GAUSS'S LAW

In the previous unit, we have focused on calculating the electric field created by either point charges or continuous charge distribution by using Coulomb's method. In this new unit, we'll describe Gauss's law as an alternative method for calculating electric fields that are produced by highly symmetric charge distributions.

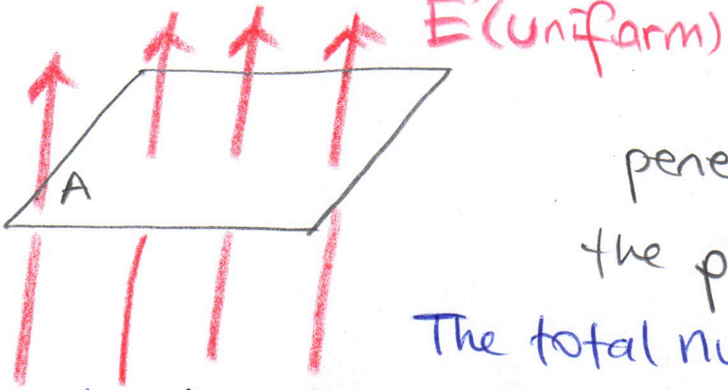
Gauss's law is more convenient and easy way for determining the electric field of highly symmetric charge distributions.

Some examples for highly symmetric charge distributions: Point charges, the charge distribution on the surface of a sphere or along the infinite length wire etc.

Gauss's law give a relation between the charge and electrix flux.

ELECTRIC FLUX

Let's consider a uniform electric field lines which are directed along the +y-axis. Then let's locate a rectangular shaped plate with the surface A which is perpendicular to the electric field lines.



Electric field lines penetrate the surface area of the plate.

The total number of electric field lines penetrating the surface area A is proportional to the product of E and A.

$\Phi_E = E.A$ → is known as electric flux that is proportional to the number of electric field lines penetrating perpendicularly to the surface.

$$\Phi_E = E\,A$$

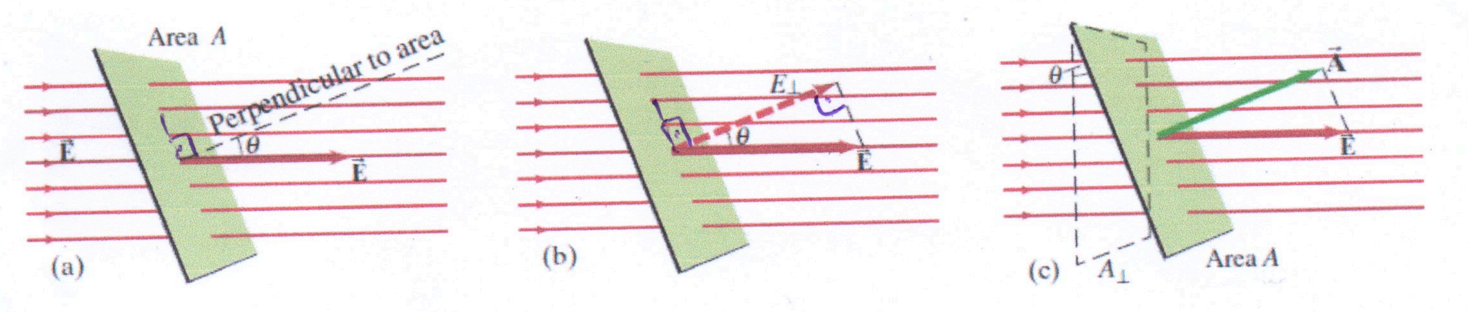
$A \rightarrow (m^2)$

$E \rightarrow (N/C)$

Electric flux is denoted by the upper case Greek letter Φ.

(Nm^2/C)

If the electric field lines do not penetrate the surface perpendicularly, how can we write electric flux?



We should find the electric field vector component which is perpendicular to the surface area A!

$E_{\perp} = E\cos\theta$ component is penetrating the surface are perpendicularly.

$\vec{A}$ = the surface area vector which is perpendicular to the surface. So $\vec{A}$ is parallel to $\vec{E}_{\perp}$

$$\underbrace{\Phi_f}_{\text{Scalar quantity}} = \vec{E} \underbrace{\cdot}_{\text{scalar product}} \vec{A} = EA\cos\theta$$

the angle between the electric field lines and surface area vector, $\vec{A}$

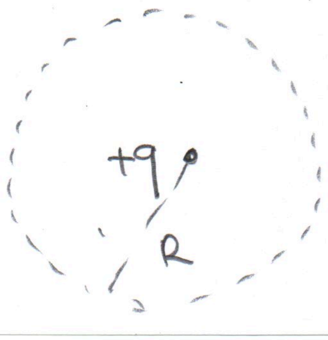
$\vec{A}$ is always perpendicular to the surface area. $\vec{A}$ is also always directed outward from the closed surface!

$\Phi_f = \vec{E} \cdot \vec{A}$ formula is valid for uniform electric field lines

$\Phi_f = \int \vec{E} \cdot \underbrace{d\vec{A}}$ formula is valid for non-uniform electric field

differential surface area element vector that is perpendicular to the surface

Exercise | A positive charge of $q = 3\mu C$ is located at the center of a sphere with radius of $0{,}2m$. Find the electric flux through the surface of the sphere.



$q = 3{,}10^{-6}C \quad R = 0{,}2m$

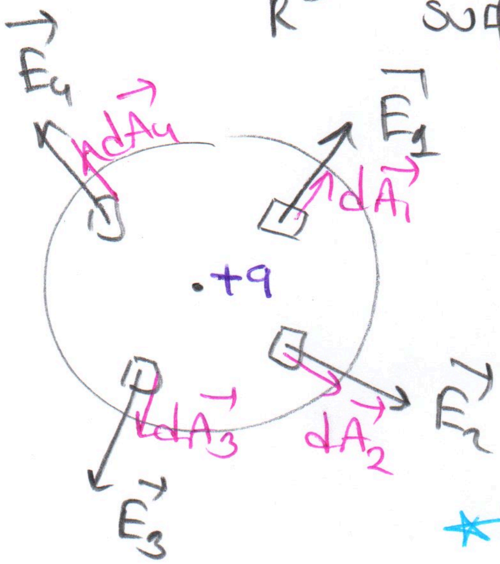
Let's define the type of electric field.

Electric field lines on the surface of

the sphere is not uniform! why?

Every point on the sphere surface has the same distance the point charge, q.

$E = k_e \frac{q}{R^2}$ (the magnitude of the electric field on the surface of the sphere)



+q creates an electric field which is radially directed outword. So every point on the surface has different electric field vector direction with the same magnitude.

