CHAPTER EIGHT

Approximate Methods of Solving First-Order Equations

In Chapter 2 we considered certain special types of first-order differential equations having closed-form solutions that can be obtained exactly. For a first-order differential equation that is not of one or another of these special types, it usually is not apparent how one should proceed in an attempt to obtain a solution exactly. Indeed, in most such cases the discovery of an exact closed-form solution in terms of elementary functions would be an unexpected luxury! Therefore, one considers the possibilities of obtaining approximate solutions of first-order differential equations. In this chapter we shall introduce several approximate methods. In the study of each method in this chapter our primary concern will be to obtain familiarity with the procedure itself and to develop skill in applying it. In general we shall not be concerned here with theoretical justifications and extended discussions of accuracy and error. We shall leave such matters, important as they are, to more specialized and advanced treatises and instead shall concentrate on the formal details of the various procedures.

8.1 GRAPHICAL METHODS

A. Line Elements and Direction Fields

In Chapter 1 we considered briefly the geometric significance of the first-order differential equation

$$\frac{dy}{dx} = f(x, y), \tag{8.1}$$

where f is a real function of x and y. The explicit solutions of (8.1) are certain real functions, and the graphs of these solution functions are curves in the xy plane called the *integral curves* of (8.1). At each point (x, y) at which $f(x, y)$ is defined, the differential equation (8.1) defines the slope $f(x, y)$ at the point (x, y) of the integral curve of (8.1) that passes through this point. Thus we may construct the tangent to an integral curve of (8.1) at a given point (x, y) without actually knowing the solution function of which this integral curve is the graph.

We proceed to do this. Through the point (x, y) we draw a short segment of the tangent to the integral curve of (8.1) that passes through this point. That is, through (x, y) we construct a short segment the slope of which is $f(x, y)$, as given by the differential equation (8.1). Such a segment is called a *line element* of the differential equation (8.1).

For example, let us consider the differential equation

$$\frac{dy}{dx} = 2x + y. \tag{8.2}$$

Here $f(x, y) = 2x + y$, and the slope of the integral curve of (8.2) that passes through the point (1, 2) has at this point the value

$$f(1, 2) = 4.$$

Thus through the point (1, 2) we construct a short segment of slope 4 or, in other words, of angle of inclination approximately 76° (see Figure 8.1). This short segment is the line element of the differential equation (8.2) at the point (1, 2). It is tangent to the integral curve of (8.2) which passes through this point.

Let us now return to the general equation (8.1). A line element of (8.1) can be constructed at every point (x, y) at which $f(x, y)$ in (8.1) is defined. Doing so for a selection of different points (x, y) leads to a configuration of selected line elements that indicates the directions of the integral curves at the various selected points. We shall refer to such a configuration as a *line element configuration.*

For each point (x, y) at which $f(x, y)$ is defined, the differential equation (8.1) thus defines a line segment with slope $f(x, y)$, or, in other words, a direction. Each such point, taken together with the corresponding direction so defined, constitutes the so-called *direction field* of the differential equation (8.1). We say that the differential equation (8.1) defines this direction field, and this direction field is represented graphically by a line element configuration. Clearly a more thorough and carefully constructed line element configuration gives a more accurate graphical representation of the direction field.

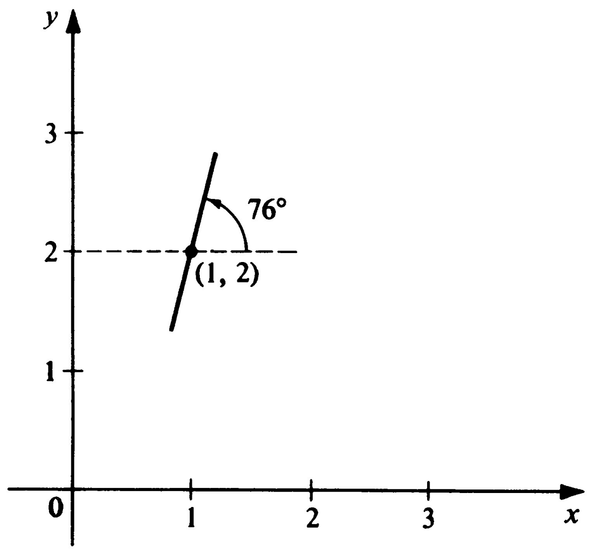

Figure 8.1 Line element of differential equation (8.2) at point (1, 2).

