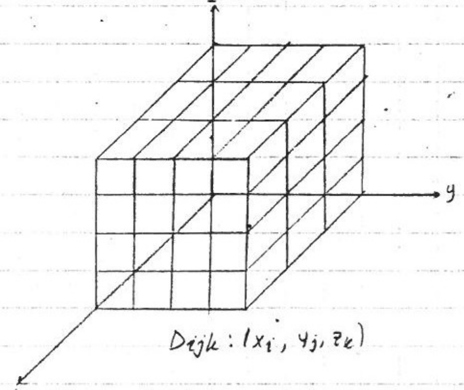
We portion a rectangular box-like region containing D into rectangular cells by planes parallel to the coordinate axis (Fig 1). We number the cells that lie completely inside D form 1 to n in some order. The k^{th} cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V = \Delta x_k \cdot \Delta y_k \cdot \Delta z_k$. We choose a point (x_k, y_k, z_k) in each cell and form the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\cdot\Delta V_k$$

$D: a \le x \le b,\ c \le y \le d,\ p \le z \le q$

F: Continuous in all points over D





$D_{ijk} : (x_i, y_j, z_k)$

$$\left.\begin{aligned} x_{i-1} &\le x \le x_i \\ y_{j-1} &\le y \le y_j \\ z_{k-1} &\le z \le z_k \end{aligned}\right\} \quad \Delta V_{ijk} = \Delta x_i\cdot\Delta y_j\cdot\Delta z_k$$

$\Delta x_i = x_i - x_{i-1}, \quad i = 1,2,\dots,m$

$\Delta y_j = y_j - y_{j-1}, \quad j = 1,2,\dots,n$

$\Delta z_k = z_k - z_{k-1}, \quad k = 1,2,\dots,r$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{r} F(x_i, y_j, z_k) \cdot \Delta V_{ijk} \quad - \quad \lim_{\substack{n,m,r\to\infty \\ (\|P\|\to 0)}} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{r} F(x,y,z)\, \Delta V_{ijk} = \iiint_D F(x,y,z)\, \underset{dxdydz}{dv}$$

Volume of a Region in Space

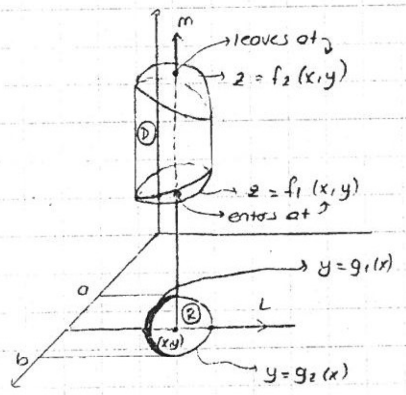
Definition: The volume of a closed bounded region D in space is

$$V = \iiint_D dV \qquad F(x,y,z) = 1$$

Finding the limits of Integration in the order $dz\,dy\,dx$

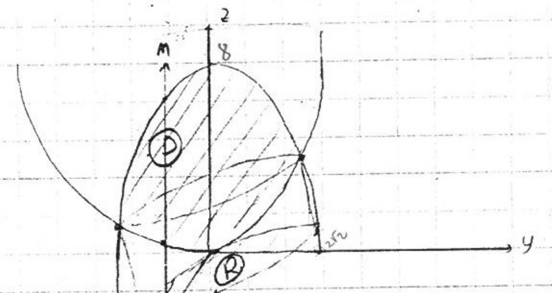
$dV = dz\,dA$
$dV = dy\,dA$
$dV = dx\,dA$

1) Sketch the region
2) Find the z-limits of integration (Line m)
3) Find the y-limits of integration (Line L)
4) Find the x-limits of integration ($x=a$, $x=b$)



$$\iiint_D F(x,y,z)\,dV = \int_{x=a}^{b}\int_{y=g_1(x)}^{g_2(x)}\int_{z=f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,dz\,dy\,dx$$

