2. Mertebeden Lineer Dif Denklemlerin Serilerle Çözümü

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$ dif denkleminin kuvvet serisi ile çözümleri incelenecektir.

Tanım: $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$

$$y'' + \frac{a_1(x)}{a_0(x)}y' + \frac{a_2(x)}{a_0(x)}y = 0$$

Eğer bir x_0 noktası için

$a_0(x_0) \neq 0$ ise x_0 noktasına adi nokta denir.

$a_0(x_0) = 0$ " " " tekil " ".

Teorem; Eğer $x = x_0$ noktası $y'' + p(x)y' + q(x)y = 0$ dif denkleminin bir adi noktası ise bu x_0 noktasının uygun bir civarında dif denklemin

$$y = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

şeklinde bir seri çözümü bulunur.

Teorem; $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ ve $\sum_{n=0}^{\infty} b_n(x - x_0)^n$ eğer bu iki kuvvet serisi ortak bir aralıkta yakınsak ise

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n + \sum_{n=0}^{\infty} b_n(x - x_0)^n = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n$$ dır.

Teorem; Eğer $\forall x$ için

1) $\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} b_n (x-x_0)^n \Rightarrow a_n = b_n$ dir

2) $\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0 \Rightarrow a_n = 0 \quad (n=0,1,2\ldots)$

Teorem; Eğer $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ kuvvet serisi bir (a,b) aralığında yakınsak ise bu aralıkta $f(x)$ in türevleri aşağıdaki gibidir.

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1} = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

$$f''(x) = \sum_{n=0}^{\infty} n(n-1) a_n (x-x_0)^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$$

ÖR/ $k \neq 0$ bir sabit olmak üzere

$y' + ky = 0$ dif denk çözünüz.

$y' = -ky \qquad \frac{dy}{dx} = -ky \qquad \frac{dy}{y} = -k\,dx$

$\ln y = -kx + \ln c$

$\ln y - \ln c = -kx$

$\ln \frac{y}{c} = -kx$

$\frac{y}{c} = e^{-kx}$

$y = ce^{-kx}$

ÖR/ $y'' - xy' - y = 0$ dif denk serilerle çözünüz.

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=3}^{\infty} (n-2) a_{n-2} x^{n-2} - \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

$$2.1.a_2 + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=3}^{\infty} (n-2) a_{n-2} x^{n-2} - a_0 - \sum_{n=3}^{\infty} a_{n-2} x^{n-2} = 0$$

$$-a_0 + 2a_2 + \sum_{n=3}^{\infty} \left[n(n-1) a_n - (n-2) a_{n-2} - a_{n-2} \right] x^{n-2} = 0$$

$2a_2 - a_0 = 0$ $\qquad n(n-1) a_n - (n-2) a_{n-2} - a_{n-2} = 0$

$a_0 = 2a_2$ $\qquad n(n-1) a_n = a_{n-2} (n-2+1)$

$$a_n = a_{n-2} \cdot \frac{\cancel{(n-1)}}{n \cancel{(n-1)}}$$

$$a_n = \frac{a_{n-2}}{n}$$

$n=3 \qquad a_3 = \dfrac{a_1}{3}$

$n=4 \qquad a_4 = \dfrac{a_2}{4} = \dfrac{a_0/2}{4} = a_0/2.4$

$n=5 \qquad a_5 = \dfrac{a_3}{5} = \dfrac{a_1/3}{5} = a_1/3.5$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$$y = a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{3} x^3 + \frac{a_0}{2.4} x^4 + \frac{a_1}{3.5} x^5 + \dots$$

$$y = a_0 \left(1 + \frac{1}{2} x^2 + \frac{1}{8} x^4 + \dots \right) + a_1 \left(x + \frac{1}{3} x^3 + \frac{1}{15} x^5 + \dots \right)$$

Şimdi bu dif denklemi kuvvet serisi yardımıyla çözelim.

$$y'+ky=0 \qquad y=\sum_{n=0}^{\infty} a_n x^n, \quad y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + k\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + k\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left((n+1) a_{n+1} + k a_n\right) x^n = 0$$

$$(n+1) a_{n+1} + k a_n = 0 \Rightarrow a_{n+1} = -\frac{k a_n}{n+1}$$

$n=0$ için $\quad a_1 = -k a_0$

$n=1$ " $\quad a_2 = -\frac{k a_1}{2} = -\frac{k}{2}(-k a_0) = \frac{1}{2} k^2 a_0$

$n=2$ " $\quad a_3 = \frac{-k a_2}{3} = -\frac{k}{3}\left[\frac{1}{2} k^2 a_0\right] = \frac{-k^3}{2.3} \cdot a_0$

$n=3$ " $\quad a_4 = \frac{-k a_3}{4} = -\frac{k}{4}\left[\frac{-k^3}{2.3} a_0\right] = \frac{k^4}{1.2.3.4} a_0$

$\vdots$

$$a_n = \frac{(-1)^n k^n}{n!} a_0$$

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n k^n}{n!} a_0 x^n = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n (kx)^n}{n!}$$

$$y = a_0 e^{-kx}$$

$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ (taylor seri e^x in) $\quad e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

$$e^{kx} = \sum_{n=0}^{\infty} \frac{1}{n!} (kx)^n$$

ÖR/ $y'' - (x-2)y' - 2y = 0$ $\quad x=2$ adi noktasında seri ile çözünüz.

$$y = \sum_{n=0}^{\infty} a_n (x-2)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} - (x-2)\sum_{n=1}^{\infty} n a_n (x-2)^{n-1} - 2\sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} - \sum_{n=1}^{\infty} n a_n (x-2)^{n} - 2\sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-2)^{n} - \sum_{n=1}^{\infty} n a_n (x-2)^{n} - 2\sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$2\cdot 1\cdot a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2}(x-2)^{n} - \sum_{n=1}^{\infty} n a_n (x-2)^{n} - 2a_0$$

$$- 2\sum_{n=1}^{\infty} a_n (x-2)^n = 0$$

$$2a_2 - 2a_0 = 0 \qquad \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - n a_n - 2a_n\right](x-2)^n = 0$$

$$2a_2 = 2a_0$$

$$a_2 = a_0$$

$$(n+2)(n+1)a_{n+2} - n a_n - 2a_n = 0$$

$$a_{n+2} = \frac{(\cancel{n+2})a_n}{(\cancel{n+2})(n+1)} = \frac{a_n}{n+1}$$

$n=1$ için $\qquad a_3 = \dfrac{a_1}{2}$

$n=2$ için $\qquad a_4 = \dfrac{a_2}{3} = \dfrac{a_0}{3}$

$n=3$ için $\qquad a_5 = \dfrac{a_3}{4} = \dfrac{a_1/2}{4} = \dfrac{a_1}{8}$

$$y = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + \dots$$

$$y = a_0 + a_1(x-2) + a_0(x-2)^2 + \frac{a_1}{2}(x-2)^3 + \frac{a_0}{3}(x-2)^4 + \dots$$

$$y = a_0\left(1 + (x-2)^2 + \frac{1}{3}(x-2)^4 + \dots\right) + a_1\left((x-2) + \frac{1}{2}(x-2)^3 + \dots\right)$$

