SUMMARY

1) obtaining DE by the General Solution

If general solution has n-constants

I. differentiate the general solution n-times

II. eliminate the n constants between (n+1) equations

III obtain nth order DE free of constant

First Order DEs

2) Separable Dif. E.

Separable DE can be written as

$$f(x)dx + g(y)dy = 0.$$

Integrate each term $\int f(x)dx + \int g(y)dy = \int 0$.

3) DE that can be transformed into separable DE

DE in the form $\frac{dy}{dx} = f(ax+by+c)$

make the change of variable $ax+by+c = u$

take the derivative $a + b\frac{dy}{dx} = \frac{du}{dx}$

Substitute these into DE

4) Homogeneous DE

Homogeneous DE $M(x,y)dx + N(x,y)dy = 0$ is written as

$$f\left(\frac{y}{x}\right)dx + g\left(\frac{y}{x}\right)dy = 0$$

By the change of variables $\frac{y}{x} = u$, $dy = u\,dx + x\,du$

One can obtain a separable DE.

or

Homogeneous DE $y' = f(x,y)$ is written as

$$y' = h\left(\frac{y}{x}\right)$$

By the change of variables $y = ux$, $y' = u'x + u$

One can obtain a separable DE.

5) DE that can be transformed into separable or homogeneous DE

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$$

If (a_1x+b_1y) and (a_2x+b_2y) are
- lin. dependent (Case 1)
- lin independent (Case 2)

then

Case 1: DE reduces separable DE by $a_1x + b_1y = u$

Case 2: DE " homogeneous DE by $\begin{cases} x = X_1 + h \\ y = Y_1 + k \end{cases}$

(h, k are constant to be determined)

6) Exact DE

$M(x,y)dx + N(x,y)dy = 0$

check exactness condition: $M_y = N_x$

Exact D.E : there exists a potantial func $\phi(x,y)$ s.t.

$$\left.\begin{array}{l} \dfrac{\partial \phi}{\partial x} = M \quad --(1) \\[2ex] \dfrac{\partial \phi}{\partial y} = N \quad --(2) \end{array}\right\}$$

Find $\phi(x,y)$

Write General solution : $\phi(x,y) = C$

7) DE that can be transformed into exact DE.

The Method of Integrating factor:

Nonexact DD_ $M(x,y)dx + N(x,y)dy = 0$ $\quad (M_y \neq N_x)$

By integrating factor

$$\lambda(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

or

$$\lambda(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

It becomes exact.

8- Linear DE

first order linear DE can be written as

$$y' + p(x)y = q(x)$$

I. Integrating Factor:

$$\lambda = \lambda(x) = e^{\int p(x)dx} \quad \text{by in}$$

By integrating factor, General solution of DE can be found.

II. Variation of Parameters:

1st Step: Assume $q(x)=0$

Solve $y' + p(x)y = 0$

replace c by $c(x)$

find $c(x)$ and then G.S.

III. Formula of General Solution:

$$y = e^{-\int p(x)dx} \left[\int q(x).e^{\int p(x)dx} . \lambda + C \right]$$

or

$$y = \frac{1}{\lambda} \left[\int q(x).\lambda.dx + C \right]$$

9 – Bernoulli DE: that can be transformed into LDE can be written as $y' + p(x)y = q(x)y^n$ (non lin

multiply by y^{-n} : $y^{-n}y' + p(x)y^{1-n} = q(x)$

make the change of variables $y^{1-n} = z$

$(1-n)\, y^{-n} \cdot y' = z'$

substitude these into DE $\frac{z'}{1-n} + p(x)z = q(x)$

write the LDE in the stondart form $z' + (1-n)p(x)z = (1-n)q(x)$

solve

last step make the reverse substitution

10 – Riccati D.E that can be transformed into Linear DE

is a equation in the form

$y' + p(x)y + q(x)y^2 = r(x)$ given particular solution y_L

by the transformation $y = y_1 + z$ it becomes Bernoulli

$y = y_1 + \frac{1}{z}$ it becomes Linear.

11- Clairaut DEs: can be written as

$$y = xy' + g(y')$$

change the variables $y' = p$

substitute this into DE $y = xp + g(p)$

Take the derivative $\cancel{y'} = \cancel{p} + xp' + p'g'(p)$

Since $y' = p$, simplfy $p'(x + g'(p)) = 0$

factoring

$p' = 0 \Rightarrow p = c$ G.S $= y = xc + g(c)$

$x + g'(p) = 0 \Rightarrow$ $\left.\begin{aligned} x &= -g'(p) \\ y &= -g'(p).p + g(p) \end{aligned}\right\}$ parametric equations of singular sol.

Elimininate the parameter p to find cartesian eq. of sing. sol.

12- **Lagrange DE** can be written as

$$y = xf(y') + g(y')$$

$y' = p$ $y = xf(p) + g(p)$

Take the derivative $\frac{dp}{dx} = ---$ DE cannot be solved

$\frac{dx}{dp} = ---$ DE can be written as linear.

Find first $x = x(p)$, then substitute it into DE $y = y(p)$

General solution is form of parametric eq.

n^{th} order Linear DE can be written in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x)$$

It has n linearly independent solution:

$$y_1, y_2, \ldots, y_n$$

Its general solution is $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$

$W(y_1, y_2, \ldots, y_n) = 0 \Leftrightarrow y_1, y_2, \ldots, y_n$ lin. independent

D = derivative operator

$D = \frac{d}{dx}$ $\quad Dy = \frac{dy}{dy} = y'$, $\quad D^2y = \frac{d^2y}{dx^2} = y''$, ...

$$a_0y''' + a_1y'' + a_2y' + a_3y = a_0D^3y + a_1D^2y + a_2Dy + a_3y = 0$$

$$(a_0D^3 + a_1D^2 + a_2D + a_3)y = 0.$$

Homogeneous, Constant-Coefficient, Linear DE

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} \cdot y' + a_n y = 0$$

Chracteristik Equation: $y^{(n)} \leftrightarrow r^n$

$\cdots$

$y' \leftrightarrow r$

$P_n(r) = 0$ $\qquad y \leftrightarrow 1$

find the roots

① $r_1 \neq r_2 \in \mathbb{R} \Rightarrow$ $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$ $\Rightarrow y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

real, distinct

② $r_1 = r_2 = r_3 \in \mathbb{R} \Rightarrow$ $y_1 = e^{r_1 x}$ $\quad y_2 = x \cdot e^{r_2 x}$ $\quad y_3 = x^2 e^{r_3 x}$

$y = (c_1 + c_2 x + c_3 x^2) e^{r_1 x}$

real, Repeated

three-fold root

③ $r_{1,2} = a \mp ib \Rightarrow$ $y_1 = e^{ax} \cos bx$, $y_2 = e^{ax} \sin bx$

conjugate complex root

$y = e^{ax} (c_1 \cos bx + c_2 \sin bx)$

④ $r_{1,2} = a \mp ib = r_{3,4}$

conj. comp. root

double root

$\Rightarrow$ $y_1 = e^{ax} \cos bx$ $\quad y_3 = e^{ax} \sin bx$

$y_2 = x . e^{ax} \cos bx$ $\quad y_4 = x . e^{ax} \sin bx$

$y = e^{ax} ((c_1 + c_2 x) \cos bx + (c_3 + c_4 x) \sin bx)$

Not that

To write CE from operator equation

replace D with r