Ex: Find the volume of the region D enclosed by the surfaces $z = x^2+3y^2$ and $z = 8-x^2-y^2$



1st way:

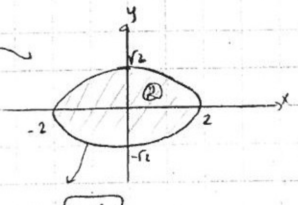
(enters at) (leaves at)

$x^2+3y^2 = 8-x^2-y^2$

$x^2+2y^2 = 4$

$\dfrac{x^2}{(2)^2}+\dfrac{y^2}{(\sqrt{2})^2} = 1$

intersection curve



$y = -\sqrt{\dfrac{4-x^2}{2}}$

$$V = \iiint_D dV = \iint_R \int dz\,dA = \iint_R \int_{z=x^2+3y^2}^{8-x^2-y^2} dz\,dA$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{z=x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} 8-2x^2-4y^2\,dy\,dx$$

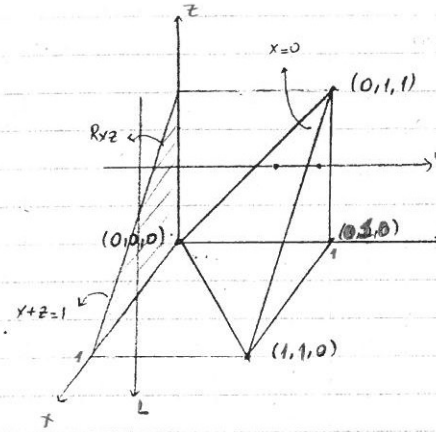
$$= \int_{-2}^{2} \left(8y-2x^2y-\frac{4}{3}y^3\right)\Bigg|_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} = 8\pi\sqrt{2} \text{ cubic units.}$$

2nd way: $\frac{x^2}{2^2} + \frac{y^2}{(\sqrt{2})^2} = 1$ $\quad$ $\left.\begin{array}{l} x = 2r\cos\theta \\ y = \sqrt{2}r\sin\theta \end{array}\right\}$ transformation $\quad$ $(x,y) \xrightarrow[J(r,\theta)]{} (r,\theta)$

$$J(r,\theta) = \begin{vmatrix} 2\cos\theta & -2r \\ \sqrt{2}\sin\theta & \sqrt{2}r\cos\theta \end{vmatrix} = 2\sqrt{2}\,r$$

$$\int_{-2}^{2}\int_{-\sqrt{\frac{4-y^2}{2}}}^{\sqrt{\frac{4-y^2}{2}}} (8-2x^2-4y^2)\,dy\,dx = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left[8-2(2r\cos\theta)^2 - 4(\sqrt{2}\,r\sin\theta)^2\right] . 2\sqrt{2}\,r\,dr\,d\theta$$

Ex: Set up the limits of integration for evaluating the triple integral of a function $F(x,y,z)$ over the tetrahedron D with vertices $(0,0,0)$, $(1,1,0)$, $(0,1,0)$ and $(0,1,1)$. Use the order of integration $dy\,dz\,dx$.



$Ax + By + Cz + D = 0 \Rightarrow$ $(0,0,0)$: $D = 0$

$(1,1,0)$: $A + B = 0 \Rightarrow A = -B$

$(0,1,1)$: $B + C = 0 \Rightarrow B = -C$

$-Bx + By - Bz = 0$

$B.(-x+y-z) = 0 \Rightarrow y = x+z$

$B \neq 0$

$$\iiint_D F(x,y,z)\,dV = \int_{x=0}^{1}\int_{z=0}^{1-x}\int_{y=x+z}^{1} F(x,y,z)\,dy\,dz\,dx$$