ÖR/ $(x^2+1)y'' - 4xy' + 6y = 0$ dif denkleminin $x=0$ civarındaki seri çözümünü bulun.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 4x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 4\sum_{n=1}^{\infty} n a_n x^n + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - 4\sum_{n=1}^{\infty} n a_n x^n + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2} x^n - 4a_1 x - 4\sum_{n=2}^{\infty} n a_n x^n$$

$$+ 6a_0 + 6a_1 x + 6\sum_{n=2}^{\infty} a_n x^n = 0$$

$$2a_2 + 6a_3 x + 2a_1 x + 6a_0 = 0$$

$$2a_2 + 6a_0 = 0 \qquad 6a_3 + 2a_1 = 0$$

$$2a_2 = -6a_0 \qquad 6a_3 = -2a_1$$

$$a_2 = -3a_0 \qquad a_3 = -\frac{1}{3}a_1$$

$$\sum_{n=2}^{\infty} \left[n(n-1)a_n + (n+2)(n+1)a_{n+2} - 4na_n + 6a_n\right]x^n = 0$$

$$\sum_{n=2}^{\infty} (n(n-1) - 4n + 6)a_n + (n+2)(n+1)a_{n+2} = 0$$

$$a_{n+2} = -\frac{(n-3)(n-2)}{(n+2)(n+1)}a_n$$

$$a_2 = -3a_0$$

$$a_3 = -\frac{1}{3}a_1$$

$n=2$ için $\quad a_4 = 0$

$n=3$ için $\quad a_5 = 0$

$n=4$ // $\quad a_6 = -\frac{2}{30}a_4$

$$y = a_0 + a_1 x + (-3a_0)x^2 - \frac{1}{3}a_1 x^3 + \cdots$$

· Farklı yöntemler ile seri uygulamalarındaki ortak noktalar ve Aynı çözümün bulunması üzerine önemli bilgiler:

Bir diferansiyel denklemi kuvvet serilerini kullanarak çözerken farklı uygulamalar yapabiliriz.
Örneğin x_0 noktası civarında kuvvet serisi çözümü aranırken

$$\left.\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x-x_0)^n \\ y' &= \sum_{n=1}^{\infty} a_n\, n\, (x-x_0)^{n-1} \\ y'' &= \sum_{n=2}^{\infty} a_n \cdot n\, (n-1)\, (x-x_0)^{n-2} \end{aligned}\right\} \text{yerine}$$

kısalık amacı ile

$$\left.\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x-x_0)^n \\ y' &= \sum_{n=0}^{\infty} a_n \cdot n\, (x-x_0)^{n-1} \\ y'' &= \sum_{n=0}^{\infty} a_n \cdot n\, (n-1)\, (x-x_0)^{n-2} \end{aligned}\right\} \text{çözümü de kullanılabilmektedir.}$$

Bu işlemler sonucunda elde edilen indirgeme formülleri her ne kadar farklı formüller gibi gözükse de indislere değer verildiğinde aynı seriyi verdikleri kolayca görülebilir.
Yine tüm terimleri ortak tek bir sigma notasyonunda toplamak için yapılan alt indis, üst indis eşitlemeleri kişisel farklar gösterse de temelde bulunan sigma notasyonunda terim ilerleterek aynı ifadenin çeşitli şekillerini elde etmek mümkündür.
Durumu bir örnek ile açıklayalım.

ÖR/ $y'' - (x-2)y' + 2y = 0$ $x=2$ civarında seri çözümü

$$y = \sum_{n=0}^{\infty} a_n (x-2)^n \qquad y' = \sum_{n=1}^{\infty} a_n\, n (x-2)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n\, n\,(n-1)(x-2)^{n-2}$$

$$\sum_{n=2}^{\infty} a_n \cdot n\,(n-1)(x-2)^{n-2} - (x-2)\sum_{n=1}^{\infty} a_n \cdot n\,(x-2)^{n-1} + 2\sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\sum_{n=2}^{\infty} a_n \cdot n\,(n-1)(x-2)^{n-2} - \sum_{n=1}^{\infty} a_n \cdot n\,(x-2)^{n} + \sum_{n=0}^{\infty} 2a_n (x-2)^n = 0$$

$(x-2)$ nin kuvvetlerini en küçükte bir araya getirelim.

$$\sum_{n=2}^{\infty} a_n \cdot n\,(n-1)(x-2)^{n-2} - \sum_{n=1+2}^{\infty} a_{n-2}(n-2)(x-2)^{n-2} + \sum_{n=0+2}^{\infty} 2a_{n-2}(x-2)^{n-2} = 0$$

şimdi alt indisler için eşitlemeye geçelim.

1. ve 3. sigma notasyonunu $n=3$ de biraraya getirelim.

$$\underbrace{\sum_{n=2}^{\infty} a_n \cdot n\,(n-1)(x-2)^{n-2}} - \sum_{n=3}^{\infty} a_{n-2}(n-2)(x-2)^{n-2} + \underbrace{\sum_{n=2}^{\infty} 2a_{n-2}(x-2)^{n-2}} = 0$$

$n=2 \Rightarrow a_2 \cdot 2(2-1)(x-2)^0$ $\qquad$ $n=2 \Rightarrow 2a_0 (x-2)^0 = 2a_0$

$$2a_2 + \sum_{n=3}^{\infty} a_n \cdot n\,(n-1)(x-2)^{n-2} - \sum_{n=3}^{\infty} a_{n-2}(n-2)(x-2)^{n-2} + 2a_0 + \sum_{n=3}^{\infty} 2a_{n-2}(x-2)^{n-2} = 0$$

$$\underbrace{2a_2 + 2a_0}_{0} + \sum_{n=3}^{\infty}\Big[\underbrace{n(n-1)a_n - (n-2)a_{n-2} + 2a_{n-2}}_{0}\Big](x-2)^{n-2} = 0$$

$$a_2 = -a_0$$

$$n(n-1)\,a_n = (n-4)\,a_{n-2}$$

$$a_n = \frac{(n-4)\,a_{n-2}}{n(n-1)}$$ $n \geq 3$ için geçerli

Eğer işlemi $(x-2)$ nin kuvvetlerini en büyükte birleştirerek başlamış olsaydık:

$$\sum_{n=2}^{\infty} a_n \cdot n(n-1)(x-2)^{n-2} - \sum_{n=1}^{\infty} a_n \cdot n (x-2)^n + \sum_{n=0}^{\infty} 2a_n (x-2)^n = 0$$

$$\sum_{n=2-2}^{\infty} a_{n+2} (n+2)(n+2-1)(x-2)^{n+2-2} - \sum_{n=1}^{\infty} a_n \cdot n (x-2)^n + \sum_{n=0}^{\infty} 2a_n (x-2)^n = 0$$

$$\underbrace{\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1)(x-2)^n}_{n=0 \Rightarrow a_2 \cdot 2 \cdot 1 \cdot (x-2)^0} - \sum_{n=1}^{\infty} a_n \cdot n (x-2)^n + \underbrace{\sum_{n=0}^{\infty} 2a_n (x-2)^n}_{n=0 \Rightarrow 2a_0} = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[a_{n+2} (n+2)(n+1) - n a_n + 2a_n \right] (x-2)^n = 0$$

$$2a_2 + 2a_0 = 0 \Rightarrow a_2 = -a_0$$

$$a_{n+2} = \frac{(n-2)a_n}{(n+2)(n+1)} \quad n \geq 1 \text{ için geçerli}$$

iki indirgeme bağıntısı da farklılık göstermektedir. hangisi doğrudur? Cevap Her ikisinde işleme göz atalım.

Her ikisinde de

$a_2 = -a_0$ var burada sorun yok

1. indirgeme formülü

$$a_n = \frac{(n-4)\, a_{n-2}}{n(n-1)}, \quad n \geq 3$$

2. İndirgeme formülü

$$a_{n+2} = \frac{(n-2)\,a_n}{(n+2)(n+1)} \quad , \quad n \geq 1$$

Bu formülde $n = n-2$ yazsak

$$a_{(n-2)+2} = \frac{[(n-2)-2]\,a_{n-2}}{(n-2+2)(n-2+1)} \quad n \geq 3$$

formülünü elde ederiz

O halde aşağıdaki indirgeme formüllerinden hangisi farklıdır

a) $a_n = \dfrac{(n-4)\,a_{n-2}}{n(n-1)}$; $n \geq 3$

b) $a_{n+2} = \dfrac{(n-2)\,a_n}{(n+2)(n+1)}$; $n \geq 1$

c) $a_{n+1} = \dfrac{(n-3)\,a_{n-1}}{(n+1)\,n}$; $n \geq 2$

d) $a_{n-2} = \dfrac{(n-6)\,a_{n-4}}{(n-2)(n+1)}$

Yine konuyu başka bir örnek ile açıklayalım.

