FIZ1112
PHYSICS 2

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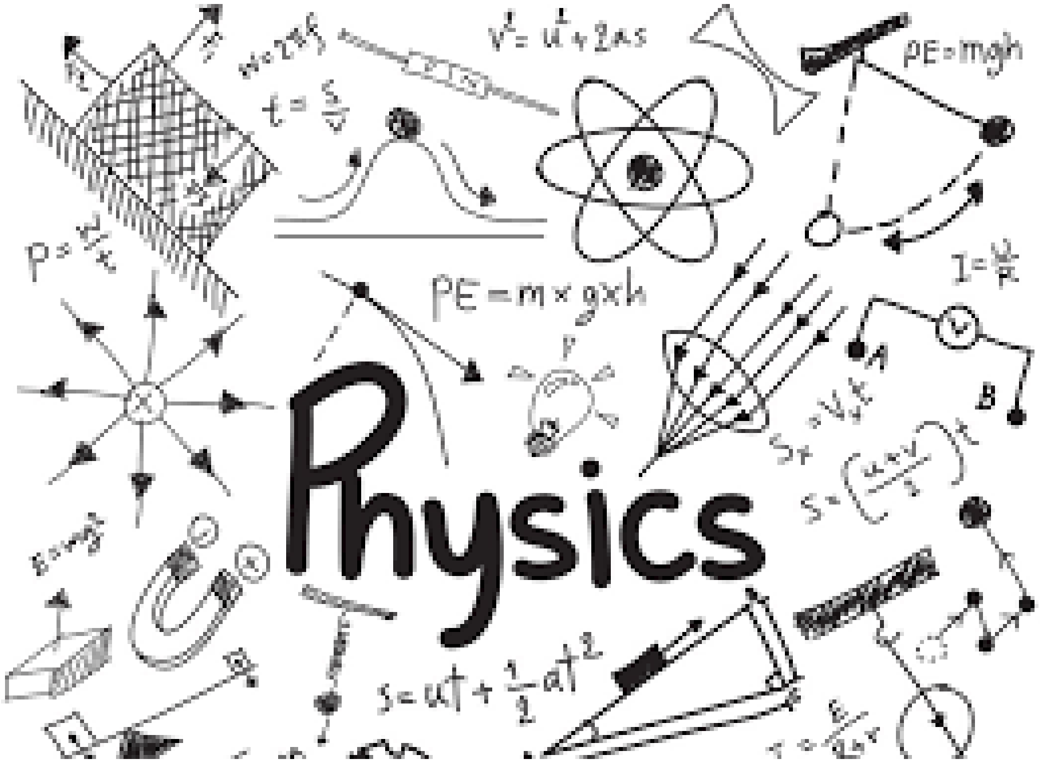
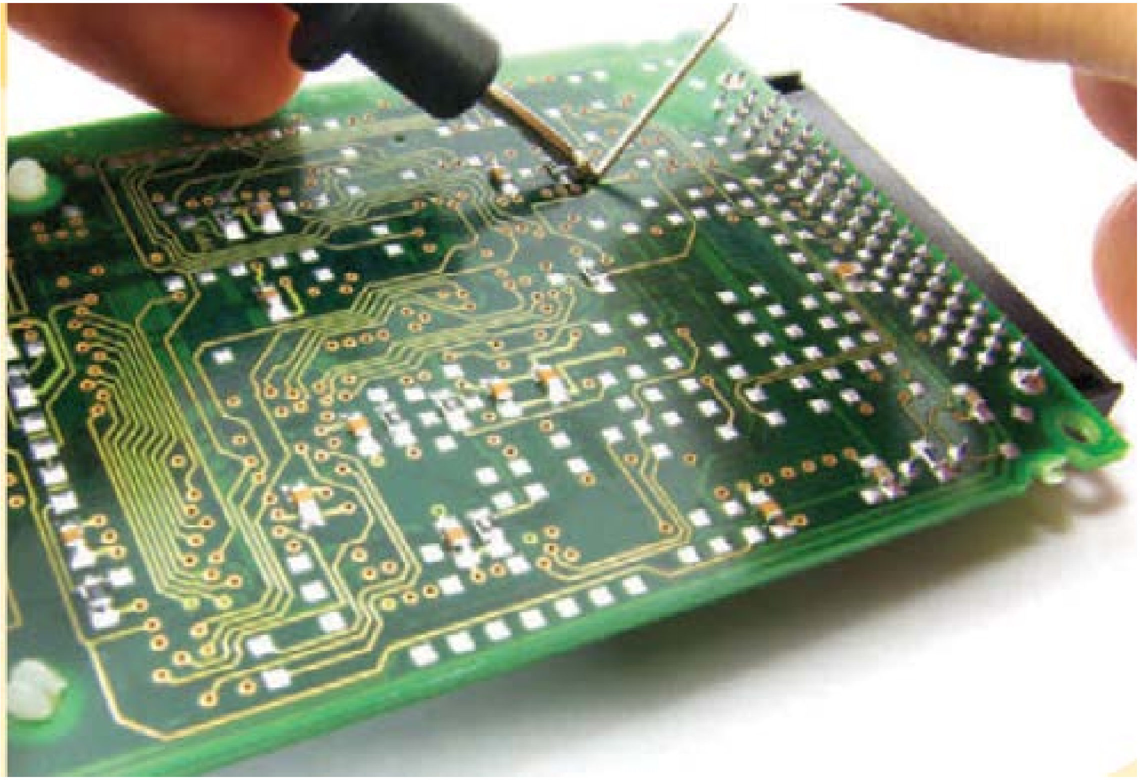
Prepared from Serway and Jewett



Chapter: 28 Direct-Current Circuits

1. Electromotive Force
2. Resistors in Series and Parallel
3. Kirchhoff's Rules
4. RC Circuits



Introduction

- In this chapter, we'll analyze simple electric circuits that contain batteries, resistors, and capacitors in various combinations.

- The analysis of more complicated circuits is simplified using *Kirchhoff rules*, which follow from the laws of conservation of energy and conservation of electric charge for isolated systems.

- Most of the circuits analyzed are assumed to be in *steady state*, which means that currents in the circuit are constant in magnitude and direction. A current that is constant in direction is called a *direct current* (DC).

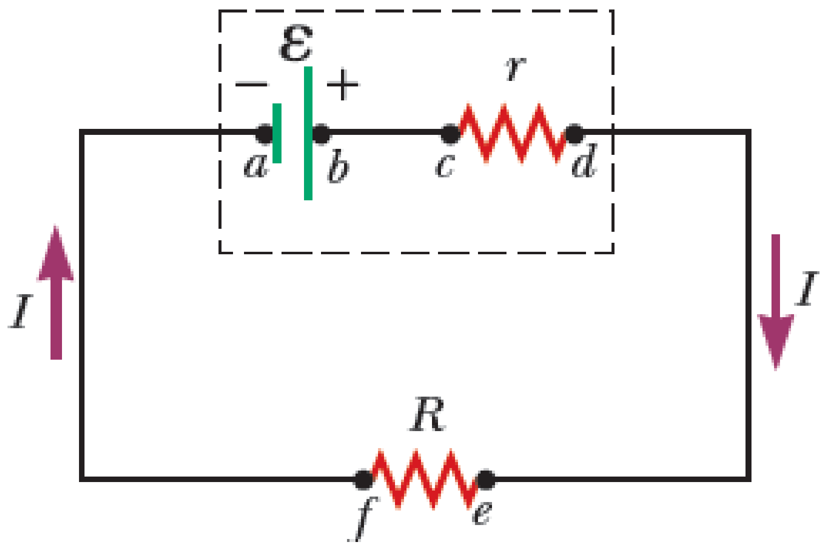
Electromotive Force

- We generally use a battery as **a source of energy for circuits**.
- Since the electric potential difference between the battery terminals is constant, the current in the circuit is constant in magnitude and direction and is called direct current (DC).
- A battery is called either a source of **e**lectro**m**otive **f**orce or, more commonly, a source of **emf**.
- **The phrase electromotive force does not describe a force, it defines a potential difference in volts.**
- **The emf of a battery is the maximum possible voltage the battery can provide between its terminals.**

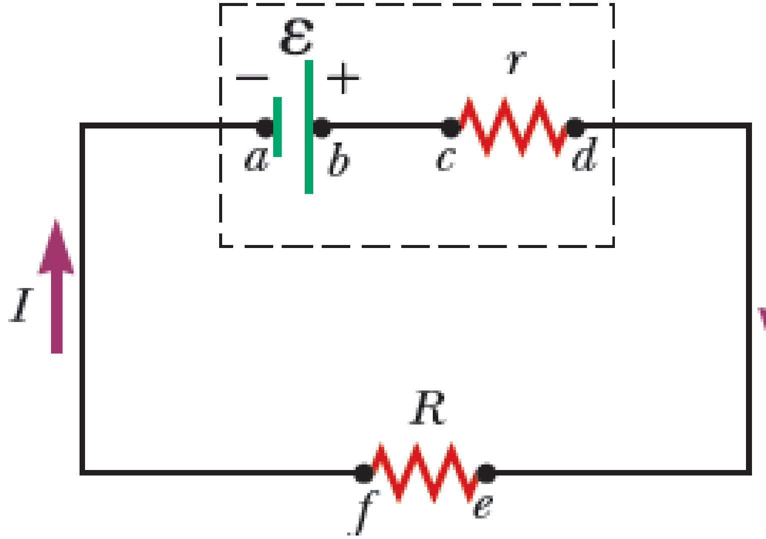
- We can think of a source of emf as a "charge pump."
- When an electric potential difference exists between two points, the source moves charges from the lower potential to the higher.
- We generally assume the connecting wires in a circuit have no resistance.
- The positive terminal of a battery is at a higher potential than the negative terminal.
- There is some resistance to the flow of charge inside the real battery. This resistance is called internal resistance r. For an idealized battery, internal resistance is zero and the potential difference across the battery equals its emf.

We model the battery as shown in the diagram; it is represented by the dashed rectangle containing an ideal, resistance-free emf (ε) in series with an internal resistance r.

A resistor of resistance R is connected across the terminals of the battery.

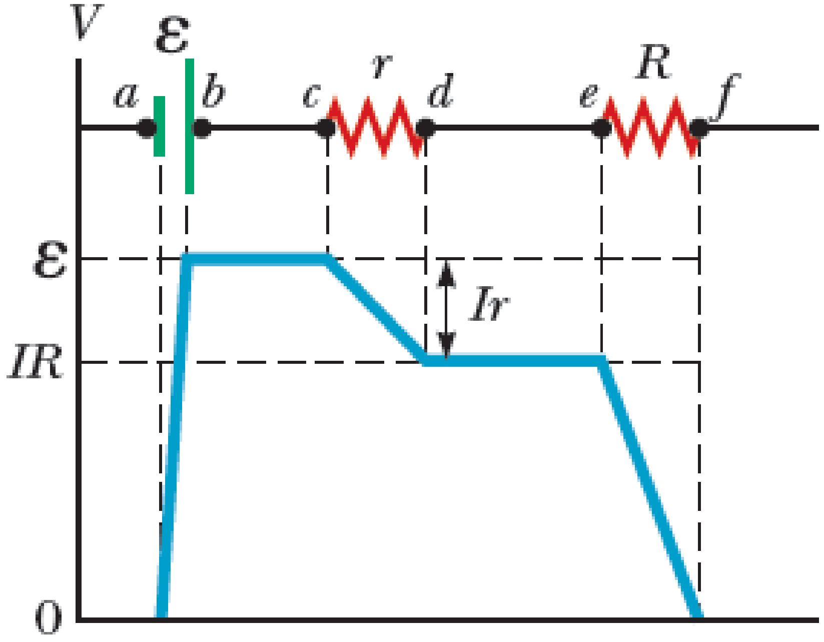


Passing from the negative terminal to the positive terminal, the potential increases by an amount ε.
As we move through the resistance *r*, however, the potential *decreases* by an amount *Ir*, where *I* is the current in the circuit. Therefore, the terminal voltage of the battery is