$F(x,y,z) = 1 \Longrightarrow \iiint_D dV = \frac{1}{6}$

Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical Coordinates

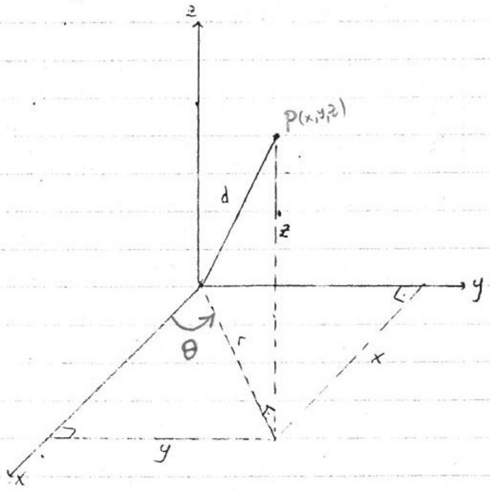


Fig.: The cylindrical coordinates of a point

$x^2+y^2 = r^2$; $\tan\theta = \frac{y}{x}$; $r \geq 0$; $0 \leq \theta \leq 2\pi$; $-\infty \leq z \leq \infty$

Cylindrical Coord. Trans.

$x = r.\cos\theta$

$y = r\sin\theta$

$z = z$

$dV = |J(r,\theta,z)|\,dr\,d\theta\,dz = r\,dr\,d\theta\,dz$

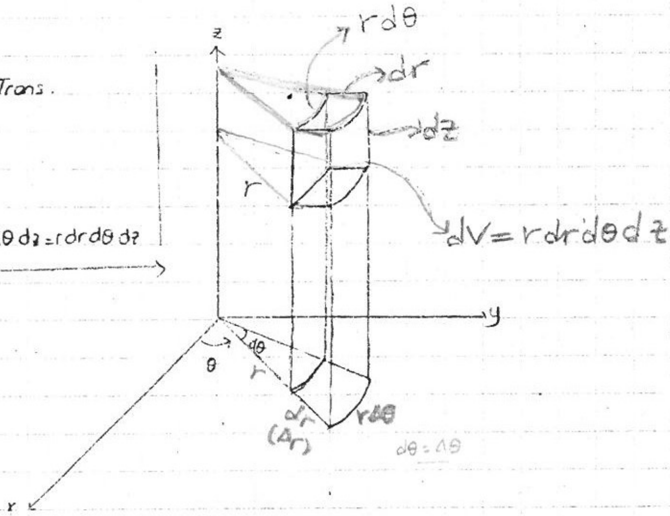


Fig.(2) Volume element to cylindrical coordinates

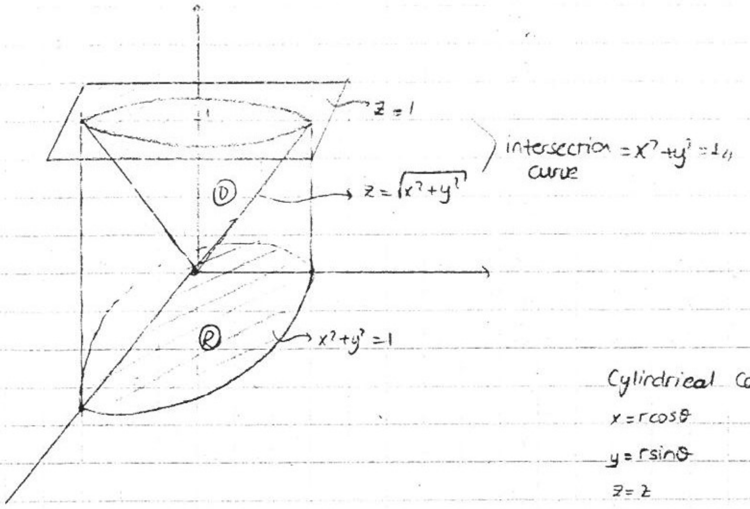
$dV = r\,dr\,d\theta\,dz = |J(r,\theta,z)|\,dr\,d\theta\,dz$