For a given differential equation of the form (8.1), let us assume that a "thorough and carefully constructed" line element configuration has been drawn. That is, we assume that line elements have been carefully constructed at a relatively large number of carefully chosen points. Then this resulting line element configuration will indicate the presence of a family of curves tangent to the various line elements constructed at the different points. This indicated family of curves is approximately the family of integral curves of the given differential equation. Actual smooth curves drawn tangent to the line elements as the configuration indicates will thus provide approximate graphs of the true integral curves.

Thus the construction of the line element configuration provides a procedure for approximately obtaining the solution of the differential equation in graphical form. We now summarize this basic graphical procedure and illustrate it with a simple example.

Summary of Basic Graphical Procedure

1. Carefully construct a line element configuration, proceeding until the family of "approximate integral curves" begins to appear.
2. Draw smooth curves as indicated by the configuration constructed in Step 1.

▶ Example 8.1

Construct a line element configuration for the differential equation

$$\frac{dy}{dx} = 2x + y, \tag{8.2}$$

and use this configuration to sketch the approximate integral curves.

Solution. The slope of the exact integral curve of (8.2) at any point (x, y) is given by

$$f(x, y) = 2x + y.$$

We evaluate this slope at a number of selected points and so determine the approximate inclination of the corresponding line element at each point selected. We then construct the line elements so determined. From the resulting configuration we sketch several of the approximate integral curves. A few typical inclinations are listed in Table 8.1 and the completed configuration with the approximate integral curves appears in Figure 8.2.

TABLE 8.1

x	y	dy/dx *(Slope)*	*Approximate inclination of line element*
$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	27°
$\frac{1}{2}$	0	1	45°
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	56°
$\frac{1}{2}$	1	2	63°
1	$-\frac{1}{2}$	$\frac{3}{2}$	56°
1	0	2	63°
1	$\frac{1}{2}$	$\frac{5}{2}$	68°
1	1	3	72°