$$\Delta V = V_d - V_a$$

$$\Delta V = \mathcal{E} - Ir$$



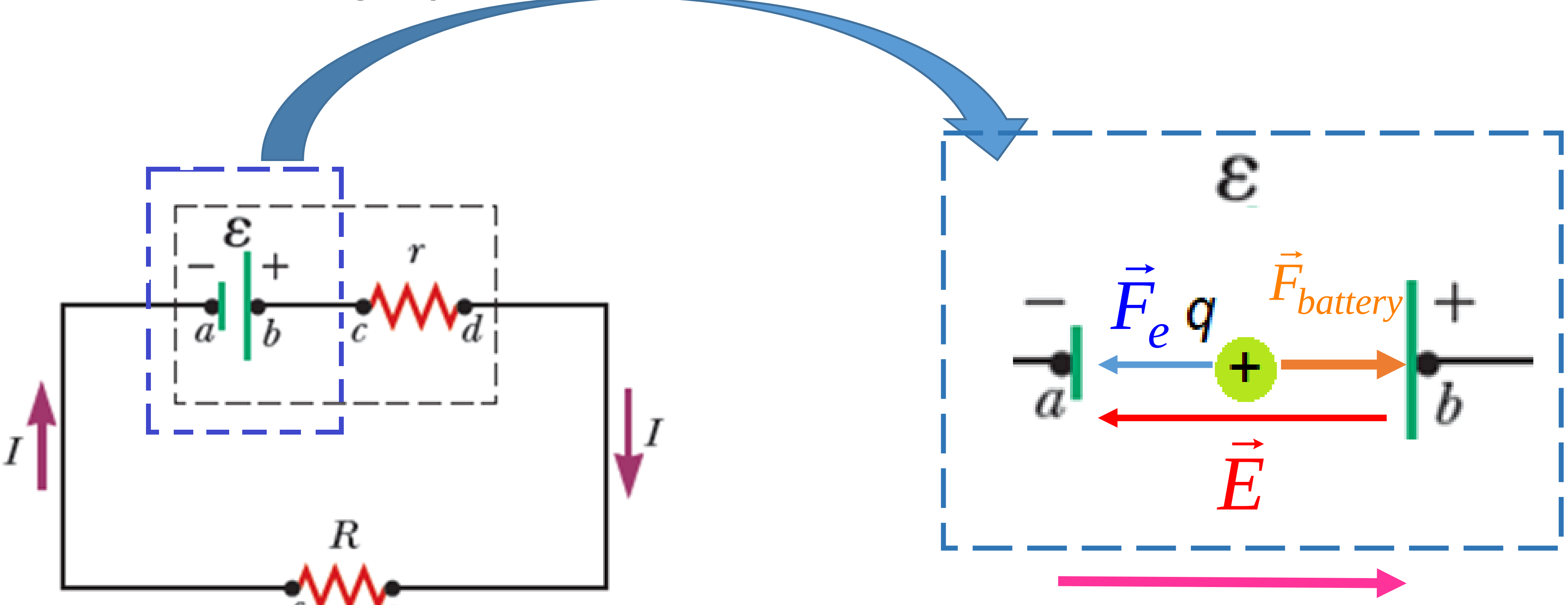
The terminal voltage ΔV must equal the potential difference across the external resistance *R*, often called the **load resistance.**

$$\Delta V = IR.$$

$$I = \frac{\mathcal{E}}{R + r}$$

Let's focus what happens inside the battery

Outside the battery, the current (I) is flowing from its «+» terminal to «-» terminal. To complete the current loop, inside the battery, the current (ie. collection of charges) must flow from its «-» terminal to «+» terminal.



Direction of charges inside the battery

$$\vec{F}_{battery} > \vec{F}_e$$

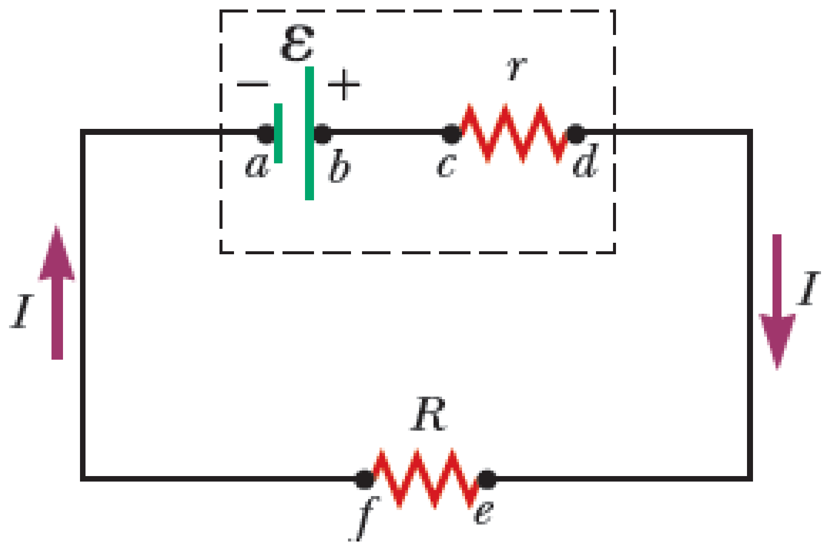
$$I = \frac{\mathcal{E}}{R + r}$$

Multiply the equation by electric current

$$\boxed{I\mathcal{E}} = \boxed{I^2R + I^2r}$$

Total power generated by the battery

Total power consumed across the resistors



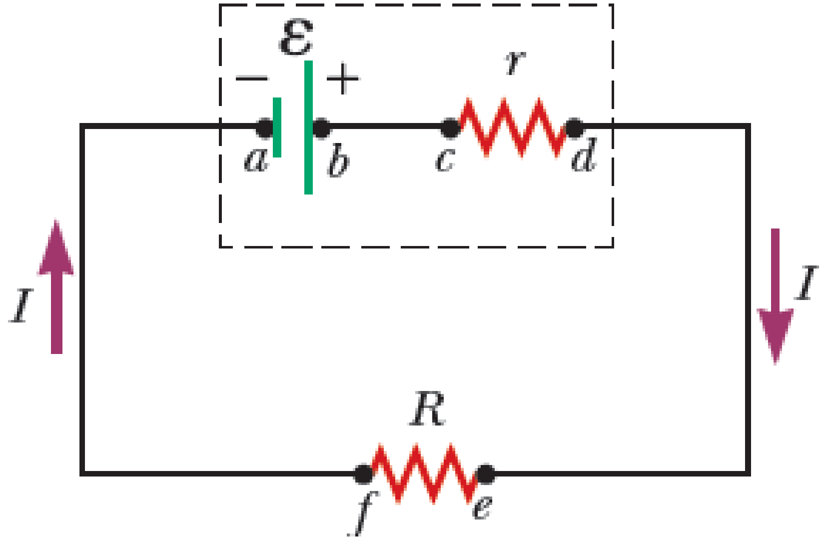
Exercise: A battery has an emf of 12 V and an internal resistance of 0.05 Ω. Its terminals are connected to a load resistance of 3 Ω .

a) Find the current in the circuit and the terminal voltage of the battery.

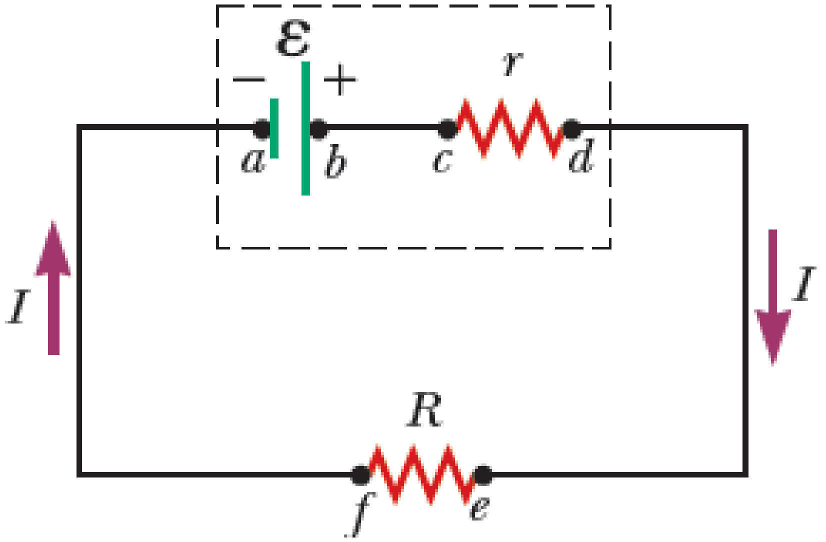
$$I = \frac{\mathcal{E}}{R + r} = \frac{12.0\ \mathrm{V}}{3.00\ \Omega + 0.050\,0\ \Omega} = \boxed{3.93\ \mathrm{A}}$$

$$\Delta V = \mathcal{E} - Ir = 12.0\ \mathrm{V} - (3.93\ \mathrm{A})(0.050\,0\ \Omega) = \boxed{11.8\ \mathrm{V}}$$

$$\Delta V = IR = (3.93\ \mathrm{A})(3.00\ \Omega) = 11.8\ \mathrm{V}$$



b) Calculate the power delivered to the load resistor, the power delivered to the internal resistance of the battery, and the power delivered by the battery.



$$P_R = I^2R = (3.93\ \text{A})^2(3.00\ \Omega) = \boxed{46.3\ \text{W}}$$

$$P_r = I^2r = (3.93\ \text{A})^2(0.050\,0\ \Omega) = \boxed{0.772\ \text{W}}$$

$$P = P_R + P_r = 46.3\ \text{W} + 0.772\ \text{W} = \boxed{47.1\ \text{W}}$$

$$P_{battery} = I\varepsilon = 3.93 \times 12 = 47.16\ \text{W}$$

c) As a battery ages, its internal resistance increases. Suppose the internal resistance of this battery rises to 2 Ω toward the end of its useful life. How does that alter the battery's ability to deliver energy?

The new current in the battery:

$$I = \frac{\mathcal{E}}{R + r} = \frac{12.0\text{ V}}{3.00\ \Omega + 2.00\ \Omega} = 2.40\text{ A}$$

The new terminal voltage: $\Delta V = \mathcal{E} - Ir = 12.0\text{ V} - (2.40\text{ A})(2.00\ \Omega) = 7.2\text{ V}$

The new powers delivered to the load resistor and internal resistance:

$$P_R = I^2R = (2.40\text{ A})^2(3.00\ \Omega) = 17.3\text{ W}$$

$$P_r = I^2r = (2.40\text{ A})^2(2.00\ \Omega) = 11.5\text{ W}$$

$$P_{battery}^{\ young} = 47.16\text{ W}$$

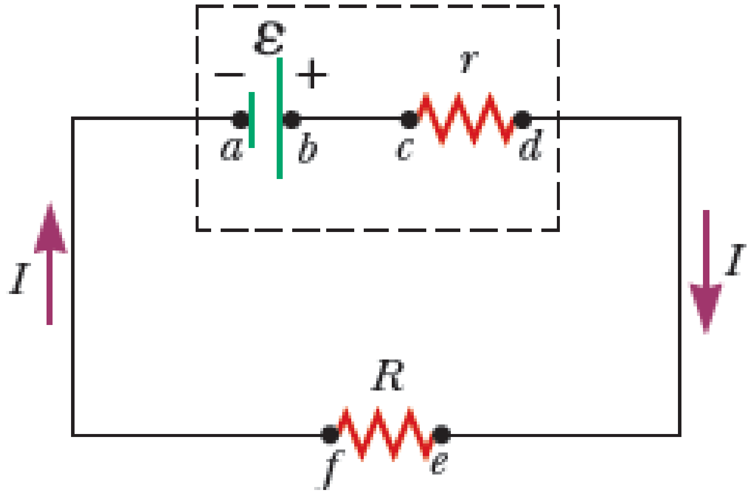
$$P_{battery}^{\ aged} = I\varepsilon = 2.4 \times 12 = 28.8\text{ W}$$

In this situation, the terminal voltage is only 60% of the emf. Notice that 40% of the power from the battery is delivered to the internal resistance when r is 2.00 Ω. When r is 0.050 0 Ω as in part (B), this percentage is only 1.6%. Consequently, even though the emf remains fixed, the increasing internal resistance of the battery significantly reduces the battery's ability to deliver energy to an external load.

Exercise: Find the load resistance R for which the maximum power is delivered to the load resistance in the Figure.

$$I = \frac{\mathcal{E}}{R + r}$$

the current in the circuit



$$P = I^2R = \frac{\mathcal{E}^2R}{(R + r)^2}$$

the power delivered to the load resistance

Let's differentiate the power with respect to the load resistance R and set the derivative equal to zero to maximize the power.

$$\frac{dP}{dR} = \frac{d}{dR}\left[\frac{\mathcal{E}^2R}{(R + r)^2}\right] = \frac{d}{dR}[\mathcal{E}^2R(R + r)^{-2}] = 0$$

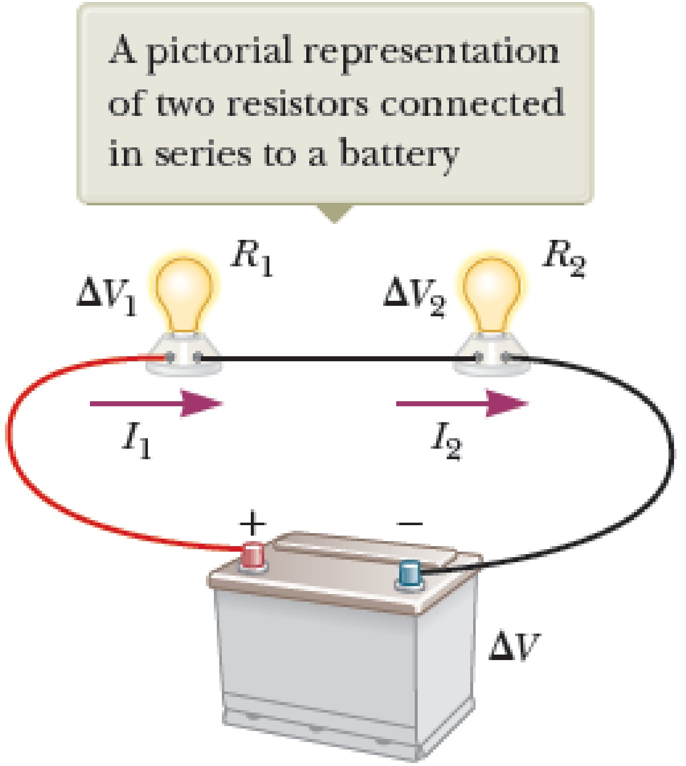
$$[\mathcal{E}^2(R + r)^{-2}] + [\mathcal{E}^2R(-2)(R + r)^{-3}] = 0$$