* Due to this reason the electric field vector is not uniform on the surface.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E \, dA \cdot \cos 0$$

integral over a closed surface, A

There is always zero degree between $d\vec{A}$ and $\vec{E}$ vectors!

$$\Phi_E = \oint E \, dA = E \boxed{\oint dA}$$

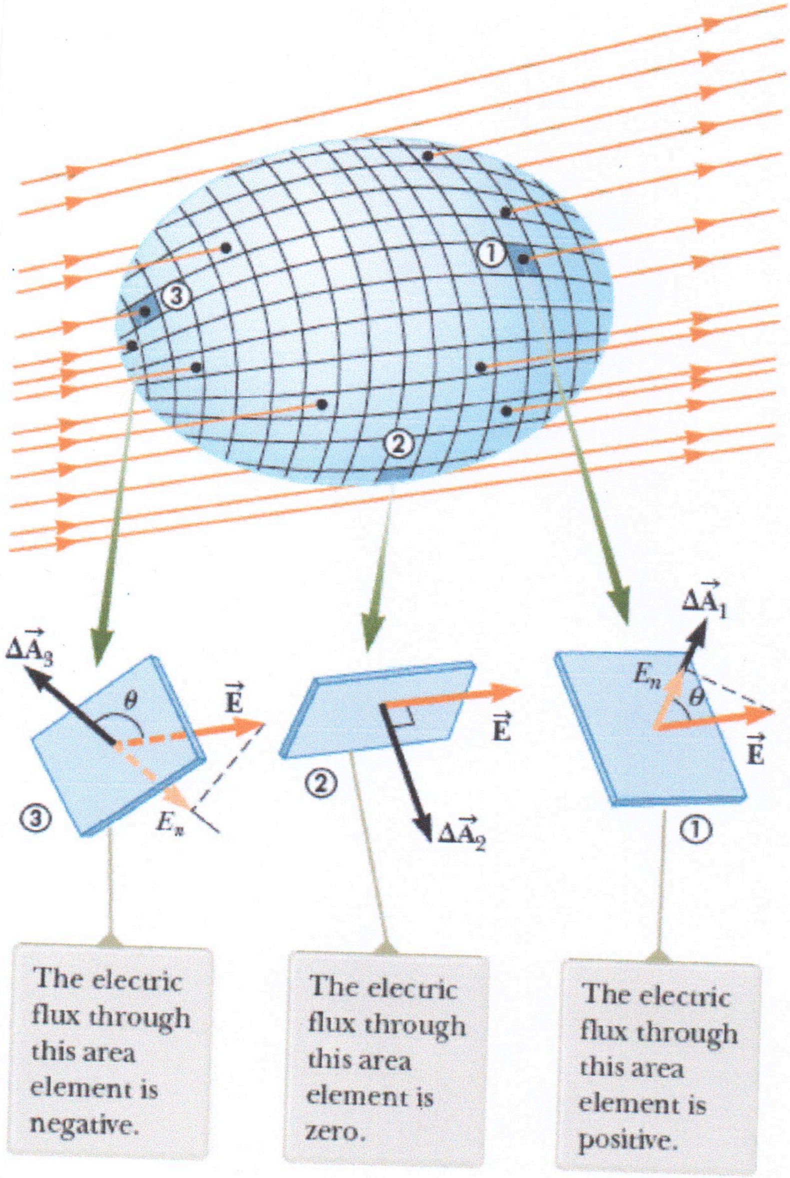
the surface area of the sphere

the magnitude of the electric field is constant.

$$\Phi_E = E \cdot 4\pi R^2 = \boxed{k_e} \frac{q}{\cancel{R^2}} 4\pi \cancel{R^2} = \frac{1}{\cancel{4\pi}\varepsilon_0} q \cancel{4\pi}$$

electric field of point charge

$$\Phi_E = \frac{q}{\varepsilon_0}$$



let's consider the closed ⑤ surface in the uniform electric field, E as shown in the figure.

let's focus on some small surface area elements ①, ②, and ③ on the surface.

① $\theta < 90°$ $\Phi_E > 0$

② $\theta = 90°$ $\vec{E} \perp \Delta\vec{A}_2$

$\Phi_E = 0$

③ $\theta > 90°$ $\cos\theta < 0$

$\Phi_E < 0$

The net flux through the surface is proportional to the net number of lines leaving the surface

$\Downarrow$

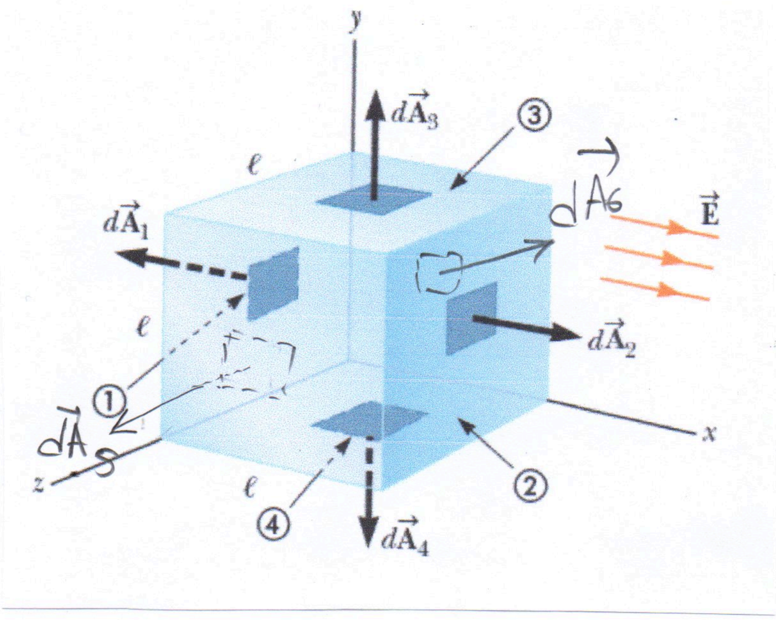
= the number of lines leaving surface − the number of lines entering the surface

If more lines are leaving than entering, $\Phi_{E_{net}} > 0$

If more lines are entering than leaving, $\Phi_{E,net} < 0$

If leaving and entering lines are the same, $\Phi_{E,net} = 0$

Exercise Consider a uniform electric field $\vec{E}$ oriented in the x-direction in empty space. A cube of edge length l is placed in the field, oriented as shown in the figure. Find the net electric flux through the surface area of the cube.



The cube has six surfaces and six of them are square. Let's calculate each surfaces electric flux separately and then obtain the net electric flux of the cube.

$$\Phi_{E,net} = \sum_{i=1}^{6} \Phi_{E,i} \qquad \vec{E} = E\hat{i}$$

$$\Phi_E = \int \vec{E} \cdot d\vec{A} = \int E\,dA\cos\theta \qquad \text{If } \theta = 90^\circ \quad \Phi_E = 0$$

For ③, ④, ⑤ and ⑥ surfaces $\Phi_E = 0$ $(\vec{E} \perp d\vec{A})$

$$\Phi_{E,net} = \Phi_{E,1} + \Phi_{E,2}$$

① st surface is placed in the yz-plane $d\vec{A} = -dA\hat{i}$

$d\vec{A}$ is perpendicular to the surface area and directed outward from the closed surface.