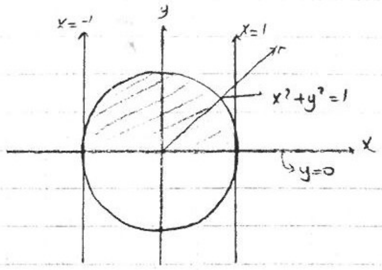
$$J(r,\theta,z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Ex: $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} z^3\,dz\,dy\,dx$, evaluate the integral by using the cylindrical coordinates

$x = -1$ $\quad$ $y = \sqrt{1-x^2}$ $\quad$ $z = \sqrt{x^2+y^2}$ (cone)

$x = 1$ $\quad$ $y = 0$ $\quad$ $z = 1$ (plane)





Cylindrical Coordinates

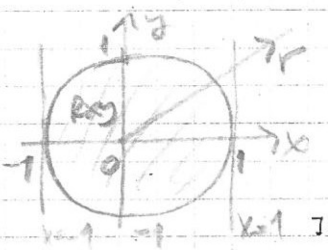
$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ dv &= r\,dz\,dr\,d\theta \end{aligned}\right\} \quad I = \int_{\theta=0}^{\pi}\int_{r=0}^{1}\int_{z=r}^{1} z^3 r\,dz\,dr\,d\theta = \pi/12$$

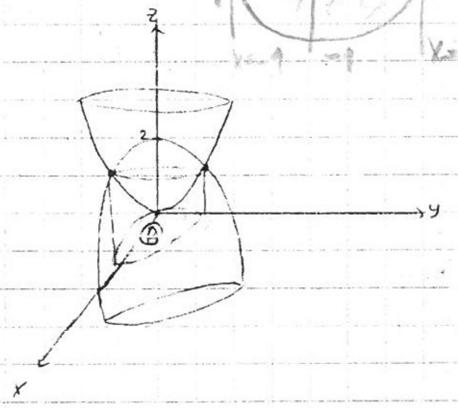
15.12.14 /

Ex: Evaluate the integral $\underbrace{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}}_{R_{xy}}\int_{x^2+y^2}^{2-x^2-y^2} \sqrt{x^2+y^2}\; dz\,dy\,dx$ by using cylindrical coord.

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ dV &= \underbrace{r}_{|J(r,\theta,z)|}dz\,dr\,d\theta \end{aligned}\right\} \quad (x,y,z) \xrightarrow[J(r,\theta,z)]{} (r,\theta,z)$$

$$\Rightarrow \underbrace{\begin{array}{c|c|c} x=-1 & y=-\sqrt{1-x^2} & z = x^2+y^2 \\ x = 1 & y=\sqrt{1-x^2} & z = 2-x^2-y^2 \\ (\text{line}) & (\text{circle}) & (\text{paraboloid}) \end{array}}_{R}$$





$$J = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r^2}^{2-r^2} r\cdot(r\,dz\,dr\,d\theta) = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (r^2\cdot z)\Big|_{z=r^2}^{2-r^2}$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (2r^2 - r^4 - r^4)\,dr\,d\theta = \int_{\theta=0}^{2\pi}\left(\frac{2}{3}r^3 - \frac{2}{5}r^5\right)\Big|_0^1$$

$$= \int_{\theta=0}^{2\pi}\frac{4}{15}\,d\theta = \frac{4\theta}{15}\Big|_0^{2\pi} = \frac{8\pi}{15}$$

Integration with Spherical Coordinates

The system of spherical coordinates related to cartesian coordinates x, y, z and cylindrical coordinates r, θ, z by the eqn.

$$x = \underbrace{\rho\cdot\sin\phi}_{r}\cos\theta$$
$$y = \underbrace{\rho\cdot\sin\phi}_{r}\cdot\sin\theta$$
$$z = \rho\cdot\cos\phi$$