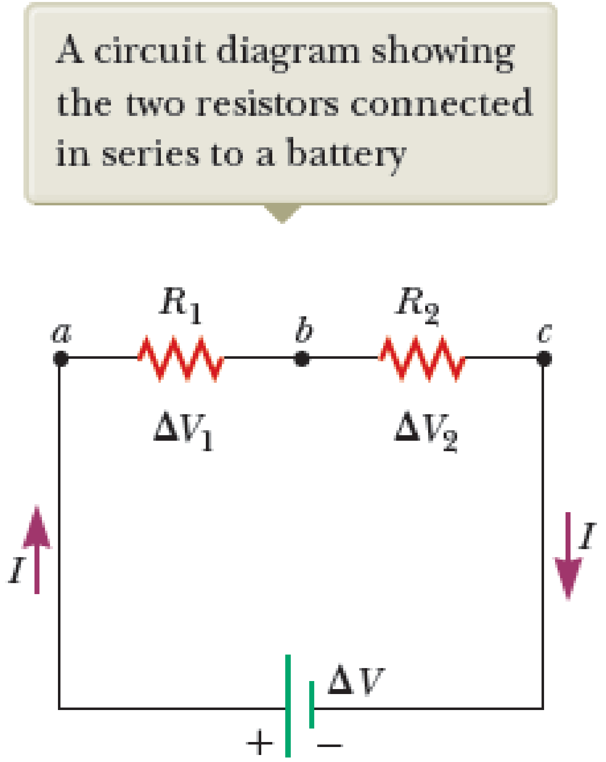
$$\frac{\mathcal{E}^2(R + r)}{(R + r)^3} - \frac{2\mathcal{E}^2R}{(R + r)^3} = \frac{\mathcal{E}^2(r - R)}{(R + r)^3} = 0$$

$$R = r$$

$$P_{max} = \mathcal{E}^2/4r.$$

A Series Combination of Resistors



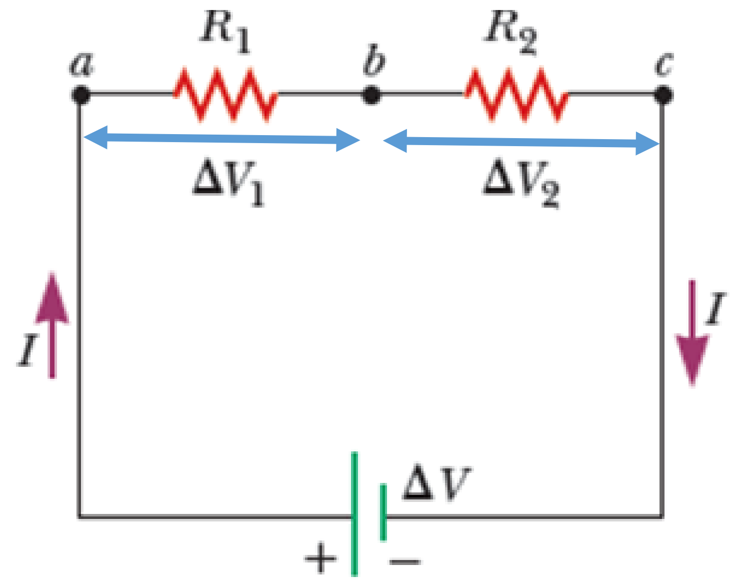


- When two or more resistors are connected together as shown in Figure, they are said to be in a series combination.
- In a series connection, if a charge Q exits resistor R_1, charge Q must also enter the second resistor R_2.
- Therefore, the same amount of charge passes through both resistors in a given time interval and the currents are the same in both resistors:

$$I = I_1 = I_2$$

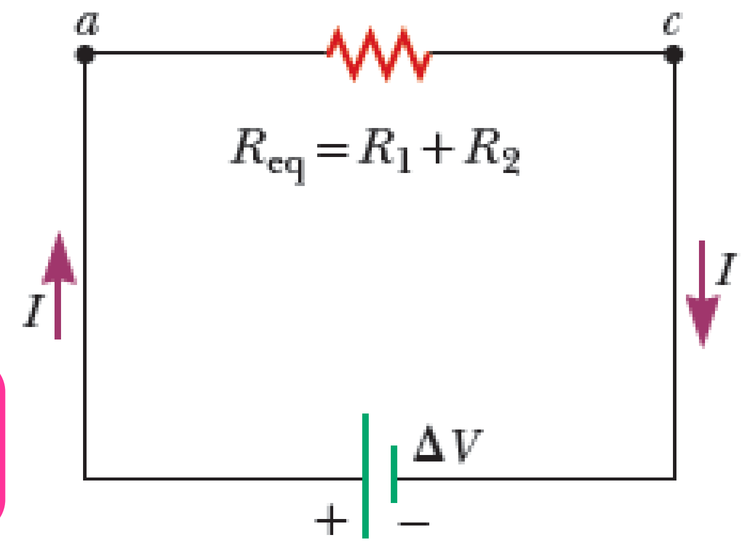
The potential difference applied across the series combination of resistors divides between the resistors.

The voltage drop from a to b equals I_1R_1 and the voltage drop from b to c equals I_2R_2, **<u>the voltage drop from a to c is:</u>**



$$\Delta V = \Delta V_1 + \Delta V_2 = I_1R_1 + I_2R_2$$

The potential difference across the battery is also applied to the equivalent resistance, R_{eq}



$$\Delta V = IR_{eq}$$

$$IR_{eq} = I_1R_1 + I_2R_2 \quad \rightarrow \quad \boxed{R_{eq} = R_1 + R_2}$$

We can replace the two resistors in series with a single equivalent resistance whose value is the sum of the individual resistances.

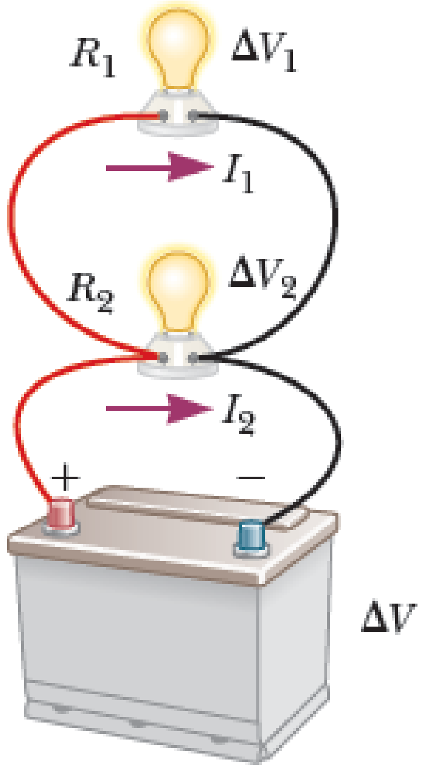
The equivalent resistance of three or more resistors connected in series is

$$R_{eq} = R_1 + R_2 + R_3 + \cdots$$

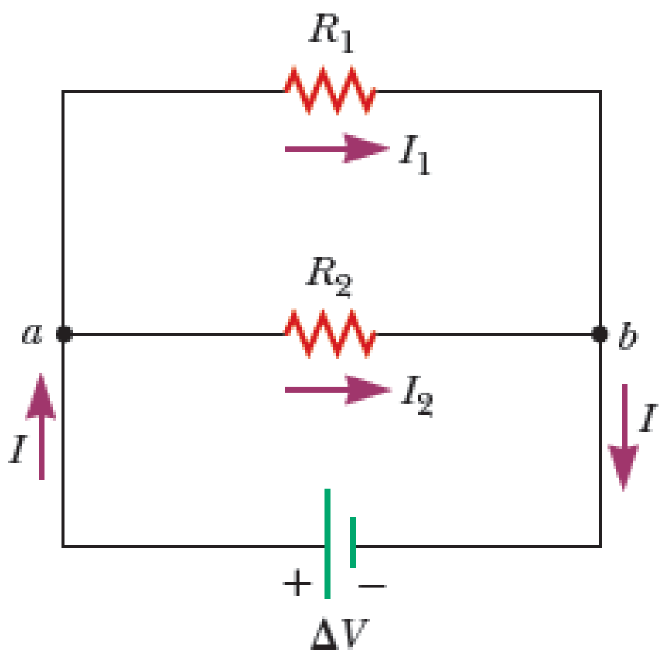
This relationship indicates that the equivalent resistance of a series combination of resistors is the numerical sum of the individual resistances and is always greater than any individual resistance.

A Parallel Combination of Resistors

A pictorial representation of two resistors connected in parallel to a battery



A circuit diagram showing the two resistors connected in parallel to a battery



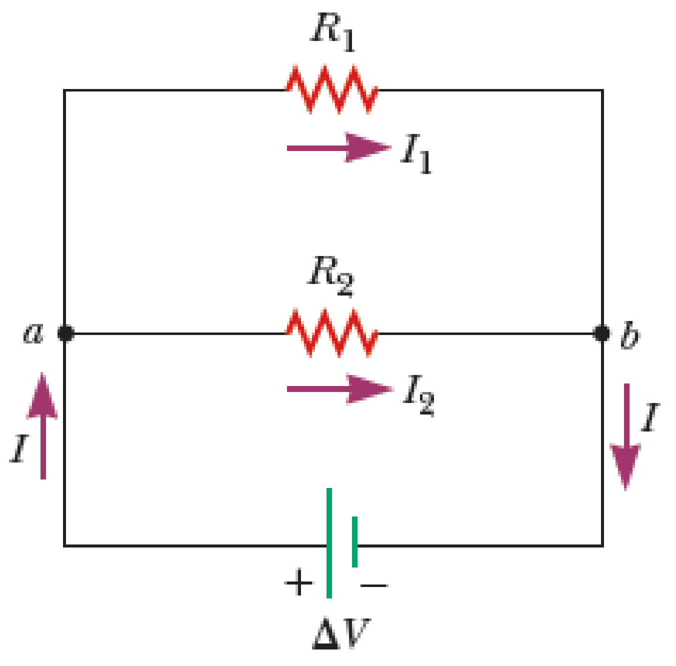
- R_1 and R_2 resistors are connected directly across the terminals of the battery. Therefore, **the potential differences across the resistors are the same.**

$$\Delta V = \Delta V_1 = \Delta V_2$$

- **When charges reach point a, they split into two parts**, with some going toward R_1 and the rest going toward R_2. A junction is any such point in a circuit where a current can split.

Because electric charge is conserved, the current I that enters point a must equal the total current leaving that point:

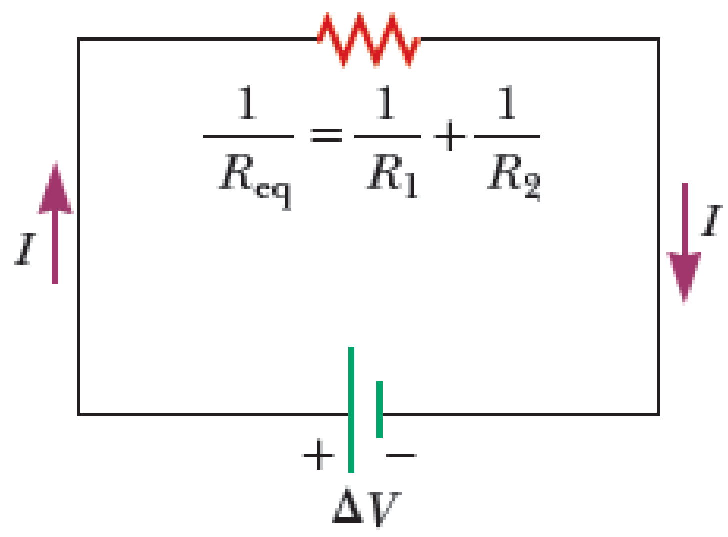
$$I = I_1 + I_2 = \frac{\Delta V_1}{R_1} + \frac{\Delta V_2}{R_2}$$



The current in the **equivalent resistance** R_{eq} is $I = \frac{\Delta V}{R_{eq}}$

$$\frac{\Delta V}{R_{eq}} = \frac{\Delta V_1}{R_1} + \frac{\Delta V_2}{R_2} \rightarrow \boxed{\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}}$$

$$\Delta V_1 = \Delta V_2 = \Delta V$$



The equivalent resistance of three or more resistors connected in parallel is

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots$$

This expression shows that the inverse of the equivalent resistance of two or more resistors in a parallel combination is equal to the sum of the inverses of the individual resistances. Furthermore, the equivalent resistance is always less than the smallest resistance in the group.