$$\Phi_{E,1} = \int \vec{E} \cdot d\vec{A}_1$$

$$\Phi_{E,1} = \int E\hat{i} \cdot (-dA\hat{i})$$

1st square surface area

$$\Phi_{E,1} = -\int E\,dA = -E\boxed{\int_1 dA} = -E\,l^2$$

$$\Phi_{E,2} = \int \vec{E} \cdot d\vec{A}_2 \qquad d\vec{A}_2 = dA\,\hat{i}$$

$$\Phi_{E,2} = \int_2 E\hat{i} \cdot dA\hat{i} = \int_2 E\,dA = E\int_2 dA = E\ell^2$$

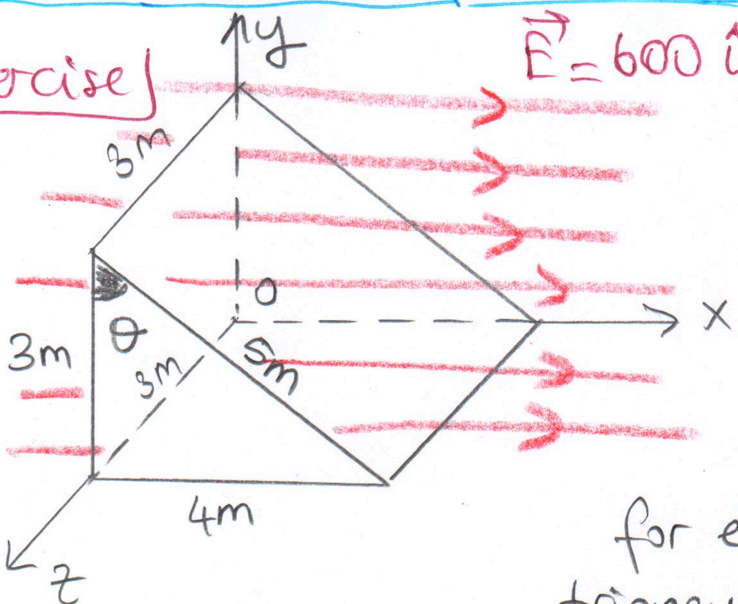
$$\Phi_{E,net} = -E\ell^2 + E\ell^2 = 0$$

We obtained a very important result from the exercise:

* When any closed surface is located in a uniform electric field, the net flux through the closed surface that surrounds no charge is zero.

** When a charged particle is placed in a closed surface or when a closed surfaced is located in a non-uniform electric field, the net flux through the closed surface cannot be zero.

Exercise



A triangular prism is located in a uniform electric field of $\vec{E} = 600\,\hat{i}$ (N/C)

a) find the electric flux for each five surfaces of the triangular prism.

For the front and back triangular surfaces in the xy-plane Φ_E must be zero!

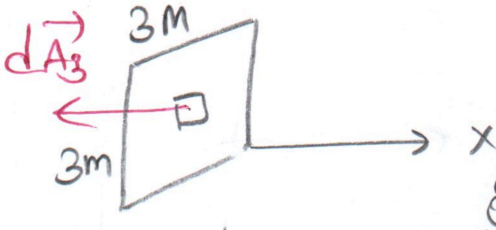
For the front triangle $d\vec{A_1} = dA\hat{k}$ and for the back triangle $d\vec{A_2} = -dA\hat{k}$

$\vec{E} = 600\hat{i}\,(N/C)$

$\vec{E} \perp d\vec{A_1}$ and $\vec{E} \perp d\vec{A_2}$

$\Phi_{E_1} = \Phi_{E_2} = 0$

(3)rd surface is $3m \times 3m$ square surface located in yz-plane

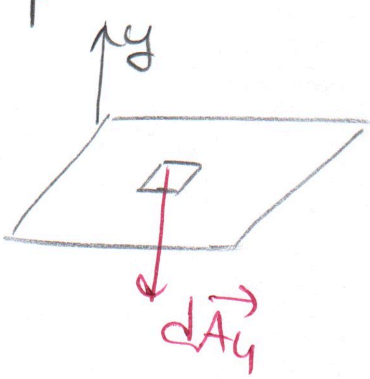


$d\vec{A_3} = -dA_3\hat{i}$

$\Phi_{E_3} = \int \vec{E}\cdot d\vec{A_3} = \int 600\hat{i}\cdot(-dA_3\hat{i})$

$\Phi_{E_3} = -600\boxed{\int dA_3} = -600\,(3\times3) = -5400\,(Nm^2/C)$

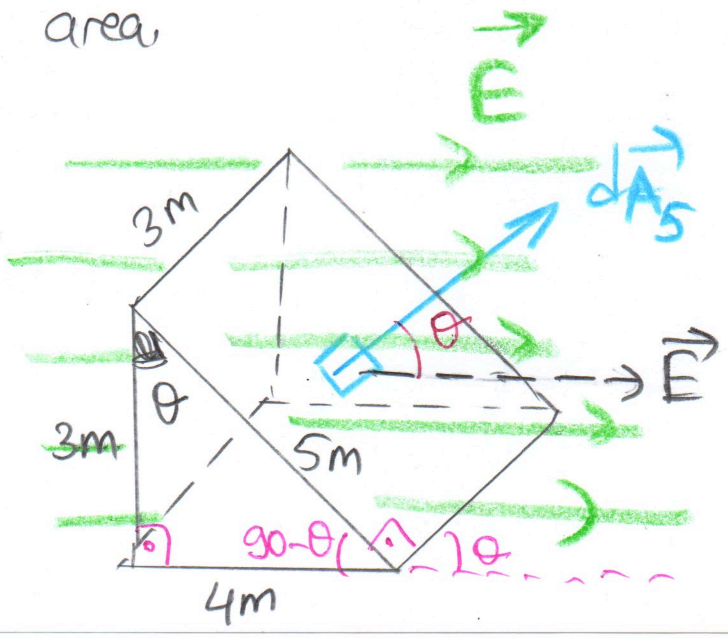
(4)rd surface is $3m \times 4m$ rectangular surface located in xz-plane



$d\vec{A_4} = -dA_4\hat{j}$

$\vec{E} \perp d\vec{A_4} \rightarrow \Phi_{E,4} = 0$

5^{th} surface is the rectangular surface with $3\times5\,m^2$ surface area.