$x^2+y^2+z^2 = \rho^2$, $\tan\phi = \frac{r}{z} = \frac{\sqrt{x^2+y^2}}{z}$, $\tan\theta = \frac{y}{x}$

$r = \sqrt{x^2+y^2} = \rho\cdot\sin\phi$, $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$

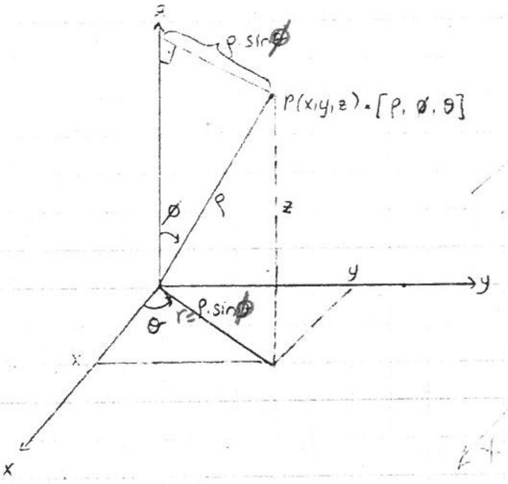
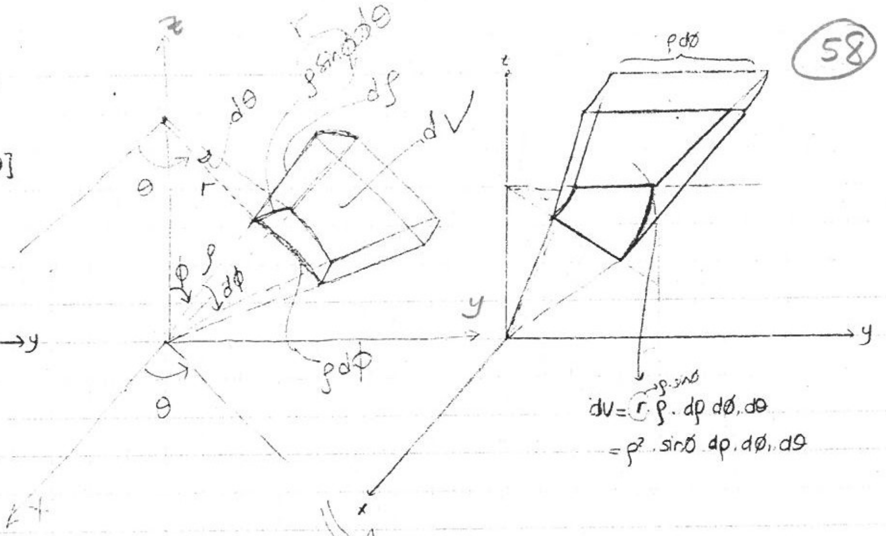


Fig 1. : The spherical coordinates of a point P



$dV = r\, \rho\, d\rho\, d\emptyset\, d\theta$ (with $r = \rho \sin\emptyset$)

$= \rho^2 \sin\emptyset\, d\rho\, d\emptyset\, d\theta$

Fig 2 : The volume element in spherical coord.

$x^2+y^2+z^2=\rho^2 \quad \tan\phi = \frac{r}{z} = \frac{\sqrt{x^2+y^2}}{z}, \tan\theta = \frac{y}{x}$

$r = \sqrt{x^2+y^2} = \rho\sin\phi, \ \rho \geq 0 \quad 0 \leq \theta \leq 2\pi, \ 0 \leq \phi \leq \pi$