Exercise:

Four resistors are connected as shown in the Figure.

a) Find the equivalent resistance between points *a* and *c*.

$$R' = 8 + 4 = 12\ \Omega \qquad \frac{1}{R''} = \frac{1}{6} + \frac{1}{3} \qquad R_{eq} = 12 + 2 = 14\ \Omega$$

$$R'' = 2\ \Omega$$

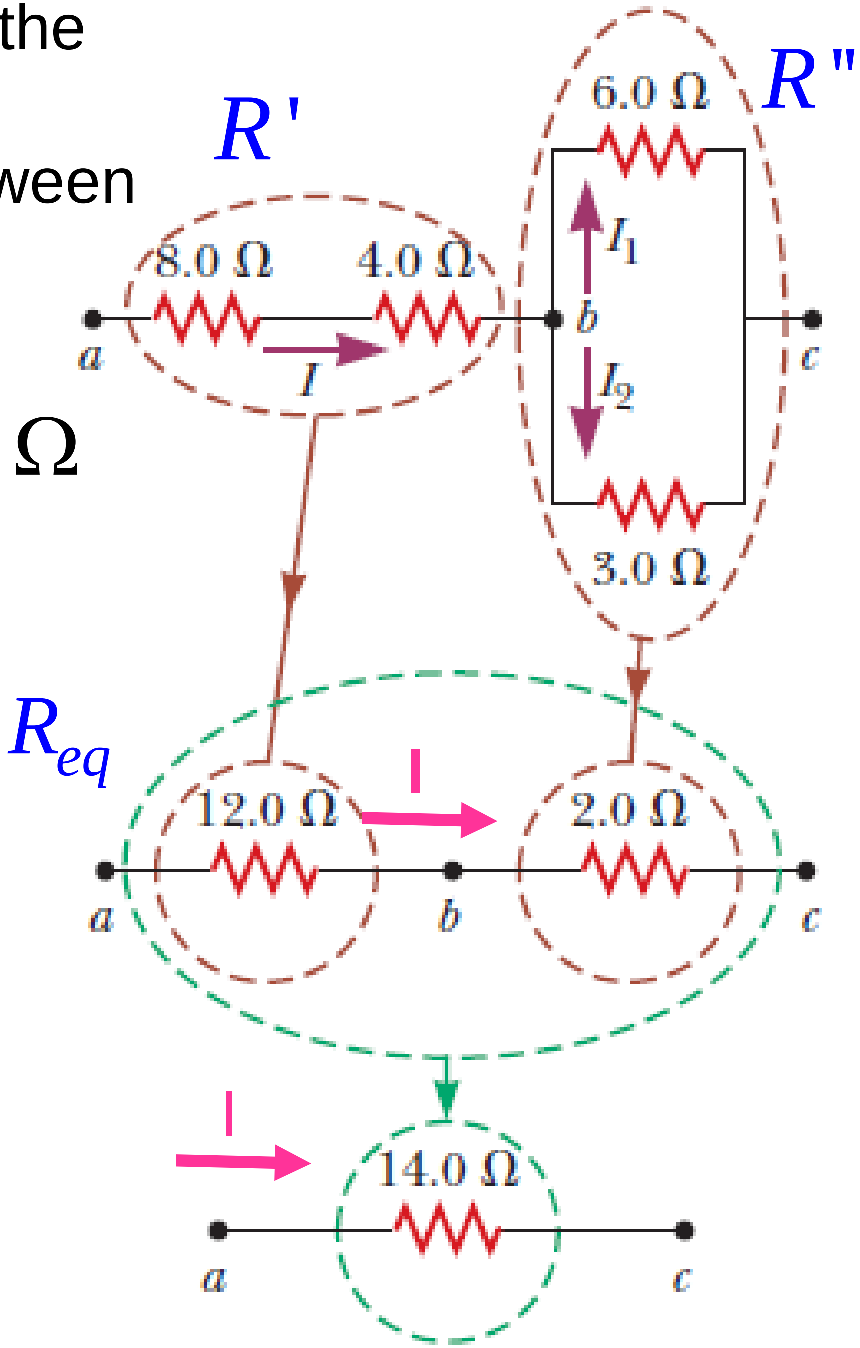
b) What is the current in each resistor if a potential difference of 42 V is maintained between a and c?

$$I = \frac{\Delta V_{ac}}{R_{eq}} = \frac{42\ \text{V}}{14.0\ \Omega} = \boxed{3.0\ \text{A}} \qquad I_1 + I_2 = 3.0\ \text{A}$$

$$\Delta V_1 = \Delta V_2 \rightarrow (6.0\ \Omega)I_1 = (3.0\ \Omega)I_2 \rightarrow I_2 = 2I_1$$

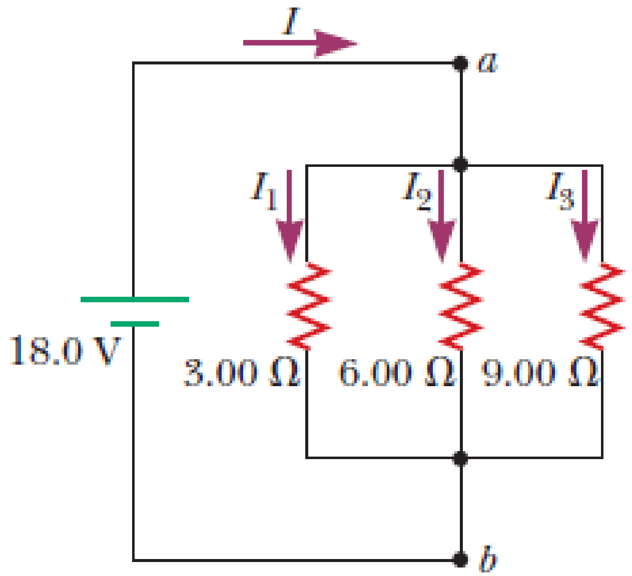
$$I_1 + I_2 = 3.0\ \text{A} \rightarrow I_1 + 2I_1 = 3.0\ \text{A} \rightarrow I_1 = \boxed{1.0\ \text{A}}$$

$$I_2 = 2I_1 = 2(1.0\ \text{A}) = \boxed{2.0\ \text{A}}$$



Exercise: Three resistors are connected in parallel as shown in the Figure. A potential difference of 18 V is maintained between points a and b.

a) Calculate the equivalent resistance of the circuit.



$$\frac{1}{R_{eq}} = \frac{1}{3.00\ \Omega} + \frac{1}{6.00\ \Omega} + \frac{1}{9.00\ \Omega} = \frac{11}{18.0\ \Omega}$$

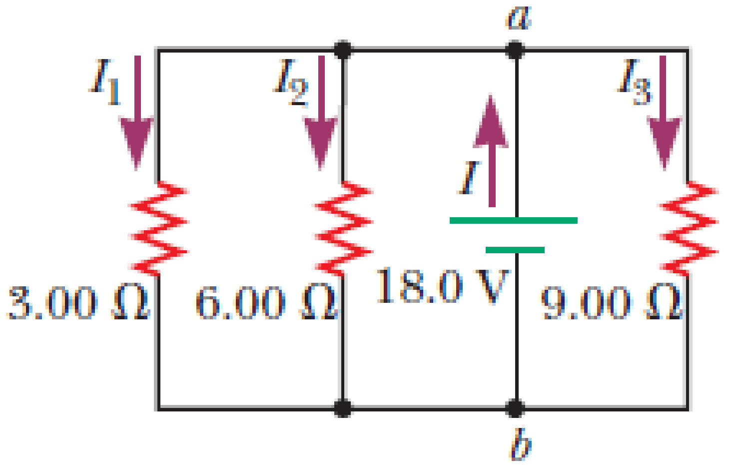
$$R_{eq} = \frac{18.0\ \Omega}{11} = \boxed{1.64\ \Omega}$$

b) Find the current in each resistor.

$$I_1 = \frac{\Delta V}{R_1} = \frac{18.0\ \text{V}}{3.00\ \Omega} = \boxed{6.00\ \text{A}}$$

$$I_2 = \frac{\Delta V}{R_2} = \frac{18.0\ \text{V}}{6.00\ \Omega} = \boxed{3.00\ \text{A}}$$

$$I_3 = \frac{\Delta V}{R_3} = \frac{18.0\ \text{V}}{9.00\ \Omega} = \boxed{2.00\ \text{A}}$$

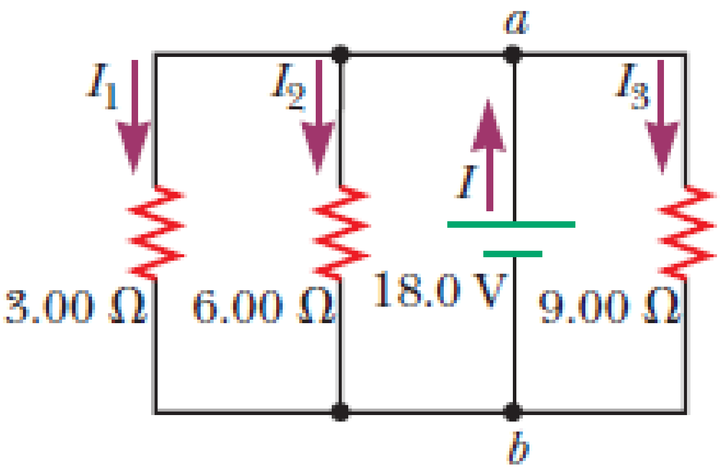


c) Calculate the power delivered to each resistor and the total power delivered to the combination of resistors.

$$3.00\text{-}\Omega: P_1 = I_1^2 R_1 = (6.00\text{ A})^2(3.00\ \Omega) = \boxed{108\text{ W}}$$

$$6.00\text{-}\Omega: P_2 = I_2^2 R_2 = (3.00\text{ A})^2(6.00\ \Omega) = \boxed{54\text{ W}}$$

$$9.00\text{-}\Omega: P_3 = I_3^2 R_3 = (2.00\text{ A})^2(9.00\ \Omega) = \boxed{36\text{ W}}$$



+

198 W

These results show that the smallest resistor receives the most power!

$$P = (\Delta V)^2/R_{eq} = (18.0\text{ V})^2/1.64\ \Omega = 198\text{ W}$$

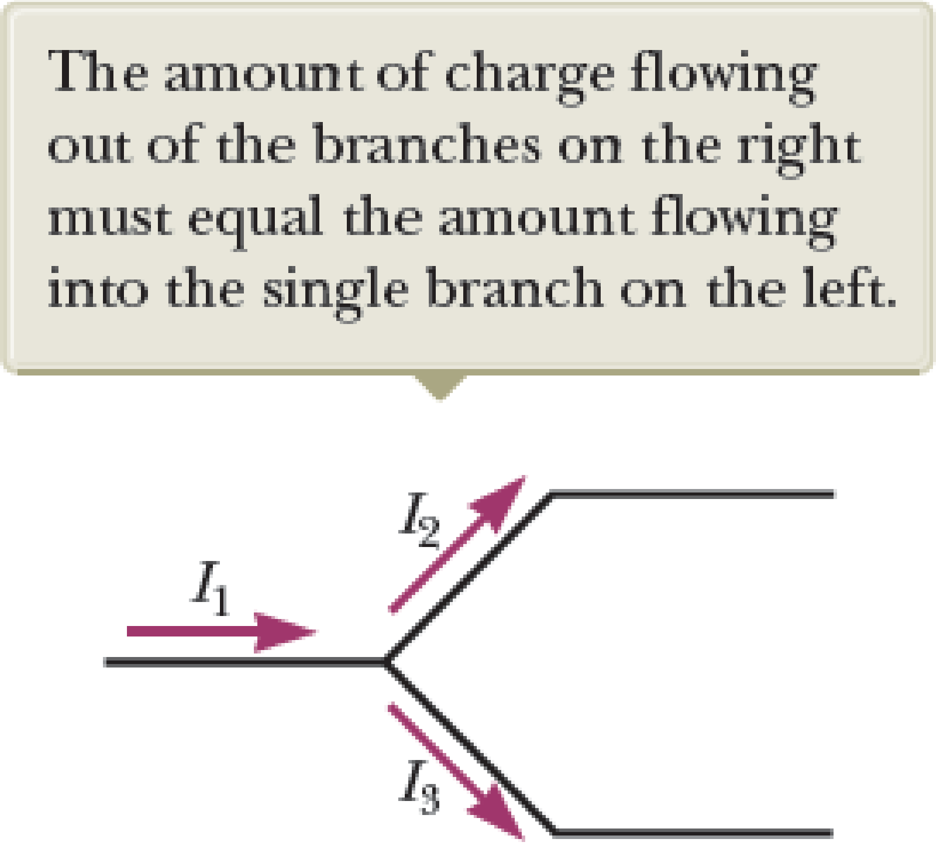
Kirchhoff's Rules

The procedure **for analyzing more complex circuits** is made possible by using the following two principles, called Kirchhoff's rules.

1. Junction rule: At any junction, the sum of the current entering the junction must equal the sum of the currents leaving that junction.

$$\sum I_{in} = \sum I_{out}$$

$$I_1 = I_2 + I_3$$



Kirchhoff's first rule is a natural consequence of conservation of electric charge.