Aşağıdakilerden hangisi

$$\sum_{n=2}^{\infty} a_n \cdot n(n-1)(x-2)^{n-2} - \sum_{n=1}^{\infty} a_n \cdot n(x-2)^n + \sum_{n=0}^{\infty} 2a_n (x-2)^n = 0$$

denkleminin farklı bir yazılışı değildir.

a) $\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-2)^n - \sum_{n=1}^{\infty} a_n \cdot n(x-2)^n + \sum_{n=0}^{\infty} 2a_n(x-2)^n = 0$

b) $\sum_{n=2}^{\infty} a_n \cdot n(n-1)(x-2)^{n-2} - \sum_{n=1}^{\infty} a_{n-2}(n-2)(x-2)^{n-2} + \sum_{n=2}^{\infty} 2a_{n-2}(x-2)^{n-2} = 0$

c) $\sum_{n=1}^{\infty} a_{n+1}(n+1)n \cdot (x-2)^{n-1} - \sum_{n=2}^{\infty} a_{n-1}(n-1)(x-2)^{n+1} + \sum_{n=1}^{\infty} 2a_{n-1}(x-2)^{n-1} = 0$

d) $\sum_{n=1}^{\infty} a_{n-1}\, n(n-1)(x-2)^{n-1} - \sum_{n=2}^{\infty} a_{n-1}(n-1)(x-2)^{n-1} + \sum_{n=1}^{\infty} 2a_{n-1}(x-2)^{n-1} = 0$

Yukarıdaki soru

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n \qquad y' = \sum_{n=0}^{\infty} a_n \cdot n(x-x_0)^{n-1} \qquad y'' = \sum_{n=0}^{\infty} a_n \cdot n(n-1)(x-x_0)^{n-2}$$

şeklinde de çözülebilir.

$$\sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2} - (x-2)\sum_{n=0}^{\infty} a_n \cdot n(x-2)^{n-1} + 2\sum_{n=0}^{\infty} a_n(x-2)^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2} - \sum_{n=0}^{\infty} a_n \cdot n(x-2)^n + \sum_{n=0}^{\infty} 2a_n(x-2)^n = 0$$

$$\underbrace{\sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2}}_{\left.\begin{matrix} n=0 \\ n=1 \end{matrix}\right\} 0} - \sum_{n=0}^{\infty} (n-2)a_{n-2}(x-2)^{n-2} + \sum_{n=0+2}^{\infty} 2a_{n-2}(x-2)^{n-2} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n (x-2)^{n-2} - \sum_{n=2}^{\infty} (n-2)\, a_{n-2} (x-2)^{n-2} + \sum_{n=2}^{\infty} 2a_{n-2} (x-2)^{n-2} = 0$$

$$\sum_{n=2}^{\infty} \left[n(n-1)a_n - (n-2)\, a_{n-2} + 2a_{n-2} \right] (x-2)^{n-2} = 0$$

$$a_n = \frac{(n-2-2)}{n(n-1)} \qquad n \geqslant 2$$

$$a_n = \frac{n-4}{n(n-1)} \; ; \quad n \geqslant 2$$

$n=2 \Rightarrow a_2 = \frac{2-4}{2(2-1)} a_0 = -a_0$

$n=3 \Rightarrow a_3 = \frac{3-4}{3(3-1)} a_1 = -\frac{1}{6} a_1$

$n=4 \Rightarrow a_4 = \frac{4-4}{4(4-1)} a_2 = 0$

$n=5 \Rightarrow a_5 = \frac{5-4}{5(5-1)} a_3 = \frac{-1}{120} a_1$

$$y = \sum_{n=0}^{\infty} a_n (x-2)^n$$

$$y = a_0 + a_1 (x-2) + a_2 (x-2)^2 + a_3 (x-2)^3 + \dots$$

$$y = a_0 + a_1 (x-2) - a_0 (x-2)^2 - \frac{1}{6} a_1 (x-2)^3 - \frac{1}{120} a_1 (x-2)^4 + \dots$$

$$y = a_0 \left[1 - (x-2)^2 + \dots \right] + a_1 \left[(x-2) - \frac{1}{6} (x-2)^3 - \frac{1}{120} (x-2)^5 + \dots \right]$$

LAPLACE DÖNÜŞÜMLERİ

Tanım; $K(x,t) = \begin{cases} 0 & , t<0 \\ e^{-st} & , t \geq 0 \end{cases}$ çekirdek fonksiyon

olmak üzere

$$\int_{-\infty}^{\infty} K(x,t) f(t)dt = \int_0^{\infty} e^{-st} f(t)dt = F(s) = \mathcal{L}\{f(t)\}$$

ye $f(t)$ fonksiyonunun Laplace Dönüşümü denir.

Bazı Elemanter Fonksiyonların Laplace Dönüşümleri

a) $f(t) = c$ (sabit) $\mathcal{L}\{c\} = ?$

$$\mathcal{L}(c) = \int_0^{\infty} c e^{-st} dt = \lim_{A\to\infty} \int_0^A c e^{-st} dt$$

$$= \lim_{A\to\infty} -\frac{c \cdot e^{-st}}{s} \Big|_0^A$$

$$= \lim_{A\to\infty} \left[\frac{-c e^{-sA}}{s} + \frac{c e^{0}}{s} \right] = \frac{c}{s}$$

$$\boxed{\mathcal{L}(c) = \frac{c}{s}}$$

b) $f(t) = t$ $\mathcal{L}\{t\} = ?$

$$\mathcal{L}(t) = \int_0^{\infty} t e^{-st} dt = \lim_{A\to\infty} \int_0^A t e^{-st} dt$$

$$u = t \quad dv = e^{-st} dt$$
$$du = dt \quad v = -\frac{1}{s} e^{-st}$$

$$\lim_{A\to\infty}\left[-\frac{t}{s}e^{-st}\Big|_0^A - \left(-\frac{1}{s}\right)\int_0^A e^{-st}dt\right.$$

$$\lim_{A\to\infty}\left[\underbrace{-\frac{A}{s}e^{-sA}+0\Big)}_{0} - \frac{1}{s^2}e^{-st}\Big|_0^A\right.$$

$$\lim_{A\to\infty} -\frac{1}{s^2}\cdot\left(e^{-sA}-e^{0}\right) = \frac{1}{s^2}$$

$$\boxed{L\{t\} = \frac{1}{s^2}}$$

$$L\{t^2\} = \frac{1\cdot 2}{s^3} \qquad L\{t^3\} = \frac{1\cdot 2\cdot 3}{s^4}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

c) $f(t) = e^{at}$ $\qquad L\{e^{at}\} = ?$ $\qquad s > a$

$$L\{e^{at}\} = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{(a-s)t}dt$$

$$= \int_0^\infty e^{-(s-a)t}dt$$

$$\lim_{A\to\infty} \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^A$$

$$= \lim_{A\to\infty} \underbrace{\cancel{\frac{e^{-(s-a)A}}{-(s-a)}}}_{0} + \frac{e^{-(s-a)\cdot 0}}{s-a}\}1$$

$$\boxed{L\{e^{at}\} = \frac{1}{s-a}}$$

4) $f(t) = \sin at \qquad (s>0)$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

5) $f(t) = \cos at$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

6) $f(t) = \sinh at = \dfrac{e^{at} - e^{-at}}{2}$

$$\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$$

7) $f(t) = \cosh at = \dfrac{e^{at} + e^{-at}}{2}$

$$\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$$

Laplace Dönüşümünün özellikleri

1) Lineerlik özelliği

$c_1, c_2 \dots c_n$ sabitler olmak üzere

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t) + \dots\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} + \dots$$

ÖR/ $\mathcal{L}\{3t^2 - 4\sin 3t + 7e^{-2t}\} = ?$

$$3\mathcal{L}\{t^2\} - 4\mathcal{L}\{\sin 3t\} + 7\mathcal{L}\{e^{-2t}\} =$$

$$= 3 \cdot \frac{2}{s^3} - 4 \cdot \frac{3}{s^2+9} + 7 \cdot \frac{1}{s-(-2)}$$

2) Kaydırma Özelliği

$\mathcal{L}\{f(t)\} = F(s)$ ise

$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ dır.