$\rho \to \rho + d\rho$

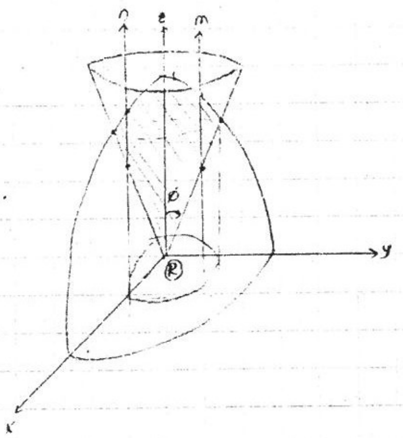
$\theta \to \theta + d\theta$

$\emptyset \to \emptyset + d\emptyset$

$dV = \rho^2 \sin\emptyset\, d\rho\, d\theta\, d\emptyset = \underbrace{|J(\rho,\emptyset,\theta)|}\, d\rho\, d\emptyset\, d\theta$

$\left[(x,y,z) \xrightarrow[J(\rho,\emptyset,\theta)]{} (\rho,\emptyset,\theta) \right]$

Ex: Evaluate the integral $I = \iiint e^{(x^2+y^2+z^2)^{3/2}} dx dy dz$ over the region enclosed by the surface $z=\sqrt{x^2+y^2}$ (cone) and above by the surface $x^2+y^2+z^2=1$ (sphere) by using the spherical coordinates.



$\iint_R \int_{z=\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} e^{(\ )} dz dA \Rightarrow$

$x = \rho \cdot \sin\emptyset \cdot \cos\theta$

$y = \rho \cdot \sin\emptyset \cdot \sin\theta$

$z = \rho\cos\emptyset$

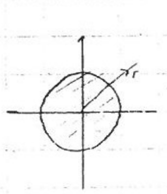
$dV = \rho^2 . \sin\emptyset\, d\rho\, d\emptyset\, d\theta$

$\emptyset$: 1st way : $z=y \Rightarrow \tan\emptyset = \frac{z}{y} = 1 \Rightarrow \tan\emptyset = 1 \Rightarrow \emptyset = \pi/4$

2nd way : $z = \rho . \cos\emptyset \to \frac{1}{\sqrt{2}} = 1 . \cos\emptyset \Rightarrow \emptyset = \pi/4$

$\Big\} \ 0 \leq \emptyset \leq \pi/4$

θ :



$\Big\} \ 0 \leq \theta \leq 2\pi \qquad \rho : 0 \leq \rho \leq 1$

$I = \iiint e^{(\rho^2)^{3/2}} \rho^2 \sin\emptyset\, d\rho\, d\emptyset\, d\theta = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 e^{\rho^3} . \rho^2 \sin\emptyset\, d\rho\, d\emptyset\, d\theta$

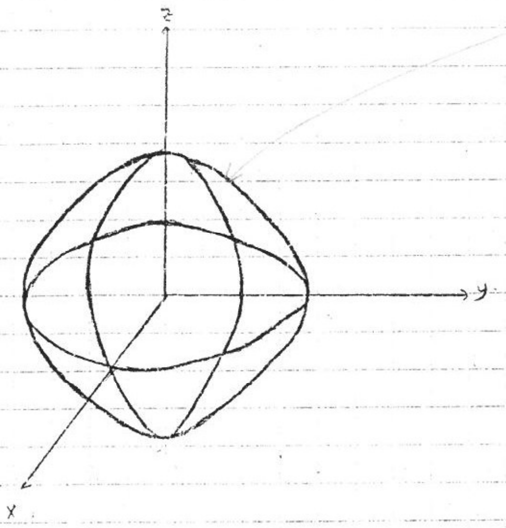
$= \int_0^{2\pi}\int_0^{\pi/4} \sin\emptyset . \frac{e^{\rho^3}}{3}\Big|_0^1 d\rho\, d\emptyset\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \sin\phi . \frac{(e-1)}{3} d\emptyset\, d\theta$

$= \frac{(e-1)}{3} . \int_0^{2\pi} (-\cos\emptyset)\Big|_0^{\pi/4} d\theta = \frac{(e-1)}{3}\int_0^{2\pi} 1 - \frac{\sqrt{2}}{2}\, d\theta = \frac{(e-1)}{3}\left(\theta - \frac{\sqrt{2}\theta}{2}\right)\Big|_0^{2\pi}$