Kirchhoff's Rules

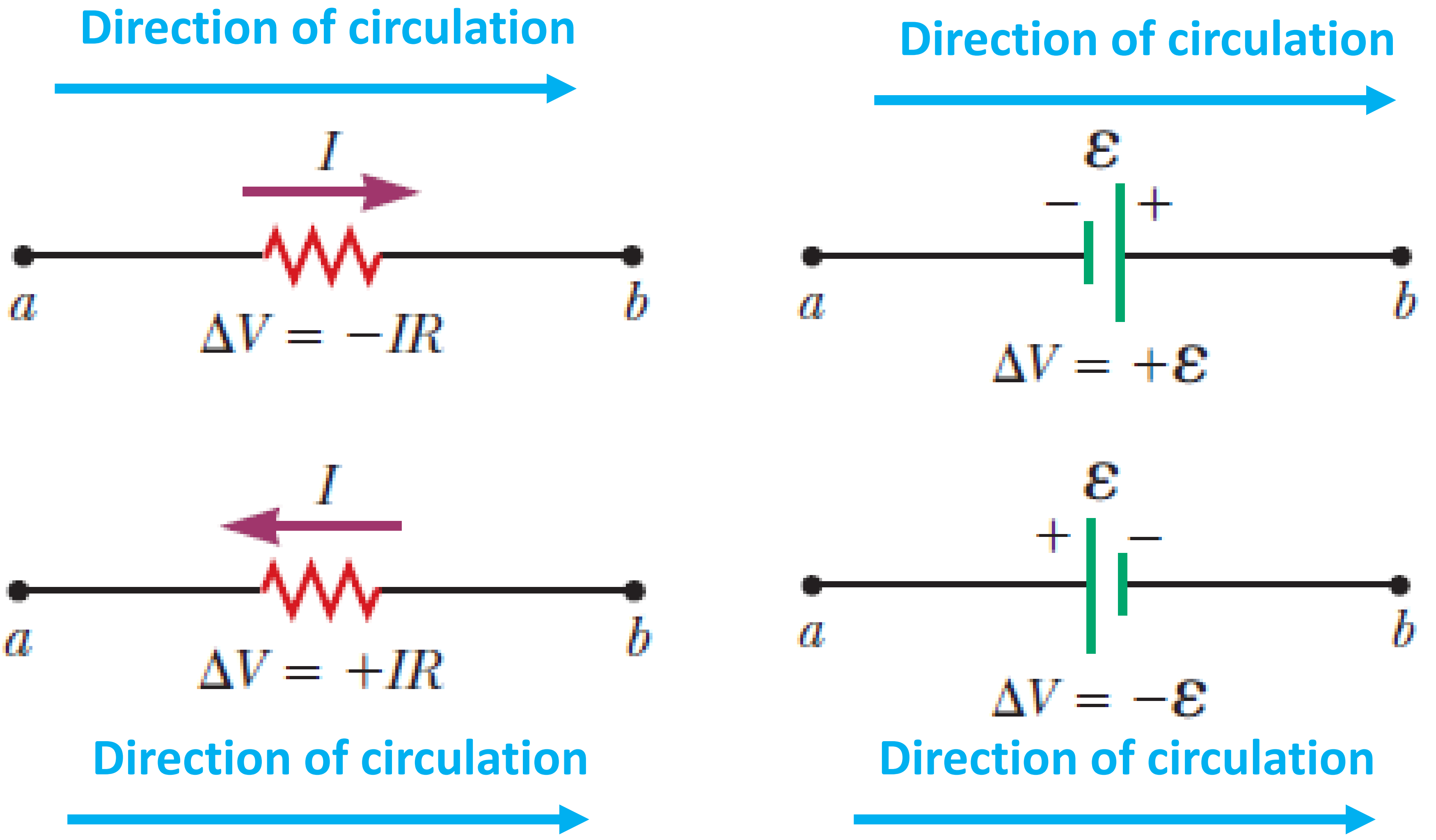
2. Loop rule: The sum of the potential differences across all elements around any closed circuit loop must be zero.

$$\sum_{\substack{\text{closed}\\ \text{loop}}} \Delta V = 0$$

Kirchhoff's second rule is a natural consequence of conservation of energy.

Let's imagine moving a charge around a closed loop of a circuit. When the charge returns to the starting point, the charge–circuit system must have the same total energy as it had before the charge was moved. The sum of the increases in energy as the charge passes through some circuit elements must equal the sum of the decreases in energy as it passes through other elements.

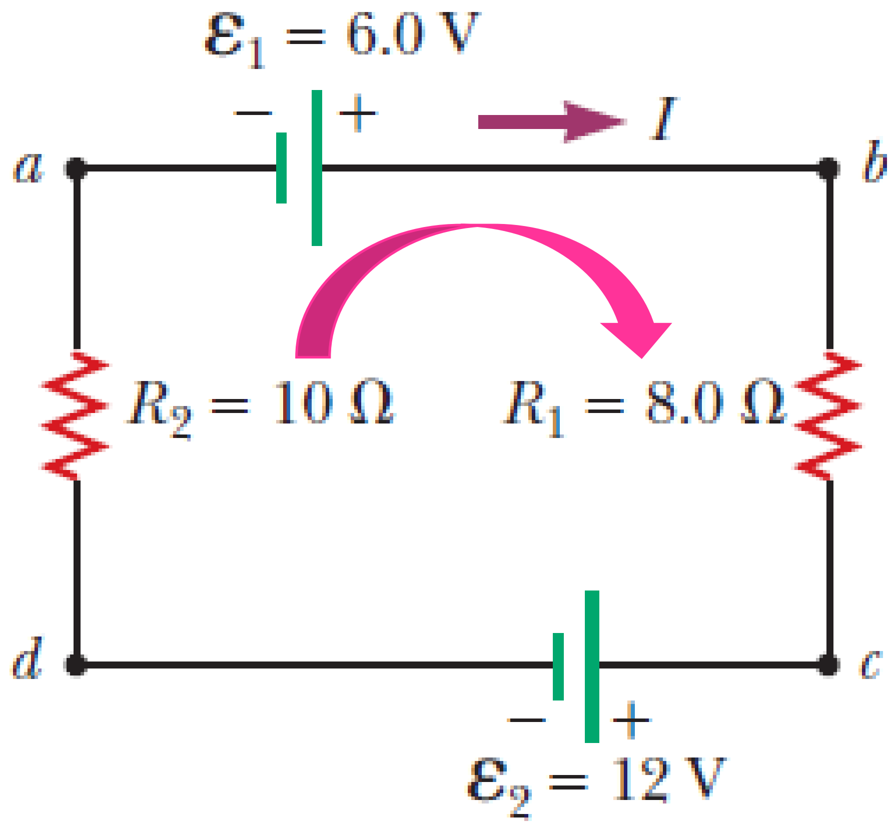
Rules for determining the signs of the potential differences across a resistor and a battery



Exercise:

A single-loop circuit contains two resistors and two batteries as shown in the Figure. (Neglect the internal resistances of the batteries.)

a) Find the current in the circuit.



Let's apply Kirchhoff's loop rule to the single loop in the circuit:

For abcda loop:

$$\sum \Delta V = 0 \;\rightarrow\; \mathcal{E}_1 - IR_1 - \mathcal{E}_2 - IR_2 = 0$$

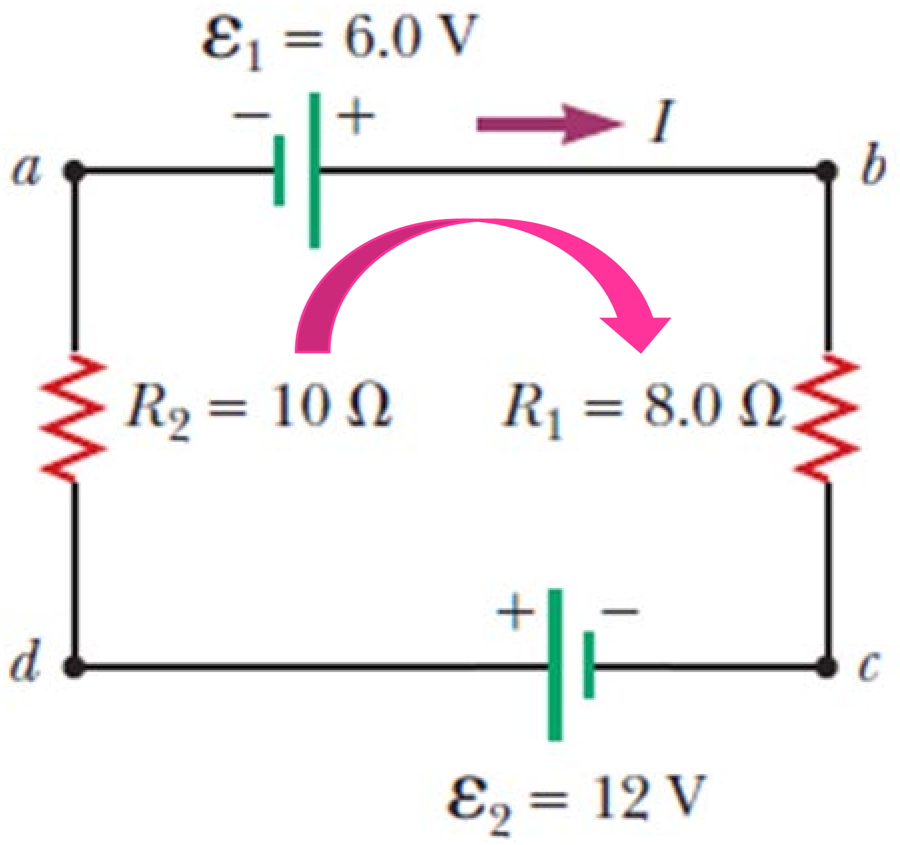
$$I = \frac{\mathcal{E}_1 - \mathcal{E}_2}{R_1 + R_2} = \frac{6.0\ \mathrm{V} - 12\ \mathrm{V}}{8.0\ \Omega + 10\ \Omega} = \boxed{-0.33\ \mathrm{A}}$$

The negative sign for electric current indicates that the direction of the current is opposite the assumed direction.

b) What if the polarity of the 12 V battery were reversed? How would that affect the circuit?

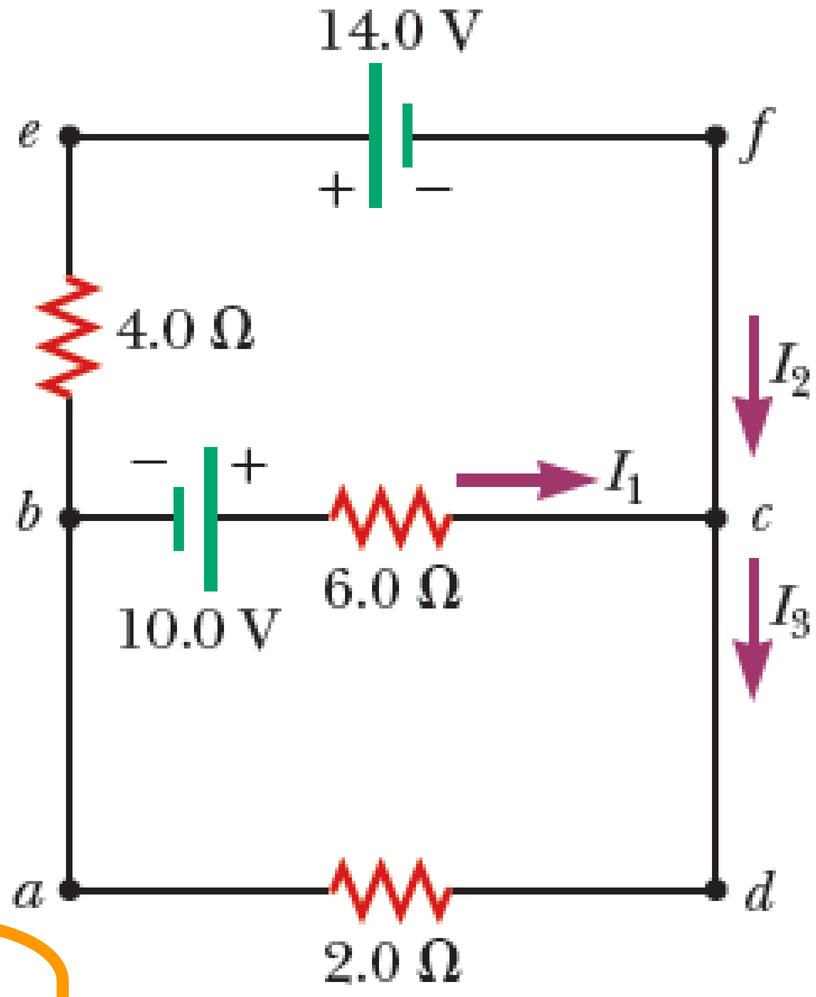
$$\sum \Delta V = 0 \;\rightarrow\; \mathcal{E}_1 - IR_1 + \mathcal{E}_2 - IR_2 = 0$$

$$I = \frac{\mathcal{E}_1 + \mathcal{E}_2}{R_1 + R_2} = \frac{6.0\text{ V} + 12\text{ V}}{8.0\ \Omega + 10\ \Omega} = 1.0\text{ A}$$



The polarity change would affect the magnitude of electric current.

Exercise: Find the currents I_1, I_2, and I_3 in the circuit shown in the Figure.