ÖR/ $f(t) = e^{-3t}\cos 2t$ $\qquad$ $\mathcal{L}\{e^{-3t}\cos 2t\} = ?$

$\mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2+4} = F(s)$ de s yerine $s-(-3)$ yazılacak

$$\mathcal{L}\{e^{-3t}\cos 2t\} = \frac{s+3}{(s+3)^2+4}$$

3) Türevin Laplace Dönüşümü

$\mathcal{L}\{f(t)\} = F(s)$ olsun.

$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$

$\vdots$

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) + \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

ÖR/ $f(t) = \cos 3t$ $\qquad$ $\mathcal{L}\{\cos 3t\} = \dfrac{s}{s^2+9} = F(s)$

1 yol: $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ $\qquad$ $f(0) = \cos 0 = 1$

$$= s\cdot\frac{s}{s^2+9} - 1$$

$$= \frac{s^2}{s^2+9} - 1 = \frac{-9}{s^2+9}$$

2 yol: $f'(t) = -3\sin 3t$

$$\mathcal{L}\{f'(t)\} = \mathcal{L}\{-3\sin 3t\} = -\frac{3\cdot 3}{s^2+9} = \frac{-9}{s^2+9}$$

4) t^n ile çarpma

$\mathcal{L}\{f(t)\} = F(s)$ olsun

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

ÖR/ $\mathcal{L}\{t^2 e^{2t}\} = ?$

$$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2} = F(s)$$

$$\mathcal{L}\{t^2 e^{2t}\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = 1 \cdot \left(\frac{2}{(s-2)^3}\right) = \frac{2}{(s-2)^3}$$

$$\frac{dF(s)}{ds} = \frac{-1}{(s-2)^2} \qquad \frac{d^2F(s)}{ds^2} = \frac{2}{(s-2)^3}$$

2 yol: Kaydırma özelliği ile çözulur.

$\mathcal{L}\{t^2 e^{2t}\} = ?$

$$\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3} = F(s) \qquad \mathcal{L}\{t^2 e^{2t}\} = \frac{2}{(s-2)^3}$$

Ters Laplace Dönüşümü;

$\mathcal{L}\{f(t)\} = F(s)$ ise $f(t)$ ye $F(s)$ fonksiyonunun ters Laplace dönüşümü denir.

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Ters Laplace Dönüşümü bağıntıları

1) $F(s) = \frac{1}{s} \Rightarrow L^{-1}\left\{\frac{1}{s}\right\} = 1$

$F(s) = \frac{1}{s^2} \Rightarrow L^{-1}\left\{\frac{1}{s^2}\right\} = t$

$F(s) = \frac{1}{s^3} \Rightarrow L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2!}$

$\vdots$

$F(s) = \frac{1}{s^{n+1}} \Rightarrow L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$

2) $F(s) = \frac{1}{s-a} \Rightarrow L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$F(s) = \frac{1}{(s-a)^2} \Rightarrow L^{-1}\left\{\frac{1}{(s-a)^2}\right\} = te^{at}$

$F(s) = \frac{1}{(s-a)^3} \Rightarrow L^{-1}\left\{\frac{1}{(s-a)^3}\right\} = \frac{t^2}{2!}e^{at}$

$\vdots$

$F(s) = \frac{1}{(s-a)^{n+1}} \Rightarrow L^{-1}\left\{\frac{1}{(s-a)^{n+1}}\right\} = \frac{t^n}{n!}e^{at}$

3) $F(s) = \frac{1}{s^2+a^2} \Rightarrow L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

4) $F(s) = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

5) $F(s) = \frac{1}{s^2-a^2} \Rightarrow L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh at}{a}$

6) $F(s) = \frac{s}{s^2-a^2} \Rightarrow L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

Ters Laplace Dönüşümünün özellikleri

1) Lineerlik özelliği;

$$L^{-1}\{c_1 F_1(s) + c_2 F_2(s) + \dots\} = c_1 L^{-1}\{F_1(s)\} + c_2 L^{-1}\{F_2(s)\} + \dots$$

ÖR/ $$L^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\} = 4L^{-1}\left\{\frac{1}{s-2}\right\} - 3L^{-1}\left\{\frac{s}{s^2+16}\right\} + 5L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t$$

2) Kaydırma özelliği;

$L^{-1}\{F(s)\} = f(t)$ ise $L^{-1}\{F(s-a)\} = e^{at} f(t)$

ÖR/ $$L^{-1}\left\{\frac{1}{s^2-2s+5}\right\} = L^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = e^{t}\frac{\sin 2t}{2}$$

ÖR/ $$L^{-1}\left\{\frac{1}{s^2-4}\right\} = L^{-1}\left\{\frac{1}{(s-2)(s+2)}\right\}$$

$$\frac{A}{s-2} + \frac{B}{s+2} = \frac{1}{s^2-4} \qquad A = 1/2 \quad B = -1/2$$

$$L^{-1}\left\{\frac{1}{s^2-4}\right\} = L^{-1}\left\{\frac{1}{2}\cdot\frac{1}{s-2} - \frac{1}{2}\cdot\frac{1}{s+2}\right\}$$

$$= \frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t} = \frac{1}{2}\sinh 2t$$

Genel Uygulama (Laplace ile ilgili)

1) $L\{t^3 e^{-3t}\} = ?$ $\qquad L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$ (Kaydırma)

$$L\{t^3 e^{-3t}\} = \frac{6}{(s+3)^4}$$

2) $\mathcal{L}\{(t-2)^2e^t\} = \mathcal{L}\{(t^2+4t+4)e^t\}$

$= \mathcal{L}\{t^2e^t\}+4\mathcal{L}\{te^t\}+4\mathcal{L}\{e^t\}$

$= \dfrac{2}{(s-1)^3} + \dfrac{4}{(s-1)^2} + \dfrac{4}{s-1}$

3) $\mathcal{L}\{e^t\cos^2 t\} = \mathcal{L}\left\{e^t\left(\dfrac{1+\cos 2t}{2}\right)\right\} = \dfrac{1}{2}\mathcal{L}\{e^t + e^t\cos 2t\}$

$= \dfrac{1}{2}\left[\left(\dfrac{1}{s-1}\right) + \dfrac{s-1}{(s-1)^2+4}\right]$

$\mathcal{L}\{e^t\cos 2t\} = ?$ (Kaydırma)

$\mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2+4} = F(s)$

$\mathcal{L}\{e^t\cos 2t\} = \dfrac{s-1}{(s-1)^2+4}$

4) $f(t) = t\cos 2t$

$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0) = s^2\cdot\left(\dfrac{s^2-4}{(s^2+4)^2}\right) - 1$

$\mathcal{L}\{t\underbrace{\cos 2t}\} = (-1)^1\dfrac{d}{ds}\left(\dfrac{s}{s^2+4}\right) = \dfrac{s^2-4}{(s^2+4)^2}$

$\mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2+4} = F(s)$

5) $\mathcal{L}^{-1}\left\{\dfrac{3}{s+4}\right\} = 3\cdot e^{-4t}$

6) $\mathcal{L}^{-1}\left\{\dfrac{1}{2s-5}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{2(s-\frac{5}{2})}\right\} = \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{1}{s-5/2}\right\}$

$= \dfrac{1}{2}\cdot e^{5/2\, t}$

7) $L^{-1}\left\{\frac{8s}{s^2+16}\right\} = 8\,L^{-1}\left\{\frac{s}{s^2+16}\right\} = 8.\cos 4t$

8) $L^{-1}\left\{\frac{6}{s^2-9}\right\} = 6.L^{-1}\left\{\frac{1}{s^2-9}\right\} = 6.\frac{\sinh .3t}{3}$

$= 2\sinh 3t$

9) $L^{-1}\left\{\frac{1}{s^{10}}\right\} = \frac{t^9}{9!}$

10) $L^{-1}\left\{\frac{2s+1}{s^2+s}\right\} = L^{-1}\left\{\frac{2s+1}{s(s+1)}\right\} = L^{-1}\left\{\frac{2}{s+1} + \frac{1}{s^2+s}\right\}$