$= \frac{(e-1)}{3} . 2\pi . \left(1 - \frac{\sqrt{2}}{2}\right)$

Ex: Find the volume of the region enclosed by the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ using the triple integral

$V=\iiint_V dV$



$x=au$
$y=bv$
$z=cw$

$u^2+v^2+w^2=1$ (sphere)

uvw = coord. plane

$(x,y,z)\xrightarrow[J(u,v,w)]{}(u,v,w) \Rightarrow J(u,v,w)=a.b.c$

$V=abc\iiint_D du\,dv\,dw=\frac{4}{3}\pi abc$

$(x)\to u=\rho.\sin\phi.\cos\theta$
$(y)\to v=\rho.\sin\phi.\sin\theta$
$(z)\to w=\rho.\cos\phi$

$dV=\rho^2.\sin\phi.d\rho.d\phi\,d\theta \Rightarrow$ $0\le\phi\le\pi$, $0\le\theta\le 2\pi$, $0\le\rho\le 1$

$$V=abc\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{1}\rho^2\sin\phi\,d\rho.d\phi\,d\theta$$

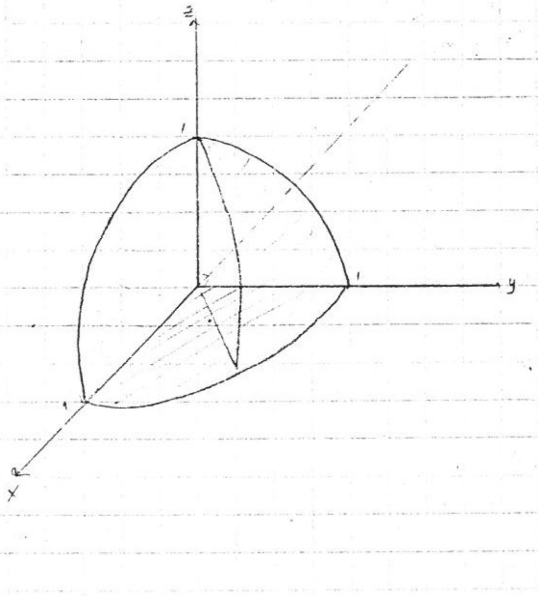
$$=\int_0^{2\pi}\int_0^{\pi}\left(\frac{\rho^3}{3}\sin\phi\right)\Big|_{\rho=0}^{1}d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi}\frac{1}{3}\sin\phi\,d\phi\,d\theta$$

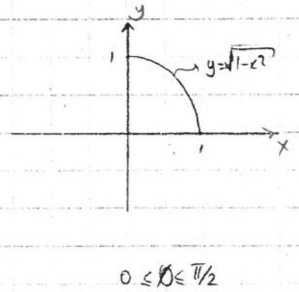
$$=\int_0^{2\pi}\frac{1}{3}(-\cos\phi)\Big|_{\phi=0}^{\pi}=\frac{1}{3}\int_0^{2\pi}\underbrace{-\cos\pi+\cos 0}_{2}=\frac{1}{3}(2\theta)\Big|_{\theta=0}^{2\pi}$$

$$V=\frac{4}{3}\pi.abc$$

Ex: Evaluate $I=\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{x^2+y^2+z^2}\,dz\,dy\,dx$ by using spherical coor.