Let's apply Kirchhoff's junction rule to junction *c* :

$$I_1 + I_2 - I_3 = 0$$

There are three loops in the circuit: abcda, befcb, and aefda.

$$\sum_{\substack{\text{closed}\\ \text{loop}}} \Delta V = 0$$

abcda: $10.0\text{ V} - (6.0\ \Omega)I_1 - (2.0\ \Omega)I_3 = 0$

befcb: $\boxed{-(4.0\ \Omega)I_2 - 14.0\text{ V} + (6.0\ \Omega)I_1 - 10.0\text{ V} = 0}$

$$10.0\text{ V} - (6.0\ \Omega)I_1 - (2.0\ \Omega)(I_1 + I_2) = 0$$

$$\boxed{10.0\text{ V} - (8.0\ \Omega)I_1 - (2.0\ \Omega)I_2 = 0}$$

Two equations and two unknowns

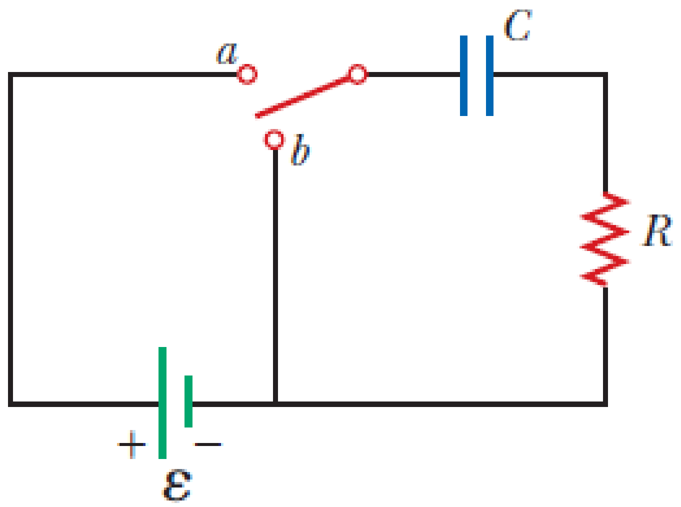
$$I_1 = \boxed{2.0\,\mathrm{A}} \qquad I_2 = \boxed{-3.0\,\mathrm{A}} \qquad I_3 = I_1 + I_2 = 2.0\,\mathrm{A} - 3.0\,\mathrm{A} = \boxed{-1.0\,\mathrm{A}}$$

Because our values for I_2 and I_3 are negative, the directions of these currents are opposite those indicated in the Figure. The numerical values for the currents are correct.

Despite the incorrect direction, we must continue to use these negative values in subsequent calculations because our equations were established with our original choice of direction.

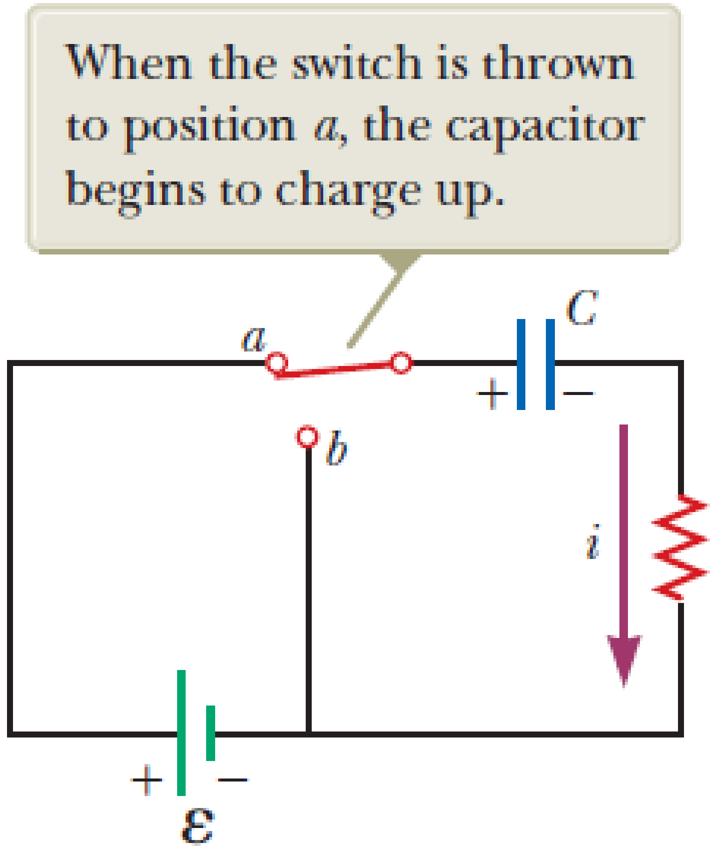
Resistor-Capacitor (RC) Circuits

A circuit containing a series combination of a resistor and a capacitor is called an **RC circuit**.



Let's assume the capacitor in this circuit is initially uncharged. There is no current while the switch is open.

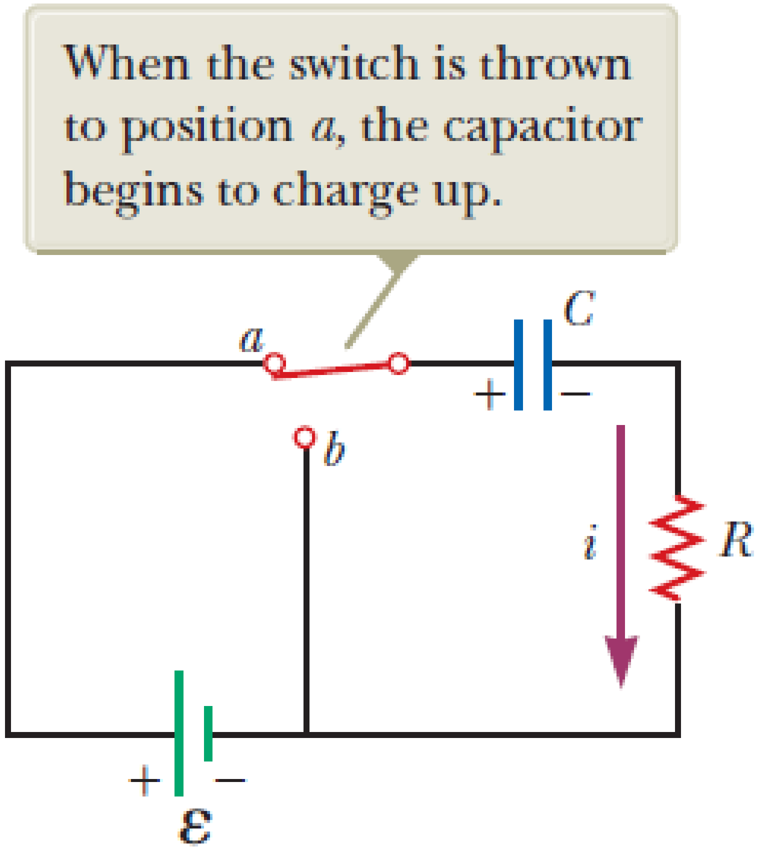
$$i = 0, \quad q = 0$$



If the switch is thrown to position a at t=0 however, **charge begins to flow, setting up a current in the circuit, and the capacitor begins to charge.**

At t=0 $i = I_{max}$

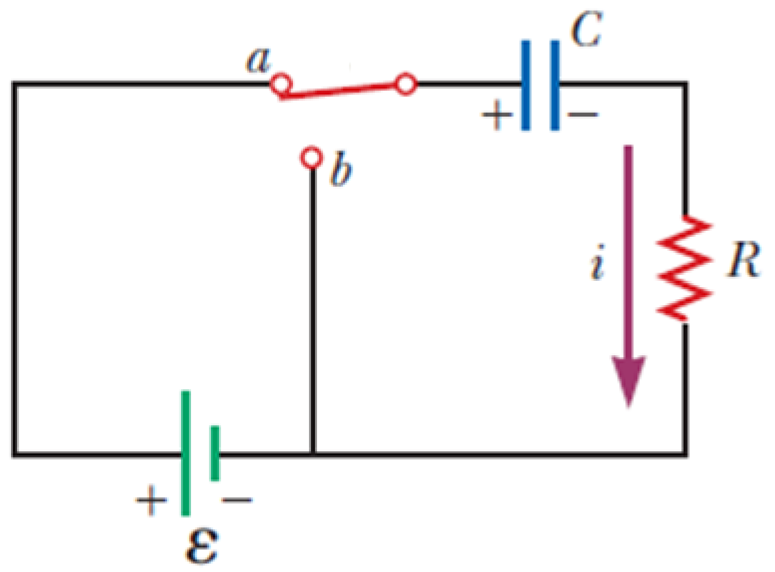
- Notice that during charging, charges do not jump across the capacitor plates because the gap between the plates represents an open circuit.
- Instead, charge is transferred between each plate and its connecting wires due to the electric field established in the wires by the battery until the capacitor is fully charged.
- As the plates are being charged, the potential difference across the capacitor increases.
- The value of the maximum charge on the plates depends on the voltage of the battery. **Once the maximum charge is reached, the current in the circuit is zero because the potential difference across the capacitor matches that supplied by the battery.**



When the capacitor is fully charged $q = Q_{max}$ $i = 0$

During the charging process,

$$q \nearrow \quad \Delta V_c \nearrow \quad \Delta V_R \swarrow \quad i \swarrow$$



When the capacitor is fully charged

$$q = Q_{\max} \quad \Delta V_c = \varepsilon \quad \Delta V_R = 0 \quad i = 0$$

Let's apply Kirchhoff's loop rule to the circuit after the switch is thrown to position *a*

$$\mathcal{E} - \frac{q}{C} - iR = 0$$

When the capacitor is fully charged, $q = Q_{\max}$ and $i = 0$

$$\mathcal{E} - \frac{q}{C} - iR = 0 \longrightarrow Q_{\max} = C\mathcal{E}$$

The capacitor was initially uncharged $q = 0$ $I_{\max} = \dfrac{\varepsilon}{R}$

$$\mathcal{E} - \frac{q}{C} - iR = 0 \longrightarrow I_{\max} = \frac{\varepsilon}{R}$$

During the charging process, there is some amount of charge on the capaitor’s plate and the electric current flows in the circuit.

$$\mathcal{E} - \frac{q}{C} - iR = 0 \qquad \frac{dq}{dt} = \frac{C\mathcal{E}}{RC} - \frac{q}{RC} = -\frac{q - C\mathcal{E}}{RC}$$

$$\frac{dq}{dt} = \frac{\mathcal{E}}{R} - \frac{q}{RC} \qquad \frac{dq}{q - C\mathcal{E}} = -\frac{1}{RC}dt$$

$$\frac{dq}{q - C\mathcal{E}} = -\frac{1}{RC}dt$$

$$\int_0^q \frac{dq}{q - C\mathcal{E}} = -\frac{1}{RC}\int_0^t dt$$

$$u = q - C\varepsilon$$

$$du = dq$$

$$q = 0 \rightarrow u = -C\varepsilon$$

$$q = q \rightarrow u = q - C\varepsilon$$

$$\int_{-C\varepsilon}^{q-C\varepsilon} \frac{du}{u} = -\frac{t}{RC}$$

$$\ln u\Big|_{-C\varepsilon}^{q-C\varepsilon} = -\frac{t}{RC}$$

The variation of the charge on the capacitor during the charging process

$$\ln\left(\frac{q - C\mathcal{E}}{-C\mathcal{E}}\right) = -\frac{t}{RC}$$

$$q(t) = C\mathcal{E}(1 - e^{-t/RC}) = Q_{\max}(1 - e^{-t/RC})$$

e is the base of the natural logarithm

$$i=\frac{dq}{dt}=\frac{d}{dt}\left(Q_{max}\left(1-e^{-t/RC}\right)\right)$$

$$i=\frac{Q_{max}}{RC}e^{-t/RC}$$

$$i=\frac{C\varepsilon}{RC}e^{-t/RC}$$

$$i(t)=\frac{\mathcal{E}}{R}e^{-t/RC}$$

$$i(t)=I_{max}e^{-t/RC}$$

Current as a function of time for a capacitor being charged

The quantity RC, which appears in the exponents of Equations, is called the **time constant** τ of the circuit.