$= 2e^{-t} + L^{-1}\left\{\frac{1}{s^2+s}\right\}$

$L^{-1}\left\{\frac{1}{s^2+s}\right\} = L^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2-\frac{1}{4}}\right\}$

$= e^{-1/2\,t}.\frac{\sinh\frac{t}{2}}{\frac{1}{2}}$

$= 2e^{-t} + e^{-t/2}\frac{\sinh t/2}{1/2}$

OR/ $L^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = L^{-1}\left\{\frac{6s-4}{(s-2)^2+16}\right\}$

$= L^{-1}\left\{\frac{6s}{(s-2)^2+16}\right\} + L^{-1}\left\{\frac{-4}{(s-2)^2+16}\right\}$

$= L^{-1}\left\{\frac{6(s-2)+12}{(s-2)^2+16}\right\} + L^{-1}\left\{\frac{-4}{(s-2)^2+16}\right\}$

$= 6L^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\} + \underbrace{12L^{-1}\left\{\frac{1}{(s-2)^2+16}\right\} - 4L^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}}_{8L^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}}$

$= 6.e^{2t}\cos 4t + 8.e^{2t}\frac{\sin 4t}{4}$

Sabit katsayılı dif denklemlerin Laplace ile Çözümü

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = \theta(t)$$

$y = y(t)$

$$y(0) = y_0 \quad y'(0) = y_1 \quad y''(0) = y_2 \cdots y^{(n-1)}(0) = y_{n-1}$$

Verilen dif denklemin her teriminin Laplace dönüşümü alınarak bilinmeyen çözüm fonksiyonunun dönüşüm cinsinden bir cebirsel denklemi elde edilir. Çözüm fonksiyonu dönüşüm parametresi cinsinden düzenlenir. Bu düzenlenen fonksiyonun ters Laplace dönüşümü alınarak gerçek çözüm fonksiyonuna ulaşılır.

ÖR/ $y'' + 4y = 0 \qquad y(0) = 1 \quad , y'(0) = -2$

başlangıç değer problemini Laplace dönüşümü kullanarak çözünüz

$$L\{y'' + 4y\} = 0 \qquad L\{y''\} + 4L\{y\} = 0$$

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = 0$$

$L\{f(t)\} = F(s)$

$L\{y(t)\} = Y(s)$

$L^{-1}\{F(s)\} = f(t)$

$L^{-1}\{Y(s)\} = y(t)$

$$s^2 Y(s) - s + 2 + 4Y(s) = 0$$

$$Y(s)\left[s^2 + 4\right] = s - 2$$

$$Y(s) = \frac{s-2}{s^2+4}$$

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{s-2}{s^2+4}\right\}$$

$$y(t) = L^{-1}\left\{\frac{s}{s^2+4}\right\} - 2L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$y(t) = \cos 2t - 2\,\frac{\sin 2t}{2}$$

$$y(t) = \cos 2t - \sin 2t$$

ÖR/ $y'' - y' - 2y = 0 \qquad y(0) = 1 \quad y'(0) = 0$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{0\} = 0$$

$$s^2 Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) = 0$$

$$s^2 Y(s) - s - sY(s) + 1 - 2Y(s) = 0$$

$$Y(s)\left[s^2 - s - 2\right] = s - 1 \Rightarrow Y(s) = \frac{s-1}{s^2-s-2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s-1}{s^2-s-2}\right\} \qquad \frac{s-1}{s^2-s-2} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$A = \frac{2}{3} \qquad B = \frac{1}{3}$$

$$= \mathcal{L}^{-1}\left\{\frac{2/3}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1/3}{s-2}\right\}$$

$$= \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$y(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$$

ÖR/ $y'' - 6y' + 9y = t^2 e^{3t} \qquad y(0) = 2, \; y'(0) = 6$

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t^2 e^{3t}\}$$

$$s^2 Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 9Y(s) = \frac{2!}{(s-3)^3}$$

$$s^2 Y(s) - 2s - 6 - 6sY(s) + 12 + 9Y(s) = \frac{2}{(s-3)^3}$$

$$Y(s)\left[s^2 - 6s + 9\right] - 2(s-3) = \frac{2}{(s-3)^3}$$

$$Y(s)\left[s^2 - 6s + 9\right] = 2(s-3) + \frac{2}{(s-3)^3}$$

$$Y(s) = \frac{2}{(s-3)^5} + \frac{2}{s-3}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^5}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= 2 \cdot \frac{t^4}{4!}e^{3t} + 2e^{3t}$$

ÖR/ $y'' + 2y' + y = te^{-t}$ $\quad y(0) = 1 \quad y'(0) = 2$

başlangıç değer probleminin çözümünü Laplace dönüşümü kullanarak bulunuz.

$$L\{y''\} + 2L\{y'\} + L\{y\} = L\{te^{-t}\}$$

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + Y(s) = \frac{1}{(s+1)^2}$$

$$(s^2 + 2s + 1)Y(s) = s + 4 + \frac{1}{(s+1)^2}$$

$$Y(s) = \frac{s+4}{(s+1)^2} + \frac{1}{(s+1)^4}$$

$$Y(s) = \frac{s+1}{(s+1)^2} + \frac{3}{(s+1)^2} + \frac{1}{(s+1)^4}$$

$$Y(s) = \frac{1}{s+1} + \frac{3}{(s+1)^2} + \frac{1}{(s+1)^4}$$

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} + 3L^{-1}\left\{\frac{1}{(s+1)^2}\right\} + L^{-1}\left\{\frac{1}{(s+1)^4}\right\}$$

$$y(t) = e^{-t} + 3te^{-t} + \frac{t^3e^{-t}}{3!}$$

ÖR/ $y''' - 3y'' + 3y' - y = t^2 e^t$ dif denkleminin

$y(0) = 1$, $y'(0) = 2$ ve $y''(0) = 3$

başlangıç değer probleminin Laplace dönüşümü

kullanarak elde edilen bilinmeyen dönüşüm

fonksiyonunu $(Y(s)) = ?$

$$\mathcal{L}\{y''' - 3y'' + 3y' - y\} = \mathcal{L}\{t^2 e^t\}$$

$$s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0) - 3\left[s^2 Y(s) - s y(0) - y'(0)\right]$$

$$+ 3\left[s Y(s) - y(0)\right] - Y(s) = \frac{2}{(s-1)^3}$$

$$s^3 Y(s) - s^2 - 2s - 3 - 3s^2 Y(s) + 3s + 6 + 3s Y(s) - 3 - Y(s) = \frac{2}{(s-1)^3}$$

$$Y(s)\left[\underbrace{s^3 - 3s^2 + 3s - 1}_{(s-1)^3}\right] - s^2 + s = \frac{2}{(s-1)^3}$$

$$(s-1)^3 Y(s) = s^2 - s + \frac{2}{(s-1)^3}$$

$$Y(s) = \frac{s^2 - s}{(s-1)^3} + \frac{2}{(s-1)^6}$$

ÖR/ $F(s) = \dfrac{s^4 - 2s^3 + 9s^2 + 8s - 12}{s^3(s^2+4)}$ fonksiyonunun ters Laplace dönüşümünü bulunuz.

$\mathcal{L}^{-1}\{F(s)\} = ?$

$$\frac{s^4 - 2s^3 + 9s^2 + 8s - 12}{s^3(s^2+4)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{Ds+E}{s^2+4}$$

$s^2+4 \quad s(s^2+4) \quad s^2(s^2+4) \quad s^3$

$$s^4 - 2s^3 + 9s^2 + 8s - 12 = As^2 + 4A + Bs^3 + 4Bs + Cs^4 + 4s^2C + Ds^4 + Es^3$$

$C + D = 1$

$B + E = -2$

$A + 4C = 9$

$4B = 8$

$4A = -12$

$B = 2$

$A = -3$

$C = 3$

$D = -2$

$E = -4$

$$F(s) = -\frac{3}{s^3} + \frac{2}{s^2} + \frac{3}{s} - \frac{2s}{s^2+4} - \frac{4}{s^2+4}$$

$$\mathcal{L}^{-1}\{F(s)\} = -3\frac{t^2}{2} + 2t + 3 - 2\cos 2t - 2\sin 2t$$

Diferansiyel Denklem Sistemleri;

Bağımsız değişken ile bu değişkenin n tane bilinmeyen fonksiyonunun ve bunların herhangi mertebeye kadar türevlerinin oluşturduğu bağıntıların meydana getirdiği topluluğa "n bilinmeyenli dif denklem sistemi" denir.