$\Phi_{E,5} = \int \vec{E}\cdot d\vec{A_5}$

$= \int E\,dA_5 . \cos\theta$

$= E\int dA_5 . \frac{3}{5}$

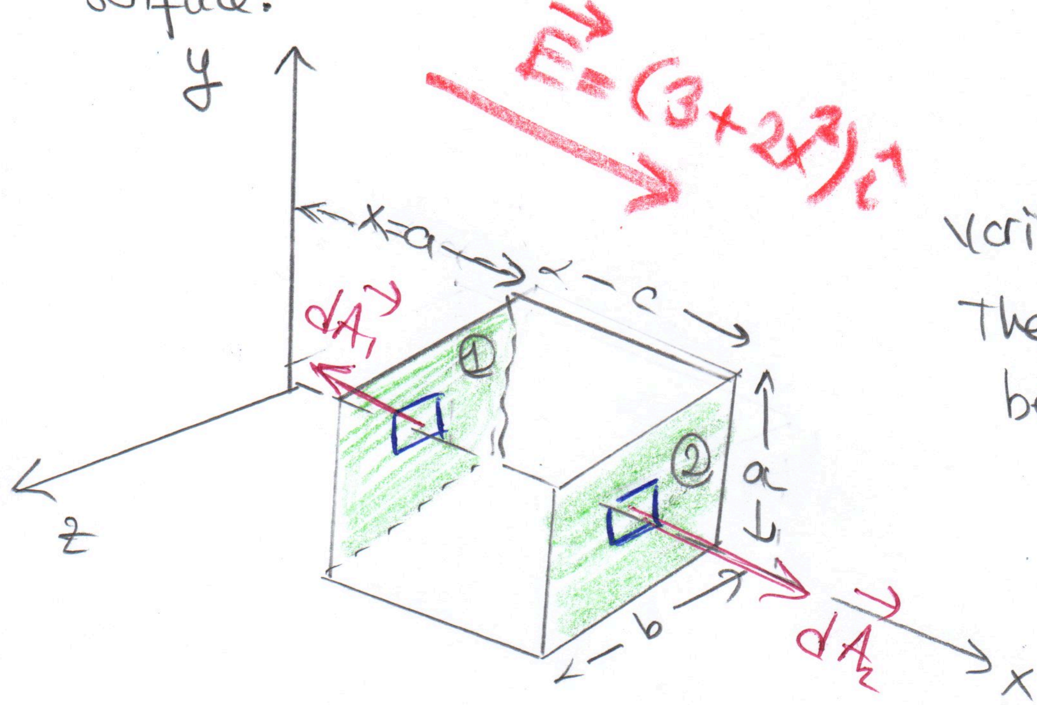
$= 600 . \frac{3}{5}\boxed{\int dA_5} \rightarrow 3\times5\,M^2$

$\cos\theta = \frac{3}{5}$ | $\Phi_{E,5} = 5400\,(Nm^2/C)$

$$\Sigma \Phi_E = 5400 - 5400 = 0$$

Exercise A closed surface with dimensions of $a = 0{,}2\,m$, $b = 0{,}3\,m$, and $c = 0{,}3\,m$ is located in the electric field of $\vec{E} = (3 + 2x^2)\hat{i}$ (N/C) where x is in meters.

Calculate the net electric flux leaving the closed surface.



The electric field is not uniform, it varies with x-coordinate. The electric flux must be calculated by using integral formula !

Nonzero electric flux is observed only for the 1st and 2nd surfaces.

1st surface is located at $x = a$ in the yz-plane.

$$d\vec{A}_1 = -dA_1\hat{i}$$

$$\Phi_{E,1} = \int \vec{E}_1 \cdot d\vec{A}_1 = \int (3 + 2a^2)\,\hat{i} \cdot (-dA_1\hat{i}) = -(3 + 2a^2)\boxed{\int dA_1}$$

The electric field for the 1st surface $(x = a)$

$\vec{E}_1 = (3 + 2a^2)\hat{i}$ = constant electric field vector

$A_1 = ab$

$$\Phi_{E,1} = -(3+2a^2)ab$$

2^{nd} surface is located at $x = a+c$ in the yz-plane.

$$d\vec{A_2} = dA_2\,\hat{i}$$

$$\Phi_{E,2} = \int \vec{E_2} \cdot d\vec{A_2}$$

$$\vec{E_2} = [3+2(a+c)^2]\,\hat{i}$$

$$\vec{E_2} = [3+2(a^2+2ac+c^2)]\,\hat{i} = \text{constant!}$$

$$\Phi_{E,2} = \int [2a^2+4ac+2c^2+3]\,\hat{i} \cdot dA_2\,\hat{i}$$

$$= (2a^2+4ac+2c^2+3)\boxed{\int dA_2} \rightarrow ab$$

$$\Phi_{E,2} = (2a^2+4ac+2c^2+3)ab$$

$$\Sigma\Phi_E = -\cancel{(3+2a^2)ab} + \cancel{(2a^2+3)ab} + (4ac+2c^2)ab$$

$$\Sigma\Phi_E = 2abc(2a+c)$$

$$\Sigma\Phi_E = 2\times 0{,}2\times 0{,}3\times 0{,}3\,[(2\times 0{,}2)+0{,}3] = \underline{0{,}2688\ (Nm^2/C)}$$

GAUSS'S LAW

Gauss's law was suggested by German mathematician and astronomer Karl Friedrich Gauss (1777-1855) in the 18th century. It relates electric charge and electric field, and is a more general and elegant form of Coulomb's law.

The net electric flux through any closed surface is given by q_{in}/ε_0 where q_{in} equals the net charge inside the closed surface known as Gaussian surface

$$\Phi_E = \oint_{\substack{\text{closed}\\ \text{surface}}} \boxed{\vec{E}} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$ Gauss's law

→ the electric field at any point on the Gaussian surface.

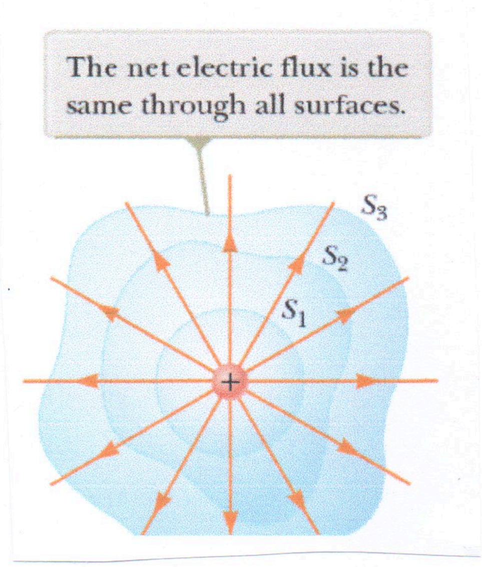
The electric field created by highly symmetric charge distributions can be calculated easily by using Gauss's law.

The way to follow when Gauss's law is applied:

1. First, a closed Gaussian surface is drawn in accordiance with the symmetry of the charge distribution.
2. The point where the electric field will be calculated must coincide with the chosen Gaussian surface
3. Then the electric field vector $\vec{E}$ and the surface area vector $d\vec{A}$ are drawn and the electric flux is written equal to $\frac{q_{in}}{\varepsilon_0}$. Hence the electric field is calculated.

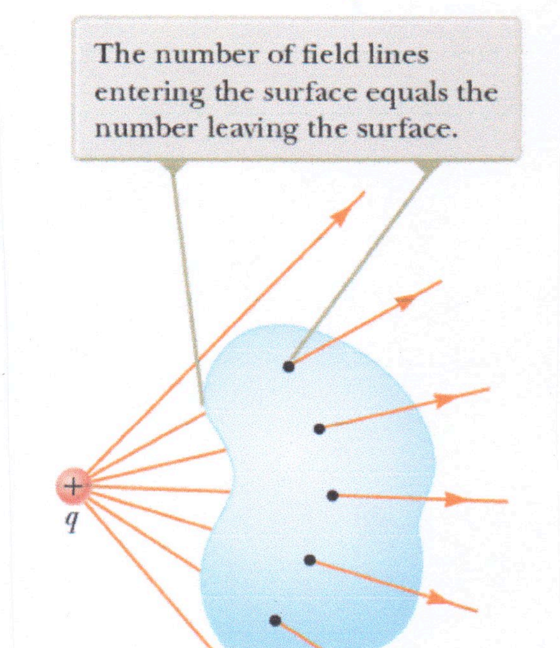
On page 6, we have calculated the electric flux of a $+q$ point charge through a spherical surface.

$$\Phi_E = \frac{q}{\varepsilon_0} \rightarrow \text{for spherical Gaussian surface.}$$



For the S_2 and S_3 non-spherical surfaces, the electric flux does not change. Because the electric flux is proportional to the number of electric field lines passing through a surface. As shown in the figure, the number of lines passing through S_1 surface is equal to the number of lines passing through non-spherical S_2 and S_3.