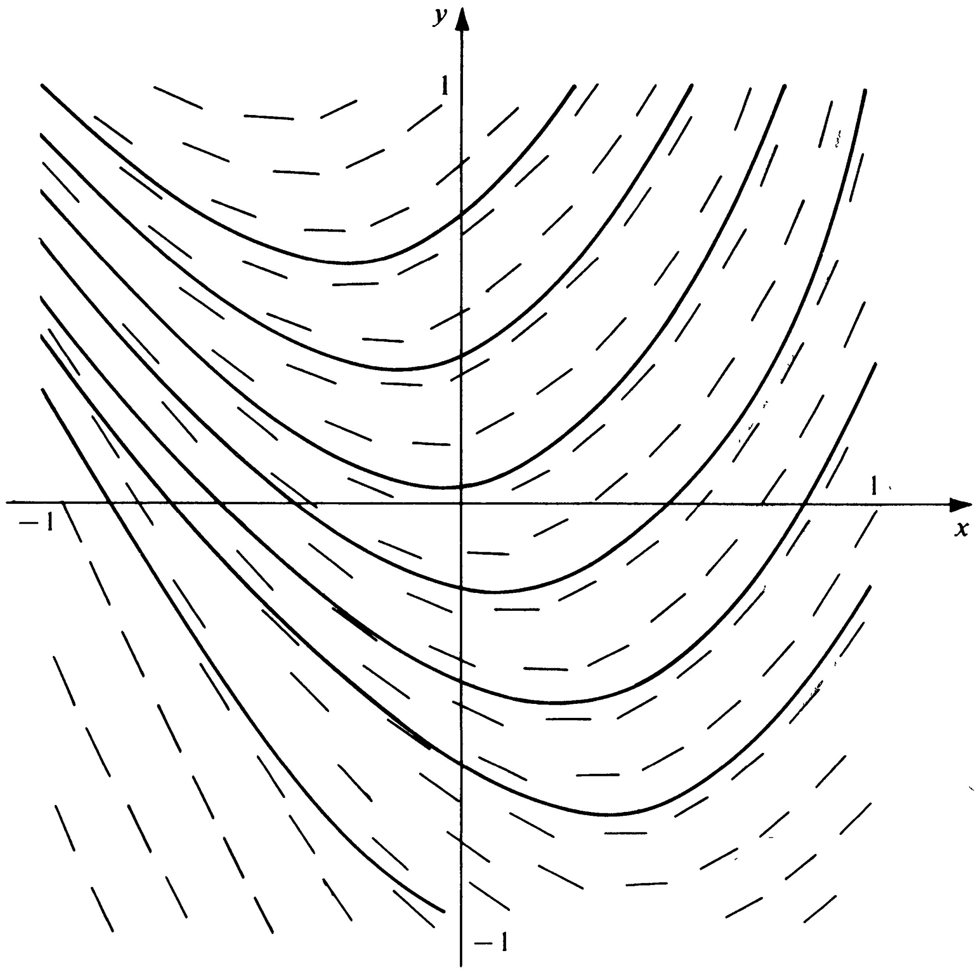

Figure 8.2

Comments. The basic graphical procedure outlined here is very general since it can be applied to any first-order differential equation of the form (8.1). However, the method has several obvious disadvantages. For one thing, although it provides the approximate graphs of the integral curves, it does not furnish analytic expressions for the solutions, either exactly or approximately. Furthermore, it is extremely tedious and time-consuming. Finally, the graphs obtained are only approximations to the graphs of the exact integral curves, and the accuracy of these approximate graphs is uncertain. Of course, apparently better approximations can be obtained by constructing more complete and careful line element configurations, but this in turn increases the time and labor involved. We shall now consider a procedure by which the process may be speeded up considerably. This is the so-called *method of isoclines.*

B. The Method of Isoclines

DEFINITION

Consider the differential equation

$$\frac{dy}{dx} = f(x, y). \tag{8.1}$$

A curve along which the slope $f(x, y)$ has a constant value c is called an isocline *of the* *differential equation (8.1). That is, the isoclines of (8.1) are the curves $f(x, y) = c$, for different values of the parameter c.*

For example, the isoclines of the differential equation

$$\frac{dy}{dx} = 2x + y \tag{8.2}$$

are the straight lines $2x + y = c$. These are of course the straight lines $y = -2x + c$ of slope -2 and y-intercept c.

Caution. Note carefully that the isoclines of the differential equation (8.1) are *not* in general integral curves of (8.1). An isocline is merely a curve along which all of the line elements have a single, fixed inclination. This is precisely why isoclines are useful. Since the line elements along a given isocline all have the same inclination, a great number of line elements can be constructed with ease and speed, once the given isocline is drawn and *one* line element has been constructed upon it. This is exactly the procedure that we shall now outline.

Method of Isoclines Procedure

1. From the differential equation

$$\frac{dy}{dx} = f(x, y) \tag{8.1}$$

determine the family of isoclines

$$f(x, y) = c, \tag{8.3}$$

and carefully construct several members of this family.

2. Consider a particular isocline $f(x, y) = c_0$ of the family (8.3). At all points (x, y) on this isocline the line elements have the same slope c_0 and hence the same inclination $\alpha_0 = \arctan c_0$, $0° \le \alpha_0 < 180°$. At a series of points along this isocline construct line elements having this inclination α_0.

3. Repeat Step 2 for each of the isoclines of the family (8.3) constructed in Step 1. In this way the line element configuration begins to take shape.

4. Finally, draw the smooth curves (the approximate integral curves) indicated by the line element configuration obtained in Step 3.