Bir dif denklem sistemindeki her bir bilinmeyenin en yüksek türev mertebelerinin toplamı sistemin mertebesini verir.

Bir dif denklem sisteminde bilinmeyen fonksiyon sayısı denklem sayısına eşit ise ve her denklem bilinmeyen fonksiyonlardan birinin en yüksek mertebeli türevine göre çözülmüşse bir sisteme "kanonik sistem" denir. Bu sistemde yalnız 1. mertebe türevler bulunuyorsa sisteme "normal sistem" denir.

Eğer sistemde bağımsız değişkenli ifadeler yoksa bu sisteme "homojen sistem" denir.

1) $$\left.\begin{aligned} \frac{dy}{dx} &= -5y - z + 1 \\ \frac{dz}{dx} &= y - 3z - e^{2x} \end{aligned}\right\}$$ 2 bilinmeyenli lineer normal sistem
homojen olmayan
$y, z \to$ bilinmeyen

2) $$\left.\begin{aligned} \frac{dy}{dx} &= -5y - z \\ \frac{dz}{dx} &= y - 3z \end{aligned}\right\}$$ homojen sistem
$y, z \to$ bilinmeyen

3) $$\frac{d^2y}{dx^2} = 2y + 3z + e^{2x}$$
$$\frac{d^2z}{dx^2} = -2z - y$$
bilinmeyen sayısı y, z
normal sistem değil
homojen sistem değil

Çözüm Yöntemleri

Türetme – Yok Etme Yöntemi

Sistemin denklemlerini türetmek ve fonksiyonlardan biri hariç diğer bütün fonksiyonları yok etmek suretiyle bilinmeyen bir tek fonksiyonun dif denklemi elde edilir. Bu denklem ile sistemin çözümü bulunur.

ÖR/
$$\frac{dy}{dx} = -5y - z + 1 + x^2$$
$$\frac{dz}{dx} = y - 3z + e^{2x}$$
dif denklem sisteminin çözümünü türetme-yok etme yöntemini kullanarak bulunuz.

1. mertebe
1. mertebe } dif denk 2-mertebeden

$$\frac{d^2y}{dx^2} = -5\frac{dy}{dx} - \frac{dz}{dx} + 2x$$

$$\frac{d^2y}{dx^2} = -5\frac{dy}{dx} - y + 3z - e^{2x} + 2x$$

$$\frac{d^2y}{dx^2} = -5\frac{dy}{dx} - y + 3\left[-\frac{dy}{dx} - 5y + 1 + x^2\right] - e^{2x} + 2x$$

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = -e^{2x} + 3x^2 + 2x + 3$$

$r^2 + 8r + 16 = 0$ (4, 4) $\quad (r+4)^2 = 0 \quad r_1 = r_2 = -4$

$$y_h = (c_1 + c_2 x)e^{-4x}$$

$$y_h = (c_1 + c_2 x)e^{-4x}$$

$$y_{ö_1} = ax^2 + bx + c \quad y_{ö_1}' = 2ax + b \quad y_{ö_1}'' = 2a$$

$$2a + 8(2ax + b) + 16(ax^2 + bx + c) = 3x^2 + 2x + 3$$

$$16a = 3 \quad a = 3/16$$

$$+16a + 16b = 2 \Rightarrow 16 \cdot \frac{3}{16} + 16b = 2$$

$$16b = 2 - 3$$

$$2a + 8b + 16c = 3$$

$$b = -1/16$$

$$y = (c_1 + c_2 x)e^{-4x} + \frac{3}{16}x^2 - \frac{1}{16}x + \frac{25}{128} - \frac{1}{36}e^{-2x} \qquad c = 25/128$$

$$y_{ö_2} = Ae^{2x}$$

$$y_{ö_2}' = 2Ae^{2x} \quad y_{ö_2}'' = 4Ae^{2x}$$

$$4Ae^{2x} + 16Ae^{2x} + 16Ae^{2x} = -e^{2x}$$

$$36A = -1 \quad A = -1/36$$

$$\frac{dy}{dx} = -5y - z + 1 + x^2$$

$$z = -5y + 1 + x^2 - \frac{dy}{dx}$$

$$z = -5\left[c_1 e^{-4x} + c_2 x e^{-4x} + \frac{3}{16}x^2 - \frac{1}{16}x + \frac{25}{128} - \frac{1}{36}e^{-2x}\right] + 1 + x^2$$

$$-\left(-4c_1 e^{-4x} + c_2 e^{-4x} - 4x c_2 e^{-4x} + \frac{6x}{16} - \frac{1}{16}\right]$$

$$z = \left[(-c_1 - c_2) - c_2 x\right]e^{-4x} + \frac{x^2}{16} - \frac{x}{16} + \frac{11}{128} + \frac{7}{36}e^{2x}$$

Operatör Yönteni / Determinant Yöntemi

Bu yöntemde dif denklem sistemi türev operatörü ile ifade edilir. Cebirsel denklemlerin çözümündeki gibi çözülür.

ÖR/ $\frac{dy}{dx} = -5y - z + 1 + x^2 \qquad D = \frac{d}{dx}$

$$\frac{dz}{dx} = y - 3z + e^{2x}$$

$$\left.\begin{aligned} \frac{dy}{dx} + 5y + z &= x^2 + 1 \\ \frac{dz}{dx} - y + 3z &= e^{2x} \end{aligned}\right\} \quad \left\{\begin{aligned} (D+5)y + z &= x^2 + 1 \\ -y + (D+3)z &= e^{2x} \end{aligned}\right\}$$

$$\Delta = \begin{vmatrix} D+5 & 1 \\ -1 & D+3 \end{vmatrix} = (D+5)(D+3) + 1 = D^{\xrightarrow{\;2\;} 2 \text{ keyfi sbt}} + 8D + 16$$

$$\Delta_1 = \begin{vmatrix} 1+x^2 & 1 \\ e^{2x} & D+3 \end{vmatrix} = (D+3)(1+x^2) - e^{2x}$$

$$= 2x + 3 + 3x^2 - e^{2x}$$

$$= 3x^2 + 2x + 3 - e^{2x}$$

$$y = \frac{\Delta_1}{\Delta} = \frac{3x^2 + 2x + 3 - e^{2x}}{D^2 + 8D + 16} \Rightarrow (D^2 + 8D + 16)y = \underbrace{3x^2 + 2x + 3 - e^{2x}}$$

$$r^2 + 8r + 16 = 0$$

$$y = (c_1 + c_2 x)e^{-4x} - \frac{3x^2}{16} - \frac{x}{16} + \frac{25}{128} - \frac{1}{36}e^{2x}$$

$$\Delta_2 = \begin{vmatrix} D+5 & 1+x^2 \\ -1 & e^{2x} \end{vmatrix} = (D+5)e^{2x} + 1 + x^2$$

$$= 2e^{2x} + 5e^{2x} + 1 + x^2$$

$$z = \frac{\Delta_2}{\Delta} = \frac{7e^{2x} + 1 + x^2}{D^2 + 8D + 16}$$

$$(D^2+8D+16)z = x^2+1+7e^{2x}$$

$$r^2+8r+16=0 \qquad r_{1,2}=-4$$

$$z_h = (c_1+c_2x)e^{-4x}$$

$$z_{ö_1} = ax^2+bx+c \qquad z_{ö_1}' = 2ax+b \qquad z_{ö_1}'' = 2a$$

$$2a+16ax+8b+16ax^2+16bx+16c = x^2+1$$

$$16a=1 \qquad a=1/16$$

$$16a+16b=0 \qquad b=-1/16 \qquad c=\frac{11}{128}$$

$$2a+8b+16c=1 \qquad z_{ö_1} = \frac{1}{16}x^2-\frac{1}{16}x+\frac{11}{128}$$

$$z_{ö_2} = ke^{2x} \qquad z_{ö_2}' = 2ke^{2x} \qquad z_{ö_2}'' = 4ke^{2x}$$

$$4ke^{4x}+16ke^{2x}+16ke^{2x} = 7e^{2x}$$

$$z_{ö_2} = \qquad 36k=7 \qquad k=7/36 \qquad z_{ö_2} = \frac{7}{36}e^{7x}$$

$$z = (c_3+c_4x)e^{-4x} + \frac{1}{16}x^2-\frac{1}{16}x+\frac{11}{128}+\frac{7}{36}e^{7x}$$

y ve z yi 1. denklemde yerine yazıp c_3 ve c_4'ü

c_1 ve c_2 cinsinden bulmalıyız.