* The net flux through any closed surface surrounding a point charge q is given by q/ε_0 and it is independent of the shape of that surface.



At this time, the point charge is outside the closed surface. So the numbers of electric field lines entering and leaving the closed surface are the same. $\Phi_E = 0$

$$\oint \vec{E} \cdot d\vec{A} = \Phi_E = \frac{q_{in}}{\varepsilon_0} = 0$$

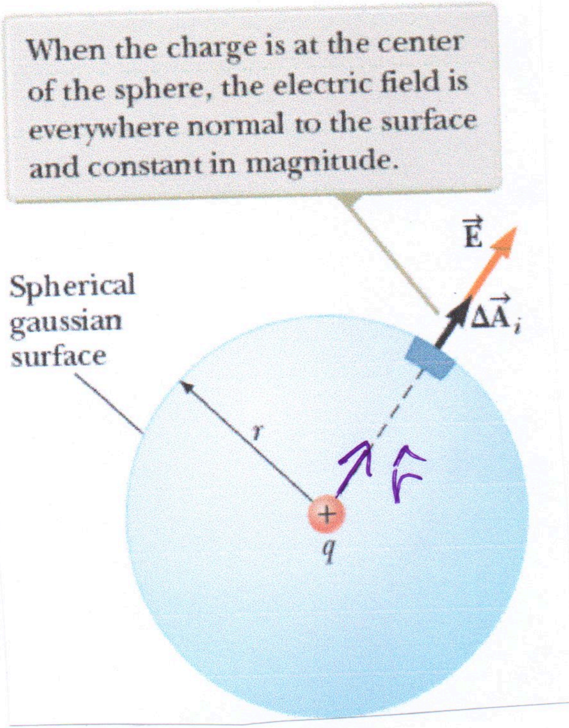
Φ_E through $S_1 = \frac{q_1}{\varepsilon_0}$

Φ_E through $S_2 = \frac{(q_2+q_3)}{\varepsilon_0}$

Φ_E through $S_3 = 0$

q_4 is outside all surfaces!

Exercise Find the electric field created by an isolated point charge of q at r distance far from the charge by using Gauss's law. (13)



Spherical Gaussian surface is suitable closed surface that surrounds the point charge of q.

You should choose a Gaussian surface such that the magnitude of the electric field at each point on the surface must be the same. [Gaussian surface is an imaginary surface!]

The electric field's magnitude is constant at every point on the Gaussian surface, because there is a constant r distance between the charge and the point on the surface

For every point on the sphere $\vec{E} \parallel d\vec{A}$ $\theta = 0°$

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$\oint E\,dA\cos\theta = \frac{q_{in}}{\varepsilon_0}$$

$$\oint E\,dA = \frac{q}{\varepsilon_0}$$

$$E \oint dA = \frac{q}{\varepsilon_0}$$

$$E\,4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

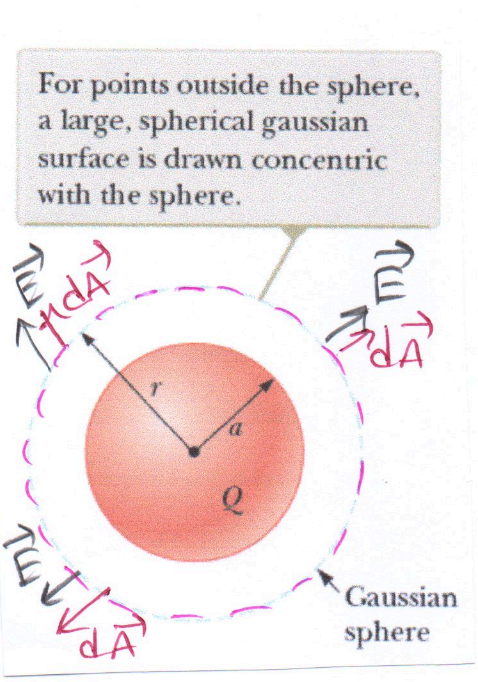
$$\vec{E} = k_e \frac{q}{r^2}\hat{r}$$

$\hat{r}$: The unit vector which is radially directed outward.

Exercise | A spherically symmetric charge distribution

An insulating solid sphere of radius a has a uniform volume charge density, ρ and carries a total positive charge Q.

a) Calculate the magnitude of the electric field at a point outside the sphere.



$r > a$ $\quad \oint \vec{E}_1 \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$

$\vec{E} \parallel d\vec{A}$ at each point on the Gaussian surface

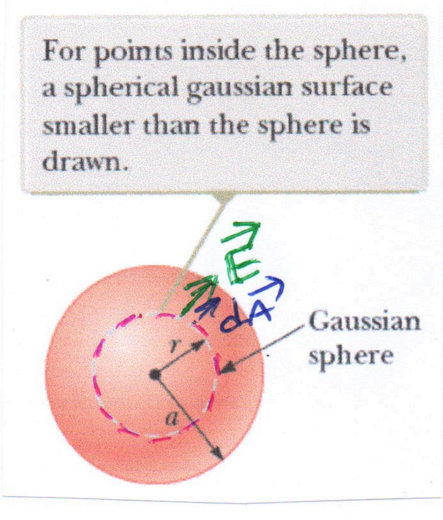
$\oint E_1 \, dA \cos 0 = \frac{q_{in}}{\varepsilon_0}$ → How much charge is trapped in the Gaussian surface?

$E_1 \oint dA = \frac{\boxed{Q}}{\varepsilon_0}$

$E_1 \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0}$

$E_1 = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} = k_e \frac{Q}{r^2}$, $r > a$

b) find the magnitude of the electric field at a point inside the sphere.



$r < a$ $\quad \oint \vec{E}_2 \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$

$\oint E_2 \, dA \cos 0 = \frac{q_{in}}{\varepsilon_0}$

$E_2 \oint dA = \frac{q_{in}}{\varepsilon_0}$

$E_2 \cdot 4\pi r^2 = \frac{q_{in}}{\varepsilon_0}$ the net charge inside the Gaussian surface must be less than total charge of Q.