$x=0$ $\quad y=0$ $\quad z=0$
$x=1$ $\quad y=\sqrt{1-x^2}$ (semi-circle) $\quad z=\sqrt{1-x^2-y^2}$ (semi-sphere)





$x=\rho.\sin\phi.\cos\theta$
$y=\rho.\sin\phi.\sin\theta$
$z=\rho.\cos\phi$
$dV=\rho^2.\sin\phi\,d\rho\,d\phi\,d\theta$

$0\le\phi\le\pi/2$
$0\le\theta\le\pi/2$
$0\le\rho\le 1$

$$I=\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{1}\frac{1}{\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{\pi/2}\int_0^{\pi/2}\sin\phi\,d\phi\,d\theta$$

$$=\int_0^{\pi/2}1\,d\theta=\theta\Big|_0^{\pi/2}=\pi/2$$

Ex: Evaluate $\int_0^3\int_0^4\int_{y/2}^{y/2+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$ by applying the transformation $u=\frac{2x-y}{2}$, $v=\frac{y}{2}$, $w=\frac{z}{3}$

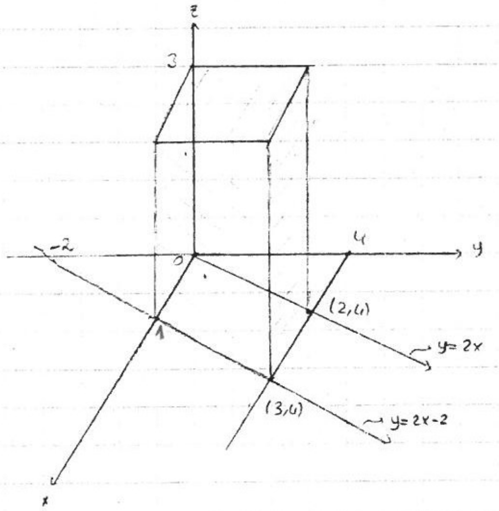
$$\left.\begin{array}{lll} z=0 & y=0 & x=y/2 \\ z=3 & y=4 & x=y/2+1 \end{array}\right\} \qquad (x,y,z) \xrightarrow[J(u,v,w)]{} (u,v,w) \quad \Rightarrow \quad J(u,v,w)=\frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}}=\frac{1}{\begin{vmatrix}1 & -1/2 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1/3\end{vmatrix}}=\frac{1}{\frac{1}{6}}=6$$

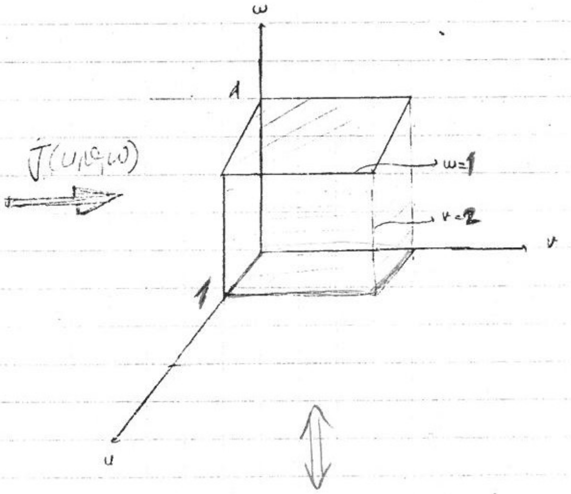
or

$$\left.\begin{array}{l} x=u+v \\ y=2v \\ z=3w \end{array}\right\} \quad J(u,v,w)=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}1 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{vmatrix}=6$$

$$I=\iiint (u+w)\,|J(u,v,w)|\,dw\,dv\,du$$

$$I=\int_{u=0}^{1}\int_{v=0}^{2}\int_{w=0}^{1} 6\cdot(u+w)\,dw\,dv\,du = 12$$





$z=0 \Rightarrow 3w=0 \Rightarrow w=0$

$z=3 \Rightarrow 3w=3 \Rightarrow w=1$

$y=0 \Rightarrow 2v=0 \Rightarrow v=0$

$y=4 \Rightarrow 2v=4 \Rightarrow v=2$

$2x-y=0 \Rightarrow 2u=0 \Rightarrow u=0$

$2x-y=2 \Rightarrow 2u=2 \Rightarrow u=1$