▶ Example 8.2

Employ the method of isoclines to sketch the approximate integral curves of

$$\frac{dy}{dx} = 2x + y. \tag{8.2}$$

Solution. We have already noted that the isoclines of the differential equation (8.2) are the straight lines $2x + y = c$ or

$$y = -2x + c. \tag{8.4}$$

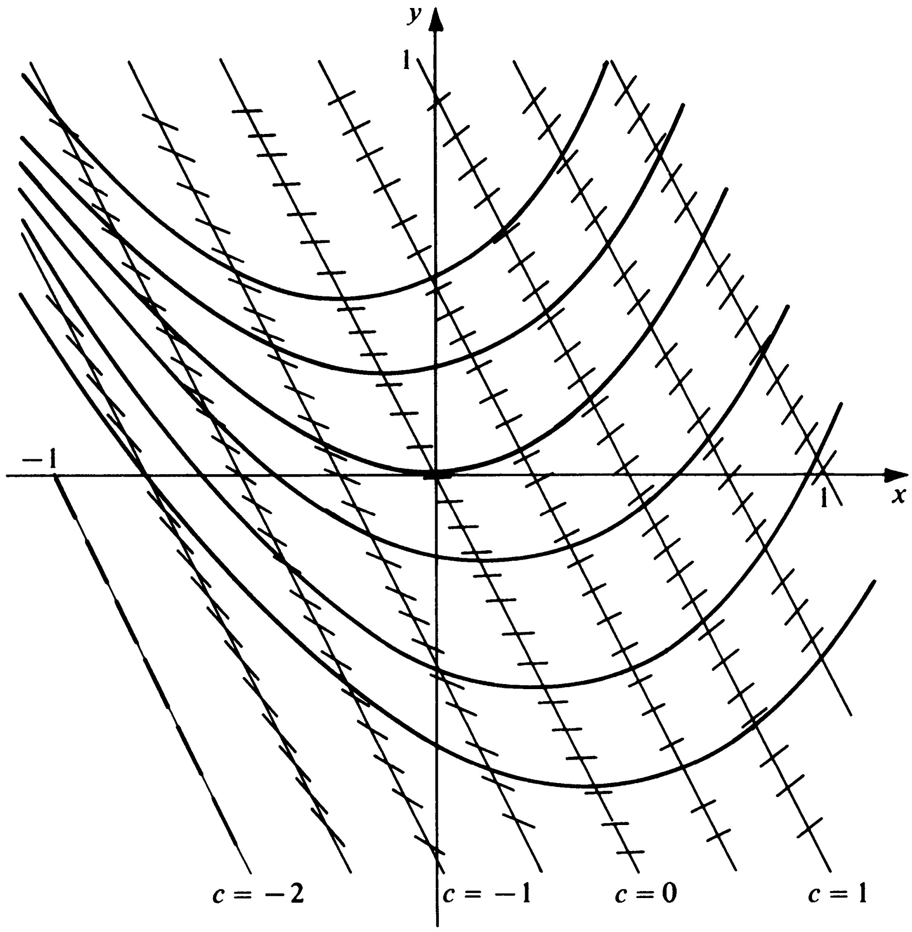

Figure 8.3

In Figure 8.3 we construct these lines for $c = -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}$, and 2. On each of these we then construct a number of line elements having the appropriate inclination $\alpha = \arctan c$, $0° \leq \alpha < 180°$. For example, for $c = 1$, the corresponding isocline is $y = -2x + 1$, and on this line we construct line elements of inclination $\arctan 1 = 45°$. In the figure the isoclines are drawn with dashes and several of the approximate integral curves are shown (drawn solidly).

► Example 8.3

Employ the method of isoclines to sketch the approximate integral curves of

$$\frac{dy}{dx} = x^2 + y^2. \tag{8.5}$$

Solution. The isoclines of the differential equation (8.5) are the concentric circles $x^2 + y^2 = c$, $c > 0$. In Figure 8.4 the circles for which $c = \frac{1}{16}, \frac{1}{4}, \frac{9}{16}, 1, \frac{25}{16}, \frac{9}{4}, \frac{49}{16}$, and 4 have been drawn with dashes, and several line elements having the appropriate inclination have been drawn along each. For example, for $c = 4$, the corresponding isocline is the circle $x^2 + y^2 = 4$ of radius 2, and along this circle the line elements have inclination $\arctan 4 \approx 76°$. Several approximate integral curves are shown (drawn solidly).