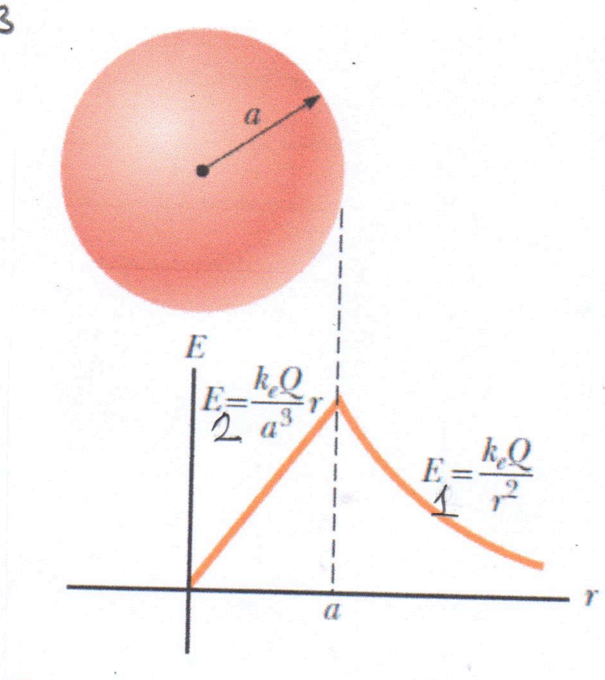
Since the charge distribution is uniform, q_{in} can be determined by constructing a direct proportion. 15

The total charge Q is distributed over $\frac{4}{3}\pi a^3$ volume

Charge of q_{in} " " " $\frac{4}{3}\pi r^3$ "

$$Q \cdot \cancel{\frac{4}{3}\pi} r^3 = q_{in} \cancel{\frac{4}{3}\pi} a^3$$

$$q_{in} = Q \frac{r^3}{a^3}$$



$$\oint \vec{E_2} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$E_2 4\pi \cancel{r^2} = \frac{Q}{\varepsilon_0} \frac{r^{\cancel{3}}}{a^3}$$

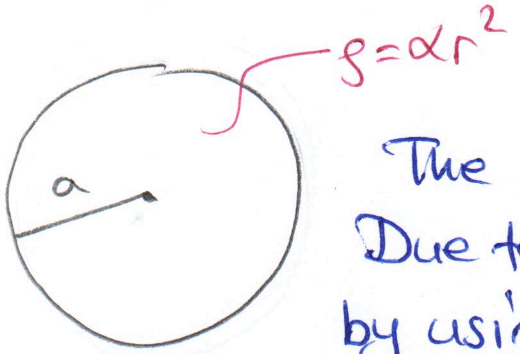
$$\boxed{E_2 = k_e Q \frac{r}{a^3} \quad , \quad r < a}$$

$r = a$ is the critical distance, $E_1 = E_2$ at $r = a$

$$E_1\Big|_{r=a} = k_e \frac{Q}{a^2} \qquad E_2\Big|_{r=a} = k_e Q \frac{a}{a^3} = k_e \frac{Q}{a^2}$$

Exercise A solid insulating sphere of radius a has a non-uniform charge density of $\rho = \alpha r^2$ where α is a positive constant and r is the radial distance from the center of the sphere.

a) Find the α constant in terms of the total charge of the sphere, Q.



$$Q = \int dq$$

The charge distribution is non-uniform. Due to this reason, the charge is calculated by using integral formula not direct proportion.

$$Q = \int dq = \int \rho \, dV$$

$$V_{sphere} = \frac{4}{3}\pi r^3$$

$$dV = \frac{4}{\cancel{3}}\pi \cancel{3} r^2 dr$$

$$\boxed{dV} = \boxed{4\pi r^2}\, dr \rightarrow \boxed{\text{Thickness}}$$

$4\pi r^2$: surface area of the sphere with radius r

dV: the volume of the spherical shell with thickness, dr

$$Q = \int \alpha r^2 4\pi r^2 dr$$

$$Q = 4\pi\alpha \int_{r=0}^{r=a} r^4 dr$$

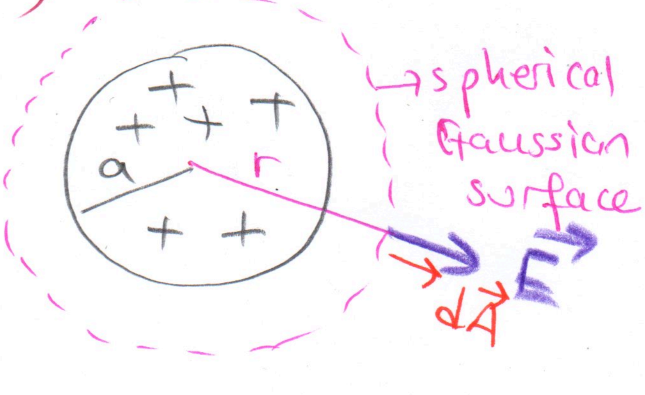
The total charge inside the sphere with radius a.

$$Q = 4\pi\alpha \left. \frac{r^5}{5} \right|_0^a$$

$$Q = 4\pi\alpha \frac{a^5}{5}$$

$$\boxed{\alpha = \frac{5Q}{4\pi a^5}}$$

b) find the electric field outside the sphere



$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}, \quad r > a$$

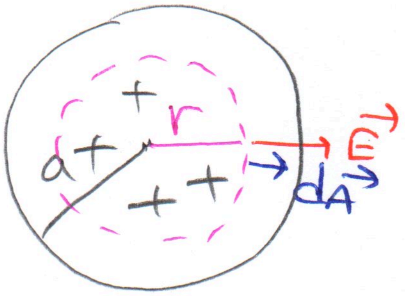
$$\oint E \, dA \cos 0 = \frac{q_{in}}{\varepsilon_0}$$

$$E \oint dA = \frac{Q}{\varepsilon_0}$$

$$E\,4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$E = k_e \frac{Q}{r^2}\ ,\ r > a$$

c) find the electric field inside the sphere.



$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}\ ,\quad r < a$$

$$\vec{E} \parallel d\vec{A}$$

$$\oint E\, dA \cos 0 = \frac{\boxed{q_{in}}}{\varepsilon_0}$$

→ the net charge inside the Gaussian sphere with radius r

$$q_{in} = \int \rho\, dV$$

$$q_{in} = \int_{r=0}^{r=r} \alpha r^2 4\pi r^2 dr$$

$$q_{in} = 4\pi\alpha \int_0^r r^4 dr$$

$$q_{in} = 4\pi \boxed{\alpha} \frac{r^5}{5} = \cancel{4\pi} \frac{\cancel{5}Q}{\cancel{4\pi} a^5} \frac{r^5}{\cancel{5}} =$$

$$q_{in} = Q \frac{r^5}{a^5}$$

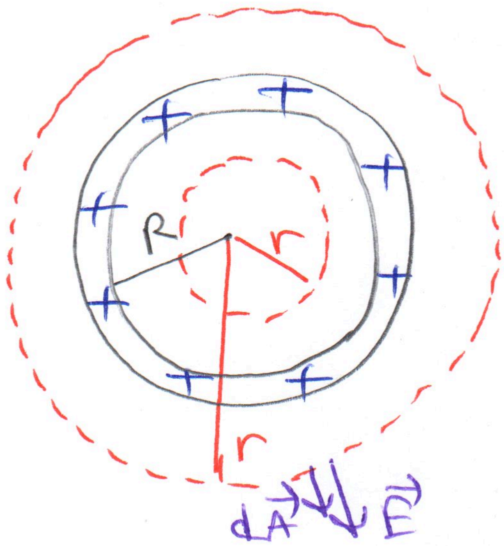
$$E \oint dA = \frac{q_{in}}{\varepsilon_0}$$

$$E\,4\pi r^2 = \frac{1}{\varepsilon_0} Q \frac{r^5}{a^5}$$

$$E = \frac{1}{4\pi\varepsilon_0} Q \frac{r^3}{a^5}$$

$$E = k_e Q \frac{r^3}{a^5}\ ,\ r < a$$

Exercise: A thin spherical shell of radius R has a total positive charge Q distributed uniformly over its surface. Find the electric field at points inside and outside the shell.