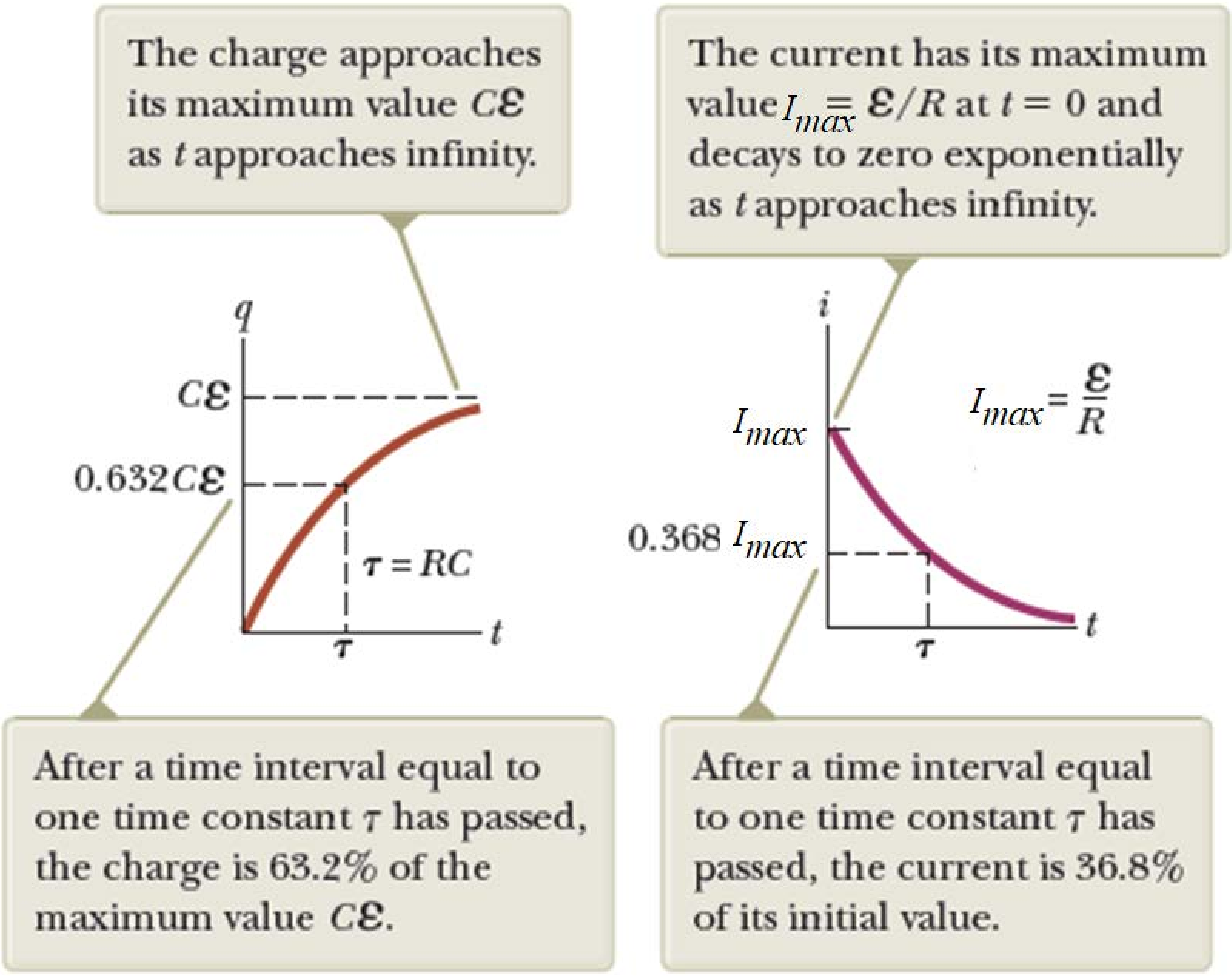
$$q(t)=Q_{max}\left(1-e^{-t/RC}\right)$$

$$q(t)=Q_{max}\left(1-e^{-t/\tau}\right)$$

$$i(t)=I_{max}e^{-t/\tau}$$

Let's prove that τ is the dimension of time:

$$\tau=RC=\frac{\Delta V}{I}\frac{Q}{\Delta V}=\frac{\Delta V}{\frac{Q}{t}}\frac{Q}{\Delta V}=t$$



When t=τ $\quad q(t) = Q_{\max}\left(1 - e^{-t/\tau}\right)$

$$q = Q_{\max}\left(1 - \frac{1}{e}\right)$$

$$q = 0.632Q_{\max}$$

When t $\to \infty$ $\quad q(t) = Q_{\max}\left(1 - e^{-t/\tau}\right)$

$$q \to Q_{\max}$$

When t=τ $\quad i(t) = I_{\max}e^{-t/\tau} = \dfrac{I_{\max}}{e}$

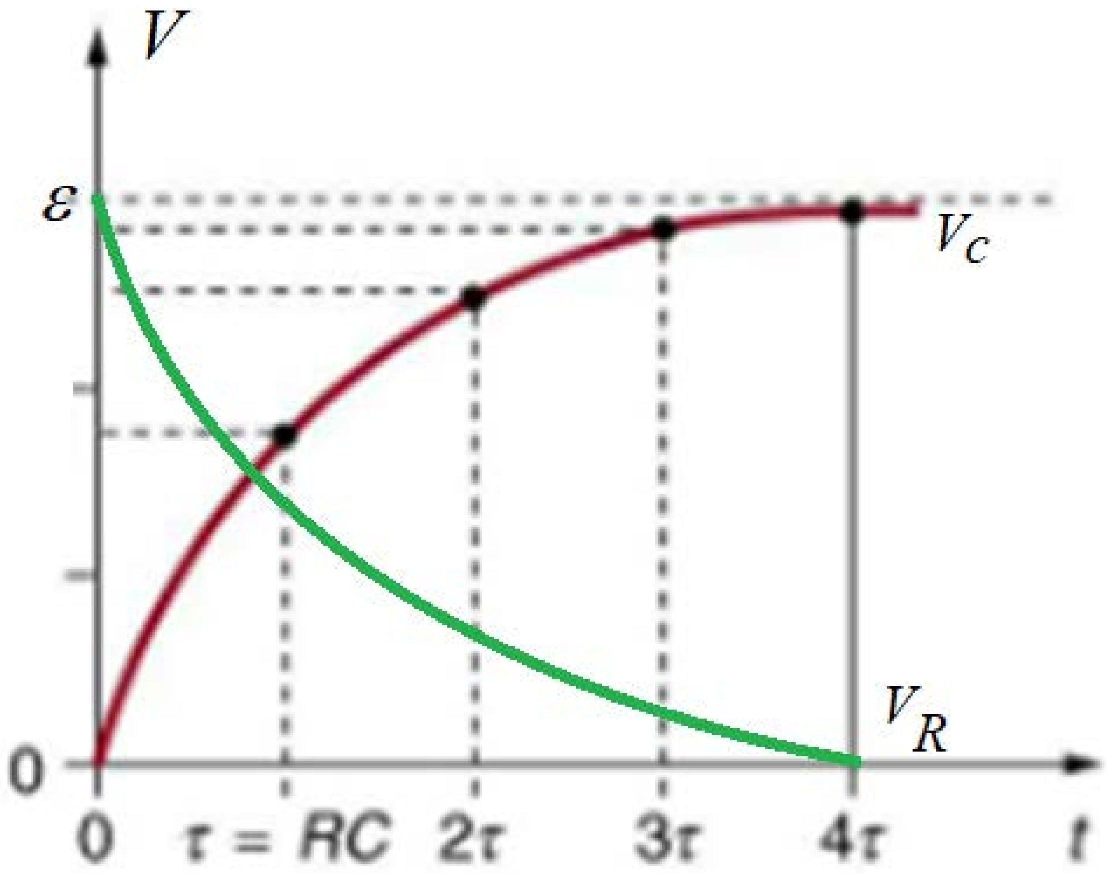
$$i = 0.368I_{\max}$$

When t $\to \infty$ $\quad i(t) = I_{\max}e^{-t/\tau}$

$$i \to 0$$

During the charging process, $\Delta V_c \nearrow \quad \Delta V_R \swarrow$

For every moment during the charging process, $$\Delta V_c + \Delta V_R = \varepsilon$$



Charging Process From Energetic Point of View

The energy supplied by the battery during the time interval required to fully charge the capacitor is

$$U = q\Delta V = Q_{\max}\varepsilon = C\varepsilon\varepsilon = C\varepsilon^2$$

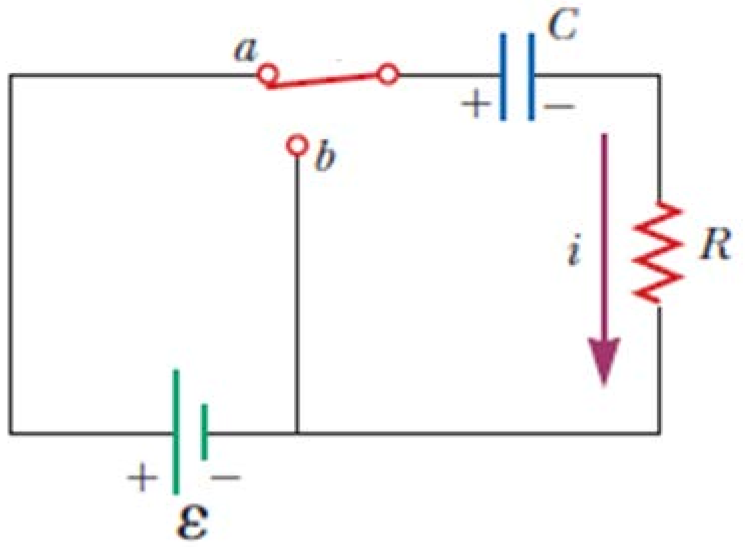
After the capacitor is fully charged, the energy stored in the capacitor is

$$U = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}C\varepsilon^2$$

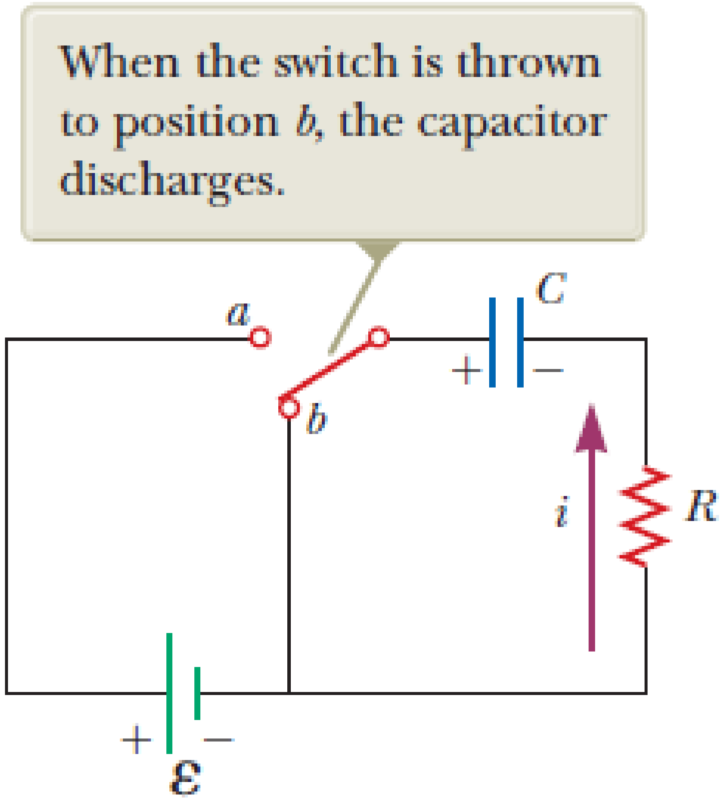
which is only half the energy output of the battery.

The remaining half of the energy supplied by the battery appears as internal energy in the resistor. This energy is consumed across the resistor.

Discharging a Capacitor

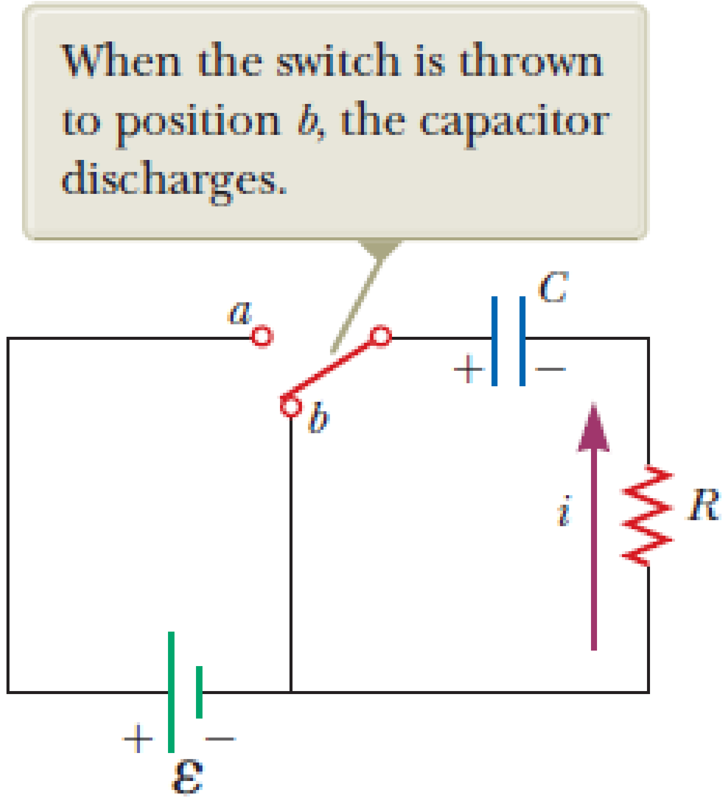


When the circuit is run for this position of the switch for a long time, the capacitor is fully charged.



Then if the switch position is changed to b, you remove the battery and discharging process starts.