Ör/ $\left.\begin{aligned} &\frac{dy}{dt} + y - x = -\sin 2t \\ &\frac{dx}{dt} + 2y - x = 0 \end{aligned}\right\}$ denk sistemini Türetme-Yoketme yönt kullanarak bulunuz.

$$\frac{dx}{dt} + 2y - x \Rightarrow \underbrace{\frac{d^2x}{dt^2} + 2\frac{dy}{dt} - \frac{dx}{dt} = 0}$$

$$\frac{dy}{dt} = x - y - \sin 2t \qquad \frac{d^2x}{dt^2} + 2x - 2y - 2\sin 2t - \frac{dx}{dt} = 0$$

$$\frac{dx}{dt} - x = -2y \qquad \frac{d^2x}{dt^2} - \frac{dx}{dt} + 2x + \frac{dx}{dt} - x - 2\sin 2t = 0$$

$$\frac{d^2x}{dt^2} + x = 2\sin 2t$$

$$r^2 + 1 = 0 \qquad r_{1,2} = \mp i$$

$$x_h = c_1 \cos t + c_2 \sin t$$

$$x_ö = A\cos 2t + B\sin 2t$$

$$x_ö' = -2A\sin 2t + 2B\cos 2t$$

$$x_ö'' = -4A\cos 2t - 4B\sin 2t$$

$$-4A\cos 2t - 4B\sin 2t + A\cos 2t + B\sin 2t = 2\sin 2t$$

$$-3A = 0 \quad A = 0 \qquad -3B = 2$$

$$B = -2/3$$

$$x_ö = -\frac{2}{3}\sin 2t$$

$$x = c_1\cos t + c_2 \sin t - \frac{2}{3}\sin 2t$$

$$y = \frac{1}{2}\left(x - \frac{dx}{dt}\right)$$

$$y = \frac{1}{2}\left(c_1\cos t + c_2 \sin t - \frac{2}{3}\sin 2t + c_1 \sin t - c_2 \cos t + \frac{4}{3}\cos 2t\right)$$

$$y = \frac{1}{2}(c_1 - c_2)\cos t + \frac{1}{2}(c_2 + c_1)\sin t - \frac{1}{3}\sin 2t + \frac{2}{3}\cos 2t$$

ÖR/ $(D-1)x + Dy = 2t+1$ $\quad D = \frac{d}{dt}$

$(2D+1)x + 2Dy = t$

$$\Delta = \begin{vmatrix} D-1 & D \\ 2D+1 & 2D \end{vmatrix} = 2D^2 - 2D - 2D^2 - D = -3D^{①}$$ → 1 tane keyfi sbt

$x = \frac{\Delta_1}{\Delta}$

$$\Delta_1 = \begin{vmatrix} 2t+1 & D \\ t & 2D \end{vmatrix}$$

$\Delta_1 = 2D(2t+1) - Dt$

$x = \frac{3}{-3D}$

$\Delta_1 = 2\,D(2t) + 2D(1) - 1$

$\Delta_1 = 2 \cdot 2 + 0 - 1 = 3$

$Dx = -1$

$\frac{dx}{dt} = -1 \qquad \int dx = \int -dt$

$x = -t + c_1$

$$\Delta_2 = \begin{vmatrix} D-1 & 2t+1 \\ 2D+1 & t \end{vmatrix}$$

$\Delta_2 = (D-1)t - (2D+1)(2t+1)$

$\Delta_2 = 1 - t - 4 - 0 - 2t - 1$

$\Delta_2 = -3t - 4$

$y = \frac{\Delta_2}{\Delta}$

$y = \frac{-3t-4}{-3D}$

$Dy = t + \frac{4}{3} \qquad \frac{dy}{dt} = t + \frac{4}{3} \qquad \int dy = \int \left(t + \frac{4}{3}\right) dt$

$y = \frac{t^2}{2} + \frac{4}{3}t + c_2$

$(D-1)x + Dy = 2t+1$

$\frac{dx}{dt} - x + \frac{dy}{dt} = 2t+1$

$-1 + t - c_1 + t + \frac{4}{3} = 2t + 1 \qquad c_1 = -2/3$

$x = -t + 2/3$

$y = t^2/2 + 4/3\,t + c_2$

ÖR/
$$\left.\begin{aligned} \frac{dx}{dt} + y &= 0 \\ \frac{dy}{dt} - x &= \cos t \end{aligned}\right\}$$ diferansiyel denklem sisteminin çözümünü türetme-yok etme yöntemini kullanarak bulunuz.

$$\frac{d^2y}{dt^2} - \frac{dx}{dt} = -\sin t$$

$$\frac{d^2y}{dt^2} + y = -\sin t$$

$$r^2 + 1 = 0 \Rightarrow r_{1,2} = \mp i$$

$$y_h = c_1 \cos t + c_2 \sin t$$

$$y_ö = (A \sin t + B \cos t)t$$

$$y_ö = (A \sin t - B \sin t)$$

$$y_ö{}' = A \sin t + B \cos t + t(A \cos t - B \sin t)$$

$$y_ö{}'' = A \cos t - B \sin t + A \cos t - B \sin t + t(-A \sin t - B \cos t)$$

$$y_ö{}'' = 2A \cos t - 2B \sin t - At \sin t - Bt \cos t$$

$$y_ö{}'' = 2A \cos t - 2B \sin t - At \sin t - Bt \cos t + At \sin t + Bt \cos t$$

$$2A \cos t - 2B \sin t = -\sin t$$

$$2A = 0 \qquad -2B = -1 \Rightarrow B = 1/2$$

$$A = 0 \qquad y_ö = \frac{1}{2} t \cos t$$

$$y = y_h + y_ö = c_1 \cos t + c_2 \sin t + \frac{1}{2} t \cos t$$

$$x = \frac{dy}{dt} - \cos t$$

$$x = -c_1 \sin t + c_2 \cos t + \frac{1}{2} \cos t - \frac{t}{2} \sin t - \cos t$$

$$x = -c_1 \sin t + c_2 \cos t - \frac{1}{2} \cos t - \frac{1}{2} t \sin t$$

ÖR/ $\left.\begin{array}{l}\dfrac{dx}{dt}+y=0\\ \dfrac{dy}{dt}-x=\cos t\end{array}\right\}$ dif denk sis çöz

$$\frac{d^2y}{dt^2}-\frac{dx}{dt}=-\sin t \Rightarrow \frac{d^2y}{dx^2}+y=-\sin t$$

$$r^2+1=0 \quad r_{1,2}=\mp i$$

$$y_h=c_1\cos t+c_2\sin t$$

$$y_ö=(A\sin t+B\cos t)t$$

$$y_{ö_1}'=A\sin t+B\cos t+t(A\cos t-B\sin t)$$

$$y_{ö_1}''=A\cos t-B\sin t+A\cos t-B\sin t+t(-A\sin t-B\cos t)$$

$$y_{ö_1}''=2A\cos t-2B\sin t-At\sin t-Bt\cos t$$

$$\frac{d^2y}{dx^2}+y=-\sin t$$

$$2A\cos t-2B\sin t-\cancel{At\sin t}-\cancel{Bt\cos t}+\cancel{At\sin t}+\cancel{Bt\cos t}=-\sin t$$

$$2A=0 \quad -2B=-1 \quad B=1/2$$

$$A=0$$

$$y_ö=\frac{1}{2}t\cos t \qquad \boxed{y=c_1\cos t+c_2\sin t+\frac{1}{2}t\cos t}$$