Inside the spherical shell, the Gaussian sphere with radius r (r<R) is selected:

$$\oint \vec{E_1}\cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0} \qquad q_{in}=0$$

$$E_1 = 0 \;, \quad r<R$$

Outside the spherical shell, the Gaussian sphere with radius r (r>R) is selected:

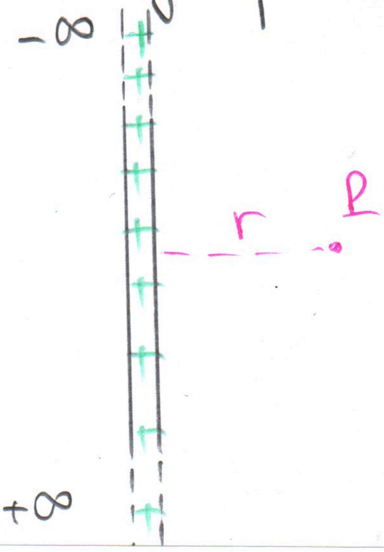
$$\oint \vec{E_2}\cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0} \qquad \vec{E} \parallel d\vec{A}$$

$$\oint E_2\, dA\cos 0 = \frac{\boxed{q_{in}}}{\varepsilon_0}$$ → total charge of the spherical shell is inside the Gaussian surface

$$E_2 \oint dA = \frac{Q}{\varepsilon_0}$$

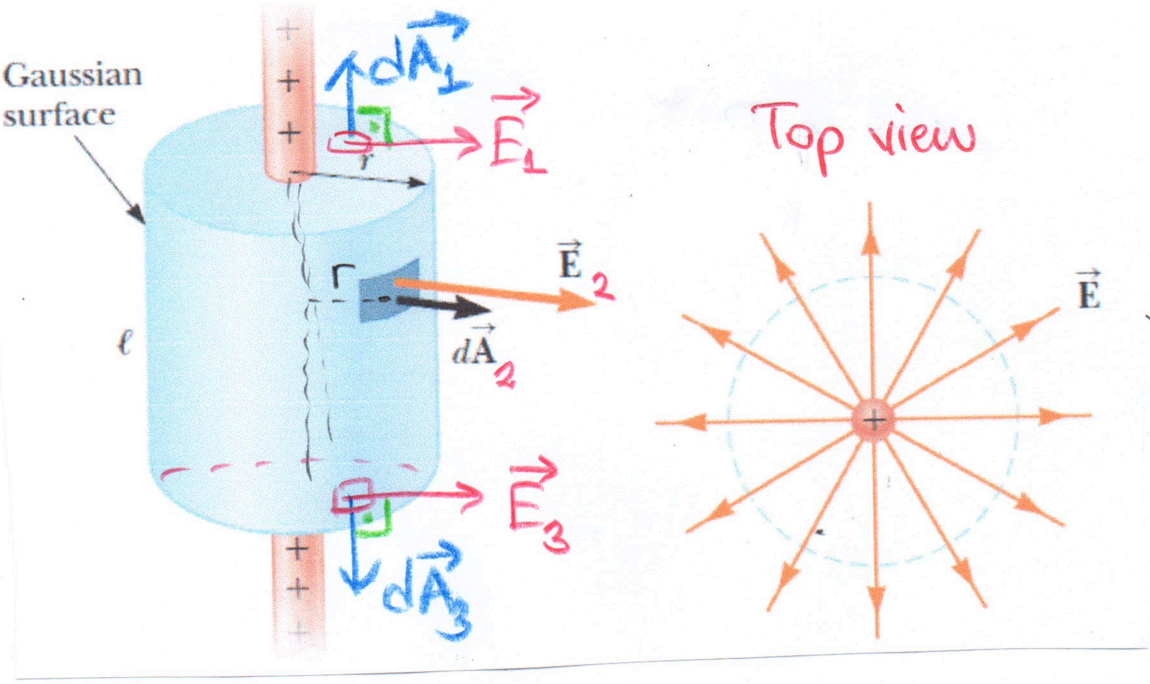
$$E_2 \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E = k_e \frac{Q}{r^2} \;, \quad r>R$$

Exercise) Find the electric field a distance r from a line of positive charge of infinite length and constant charge per unit length, λ.



First step: Decide to the suitable geometry for the Gaussian surface (GS)

Second step: P point must be on the GS and every point on the GS must have the same magnitude of electric field.



The most suitable GS that surrounds the linear charge density is cylindrical G.S. with radius r and length ℓ.

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

closed surface

The closed cylindrical GS consists of three surfaces. The two flat ends of the cylinder and the curved lateral surface

$$\oint \vec{E}\cdot d\vec{A} = \underbrace{\cancel{\int \vec{E}_1 \cdot d\vec{A}_1}}_{\text{upper flat end surface}}^{0} + \underbrace{\int \vec{E}_2 \cdot d\vec{A}_2}_{\text{lateral surface}} + \underbrace{\cancel{\int \vec{E}_3 \cdot d\vec{A}_3}}_{\text{lower flat end surface}}^{0} = \frac{q_{in}}{\varepsilon_0}$$

$\vec{E}_1 \perp d\vec{A}_1$ $\quad$ $\vec{E}_2 \parallel d\vec{A}_2$ $\quad$ $\vec{E}_3 \perp d\vec{A}_3$

the electric field magnitude is same for all points on the lateral surface and $\vec{E}$ is always parallel to the $d\vec{A}$

$$\int_{\text{lateral surface}} E\, dA \cos 0 = \frac{q_{in}}{\varepsilon_0} \Rightarrow E \boxed{\int_{\text{lateral surface}} dA} = \frac{q_{in}}{\varepsilon_0}$$

$$E \cdot 2\pi r l = \frac{\boxed{q_{in}}}{\varepsilon_0}$$

→ the charge surrounded by the cylinder with radius r and length l.