At t=0 $i = 0$, $q = Q_{\max}$, $\Delta V_c = \varepsilon = \dfrac{Q_{\max}}{C}$



During the discharging process, current *i* starts to flow in the circuit and the charge on capacitor's plate decreases.

$$q \swarrow \quad \Delta V_c \swarrow \quad i \nearrow$$

According to Kirchhoff's loop rule, during the discharging process:

$$-\frac{q}{C} - iR = 0$$

$$i = \frac{dq}{dt}$$

$$-R\frac{dq}{dt} = \frac{q}{C}$$

$$\frac{dq}{q} = -\frac{1}{RC}dt$$

$$\int_{Q_{max}}^{q} \frac{dq}{q} = -\int_0^t \frac{dt}{RC}$$

$$\ln\left(\frac{q}{Q_{max}}\right) = -\frac{t}{RC}$$

$$q(t) = Q_{max} e^{-t/RC}$$

$$q(t) = Q_{max} e^{-t/\tau}$$

Charge as a function of time for a discharging capacitor

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(Q_{max} e^{-t/RC}\right)$$

$$i = -\frac{Q_{max}}{RC} e^{-t/RC}$$

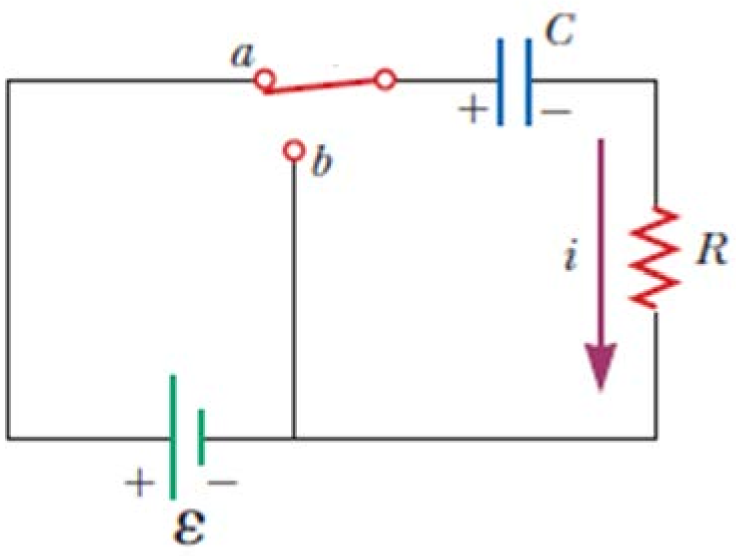
$$i = -\frac{C\varepsilon}{RC} e^{-t/RC}$$

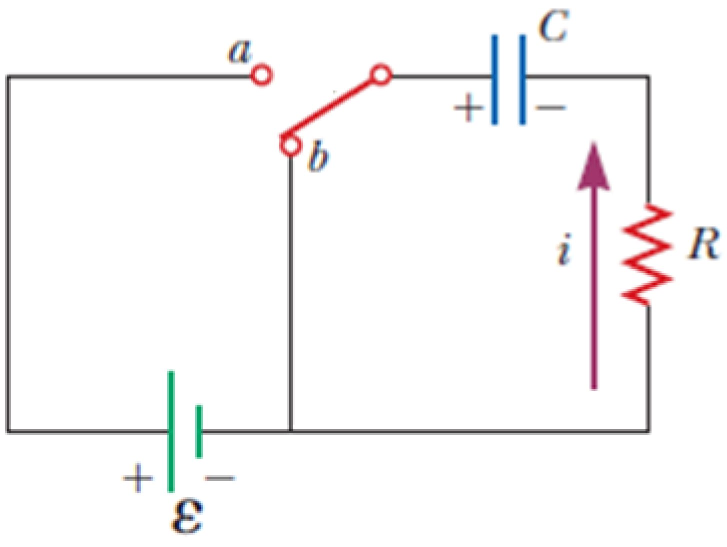
$$i(t) = -\mathrm{I}_{max}\, e^{-t/\tau}$$

Current as a function of time for a capacitor being discharged

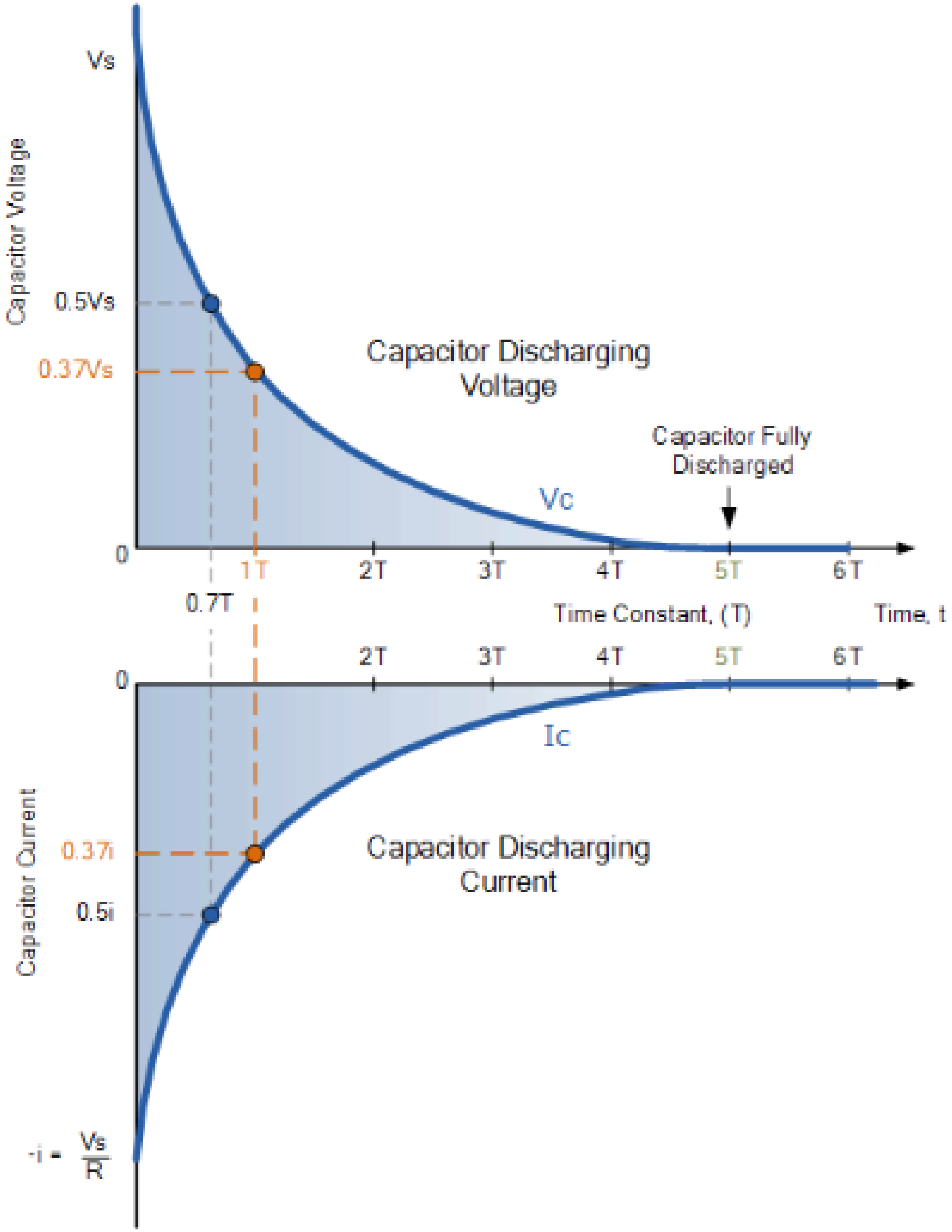
$$i(t) = -\mathrm{I}_{\max}\, e^{-t/\tau}$$

The negative sign indicates that as the capacitor discharges, the current direction is opposite its direction when the capacitor was being charged.





$$V_s = \varepsilon$$

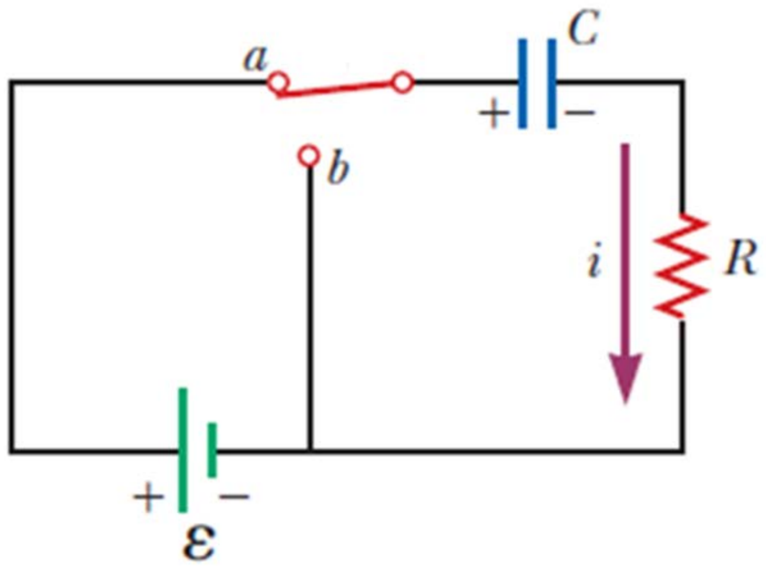


Exercise:

An uncharged capacitor and a resistor are connected in series to a battery as shown in the Figure where ε=12 V, C=5 μF, and R=8x10^5 Ω. The switch is thrown to position a.

Find the time constant of the circuit, the maximum charge on the capacitor, the maximum current in the circuit, and the charge and current as functions of time.

$$\tau = RC = (8.00 \times 10^5\ \Omega)(5.00 \times 10^{-6}\ \mathrm{F}) = \boxed{4.00\ \mathrm{s}}$$



$$Q_{\max} = C\mathcal{E} = (5.00\ \mu\mathrm{F})(12.0\ \mathrm{V}) = \boxed{60.0\ \mu\mathrm{C}}$$

$$I_i = \frac{\mathcal{E}}{R} = \frac{12.0\ \mathrm{V}}{8.00 \times 10^5\ \Omega} = \boxed{15.0\ \mu\mathrm{A}}$$

$$q(t) = \boxed{60.0(1 - e^{-t/4.00})\ \mu\mathrm{C}}$$

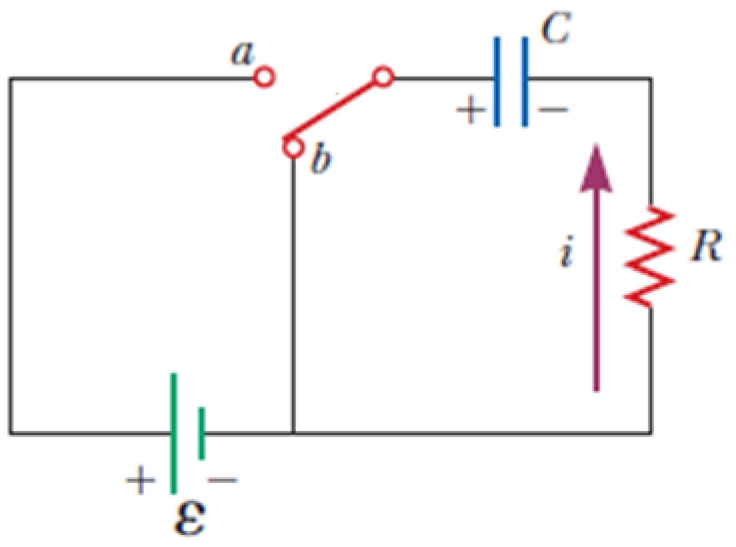
$$i(t) = \boxed{15.0e^{-t/4.00}\ \mu\mathrm{A}}$$

Exercise:

Consider a capacitor of capacitance C that is being discharged through a resistor of resistance R as shown in the Figure.

a) After how many time constants is the charge on the capacitor one-fourth its initial value?

$$q(t) = Q_{\max} e^{-t/RC}$$



$$\frac{Q_{\max}}{4} = Q_{\max} e^{-t/RC}$$

$$\frac{1}{4} = e^{-t/RC}$$

$$\ln\left(\frac{1}{4}\right) = -\frac{t}{RC} \rightarrow t = -RC\ln\left(\frac{1}{4}\right)$$

$$t = RC\ln 4 = 1.39RC = 1.39\tau$$

b) The energy stored in the capacitor decreases with time as the capacitor discharges. After how many time constants is this stored energy one-fourth its initial value?

$$U_i = \frac{q^2}{2C} = \frac{Q_{\max}^2}{2C}$$

$$U_f = \frac{q^2}{2C} = \frac{1}{2C}\left(Q_{\max} e^{-t/RC}\right)^2$$

$$U_f = \frac{Q_{\max}^2}{2C} e^{-2t/RC} = U_i e^{-2t/RC}$$

$$U_f = \frac{U_i}{4} = U_i e^{-2t/RC}$$

$$\frac{1}{4} = e^{-2t/RC}$$

$$\ln\left(\frac{1}{4}\right) = -\frac{2t}{RC}$$

$$t = -\frac{RC}{2}\ln\left(\frac{1}{4}\right) = \frac{RC}{2}\ln 4 = \frac{1.39}{2}RC$$

$$t = 0.693\tau$$

Exercise:

A 5 μF capacitor is charged to a potential difference of 800 V and then discharged through a resistor. How much energy is delivered to the resistor in the time interval required to fully discharge the capacitor?

The energy delivered to the resistor by integrating the power over all time because it takes an infinite time interval for the capacitor to completely discharge.

$$P = \frac{dE}{dt} \rightarrow E_R = \int_0^\infty P\,dt$$

$$E_R = \int_0^\infty i^2 R\,dt$$

$$E_R = \int_0^\infty \left(-\frac{Q_i}{RC} e^{-t/RC}\right)^2 R\,dt = \frac{Q_i^2}{RC^2}\int_0^\infty e^{-2t/RC}\,dt = \frac{\mathcal{E}^2}{R}\int_0^\infty e^{-2t/RC}\,dt$$

$$E_R = \frac{\mathcal{E}^2}{R}\left(\frac{RC}{2}\right) = \tfrac{1}{2}C\mathcal{E}^2 \qquad E_R = \tfrac{1}{2}(5.00 \times 10^{-6}\text{ F})(800\text{ V})^2 = \boxed{1.60\text{ J}}$$