Şimdi 1inci denklemi türetip x bulacağız.

$$\frac{d^2x}{dt^2}+\frac{dy}{dt}=0$$

$$\frac{d^2x}{dt^2}+x+\cos t=0 \qquad \frac{d^2x}{dt^2}+x=-\cos t \qquad r^2+1=0$$

$$r_{1,2}=\mp i$$

$$x_h=c_3\cos t+c_4\sin t$$

$$x_ö=(A\sin t+B\cos t)t$$

$x_0^{u\,\prime} = A\sin t + B\cos t + t\,(A\cos t - B\sin t)$

$x_0^{u\,\prime\prime} = A\cos t - B\sin t + A\cos t - B\sin t + t\,(-A\sin t - B\cos t)$

$x_0^{u\,\prime\prime} = 2A\cos t - 2B\sin t - At\sin t - Bt\cos t$

$\dfrac{d^2x}{dt^2} + x = -\cos t$

$2A\cos t - 2B\sin t - \cancel{At\sin t} - \cancel{Bt\cos t} + \cancel{At\sin t} + \cancel{Bt\cos t} = -\cos t$

$2A\cos t - 2B\sin t = -\cos t$

$2A = -1 \quad A = -1/2 \quad B = 0$

$x_0^u = -\dfrac{1}{2} t\sin t$

$\boxed{x = c_3\cos t + c_4\sin t - \dfrac{1}{2}t\sin t}$

$c_3, c_4 \longrightarrow c_1, c_2$ cinsinden yazılmalı

$\dfrac{dx}{dt} + y = 0$ da yazalım.

$-c_3\sin t + c_4\cos t - \dfrac{1}{2}\sin t - \dfrac{1}{2}\cancel{t\cos t} + c_1\cos t + c_2\sin t$

$+ \dfrac{1}{2}\cancel{t\cos t} = 0$

$\left(-c_3 - \dfrac{1}{2} + c_2\right)\sin t + (c_4 + c_1)\cos t = 0$

$-c_3 + c_2 - \dfrac{1}{2} = 0 \Rightarrow -c_3 + c_2 = \dfrac{1}{2} \Rightarrow \underline{c_3 = c_2 - \dfrac{1}{2}}$

$c_4 + c_1 = 0 \quad \Rightarrow \underline{c_4 = -c_1}$

$x = c_3\cos t + c_4\sin t - \dfrac{1}{2}t\sin t$

$x = \left(c_2 - \dfrac{1}{2}\right)\cos t - c_1\sin t - \dfrac{1}{2}t\sin t$

$x = -c_1\sin t + c_2\cos t - \dfrac{1}{2}\cos t - \dfrac{1}{2}t\sin t$ bulunur.

ÖR/ $(D-2)x+(D-4)y=e^{t}$
$Dx+(D-1)y=e^{4t}$ denklem sistemini çözünüz.

$$\Delta=\begin{vmatrix} D-2 & D-4 \\ D & D-1 \end{vmatrix}=D^2-3D+2-D^2+4D=D+2$$

$$\Delta_1=\begin{vmatrix} e^t & D-4 \\ e^{4t} & D-1 \end{vmatrix}=(D-1)e^t-(D-4)e^{4t}=e^t-e^t-4e^{4t}+4e^{4t}=0$$

$x=\frac{\Delta_1}{\Delta} \Rightarrow (D+2)x=0 \quad \frac{dx}{dt}+2x=0 \quad \int\frac{dx}{x}=\int-2dt$

$\ln x-\ln c_1=-2t$

$\frac{x}{c_1}=e^{-2t}$

$x=c_1e^{-2t}$

$$\Delta_2=\begin{vmatrix} D-2 & e^t \\ D & e^{4t} \end{vmatrix}=(D-2)e^{4t}-De^t=4e^{4t}-2e^{4t}-e^t$$
$$=2e^{4t}-e^t$$

$y=\frac{\Delta_2}{\Delta} \Rightarrow y=\frac{2e^{4t}-e^t}{D+2} \quad (D+2)y=2e^{4t}-e^t$

$\frac{dy}{dt}+2y=2e^{4t}-e^t$ Lineer dif denk

$\lambda=e^{\int 2dt}=e^{2t}$

denklem ile çarp

$y=c_2e^{-2t}+\frac{1}{3}e^{4t}-\frac{1}{3}e^t$

$Dx+(D-1)y=e^{4t}$ de yerine yazalım.

$-2c_1e^{-2t}+2c_2e^{-2t}+\cancel{\frac{4}{3}e^{4t}}-\cancel{\frac{1}{3}e^t}-c_2e^{-2t}-\cancel{\frac{1}{3}e^{4t}}+\cancel{\frac{1}{3}e^t}=\cancel{e^{4t}}$

$(-2c_1-3c_2)e^{-2t}=0 \qquad -2c_1-3c_2=0$

$2c_1=-3c_2$

$c_1=-\frac{3}{2}c_2$

$$\begin{cases} x=-\frac{3}{2}c_2e^{-2t} \\ y=c_2e^{-2t}+\frac{1}{3}e^{4t}-\frac{1}{3}e^t \end{cases} \quad \text{veya} \quad \begin{cases} x=c_1e^{-2t} \\ y=-\frac{2c_1}{3}e^{-2t}+\frac{1}{3}e^{4t}-\frac{1}{3}e^t \end{cases}$$

şeklinde çözümler bulunur.

ÖR/

$$\left.\begin{aligned} \frac{dx}{dt} + 2x + 3y = 0 \\ \frac{dy}{dt} + 2y + 3x = 2e^{2t} \end{aligned}\right\}$$ denklem sistemini çözünüz.

$$D = \frac{d}{dt}$$

$$-3 / \quad (D+2)x + 3y = 0$$

$$D+2 / \quad 3x + (D+2)y = 2e^{2t}$$

$$+ \quad (D+2)^2 y - 9y = \underbrace{D(2e^{2t})}_{4e^{2t}} + 4e^{2t}$$

$$D^2 y + 4Dy + 4y - 9y = 8e^{2t}$$

$$(D^2 + 4D - 5)y = 8e^{2t}$$

$$r^2 + 4r - 5 = 0 \qquad r_1 = 1 \quad r_2 = -5$$

$$-1 \quad 5$$

$$y_h = c_1 e^t + c_2 e^{-5t}$$

$$y_ö = Ae^{2t} \qquad y_ö' = 2Ae^{2t} \qquad y_ö'' = 4Ae^{2t}$$

$$4Ae^{2t} + 8Ae^{2t} - 5Ae^{2t} = 8e^{2t}$$

$$7Ae^{2t} = 8e^{2t} \Rightarrow 7A = 8 \quad A = 8/7$$

$$y_ö = \frac{8}{7} e^{2t} \qquad y = y_h + y_ö = c_1 e^t + c_2 e^{-5t} + \frac{8}{7} e^{2t}$$

$3x + (D+2)y = 2e^{2t}$ de yerine yazalım.

$$3x + c_1 e^t - 5c_2 e^{-5t} + \frac{16}{7} e^{2t} + 2c_1 e^t + 2c_2 e^{-5t} + \frac{16}{7} e^{2t} = 2e^{2t}$$

$$3x + 3c_1 e^t - 3c_2 e^{-5t} + \frac{32}{7} e^{2t} = 2e^{2t}$$

$$x = -c_1 e^t + c_2 e^{-5t} - \frac{6}{7} e^{2t}$$