$\Downarrow$

Linear charge distribution → 1D

$q_{in} = \lambda . l$

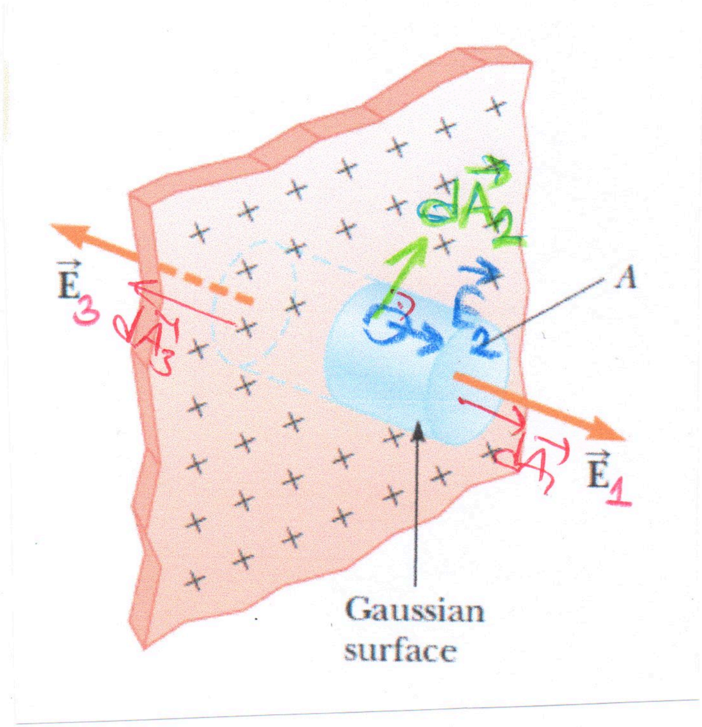
$$E . 2\pi r . \cancel{l} = \frac{\lambda \cancel{l}}{\varepsilon_0}$$

$$E = \frac{1}{2\pi\varepsilon_0} \frac{\lambda}{r} = 2 k_e \frac{\lambda}{r}$$

Compare the result with chapter 23 pages 25, 26 notes!

* If the linear charge distribution were of finite length, the electric field would not be calculated by Gauss's law. Because a finite length charge does not have sufficient symmetry for using Gauss's law.

Exercise | Find the electric field due to an infinite plane of positive charge with uniform surface charge density, σ.



Similar to our previous exercise we can select a cylindrical Gaussian surface that includes small portions of the charge distribution.

$$\oint \vec{E} . d\vec{A} = \int \vec{E}_1 . d\vec{A}_1 + \int \vec{E}_2 . d\vec{A}_2 + \int \vec{E}_3 . d\vec{A}_3$$

right flat end surface — $\vec{E}_1 \parallel d\vec{A}_1$

lateral surface — $\vec{E}_2 \perp d\vec{A}_2$

left flat end surface — $\vec{E}_3 \parallel d\vec{A}_3$

$$\oint \vec{E} \cdot d\vec{A} = \int E\, dA \cos 0 + \int E\, dA \cos 0$$

The electric field's magnitude will be same for the right and left ends. Because the distances between the charge distribution and the ends are the same!

$$\oint \vec{E} \cdot d\vec{A} = 2\int E\, dA = \frac{q_{in}}{\varepsilon_0}.$$

$$2E \underbrace{\boxed{\int dA}}_{A} = \frac{\boxed{q_{in}}}{\varepsilon_0}$$ → the charge inside the GS.

$q_{in} = \sigma A$

A is the surface area of the flat ends.

$$2E\cancel{A} = \frac{\sigma \cancel{A}}{\varepsilon_0}$$

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}$$ ✶ Compare the result with pages 29 and 30 in the lecture notes of Chapter 23.

CONDUCTORS IN ELECTROSTATIC EQUILIBRIUM

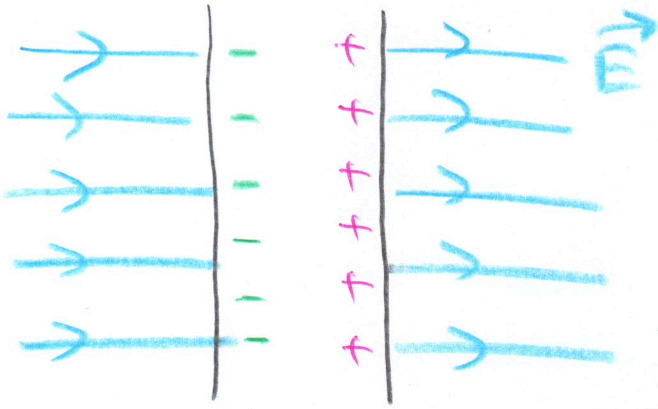
Uptill now, we have focused on the calculation of electric fields due to insulating materials. The charge carriers cannot move in the insulators but they can freely move within the conducting materials.

When there is no net motion of charge within a conductor, the conductor said to be in electrostatic equilibrium.

In the Gauss problems, we will often face with the conductors that are in electrostatic equilibrium

A conductor in electrostatic equilibrium must satisfy the following properties:

1. The electric field is zero everywhere inside the conductor, whether the conductor is solid or hollow.

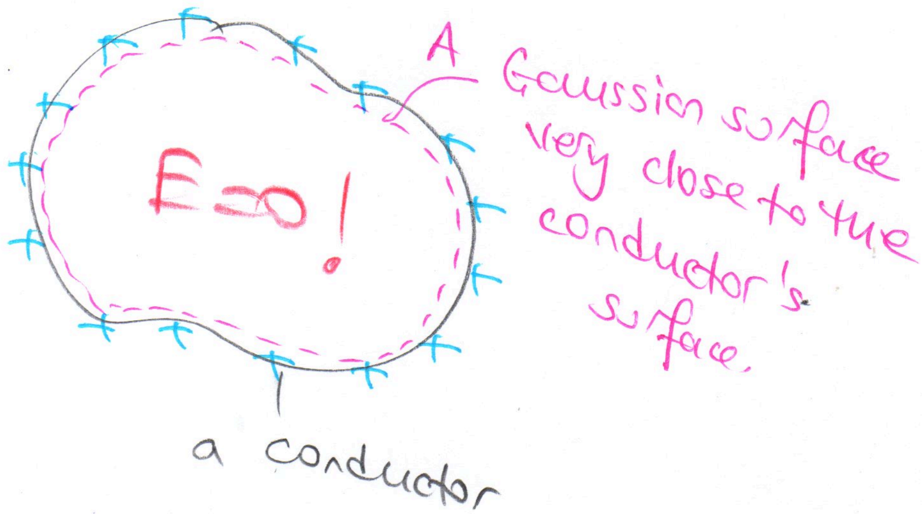


Let's locate a conducting slab in an external electric field $\vec{E}$. The free electrons accelerates to the left and the motion of the electrons causes a positively charged right surface.

So an opposite electric field to the external electric field $\vec{E}$ is established inside the conductor. The motion of the electrons continues until the electric field inside conductor balances an external electric field vector $\vec{E}$. Then the net electric field inside the conductor becomes zero. It requires 10^{-16} s for good conductors.

Additionally, if the electric field were not zero inside the conductor, the free electrons would experience an $e\vec{E}$ electric forces on themselves and they would accelerate. Their motion would not consistent with the electrostatic equilibrium.

2. If an isolated conductor carries a charge, the charge resides on its surface.

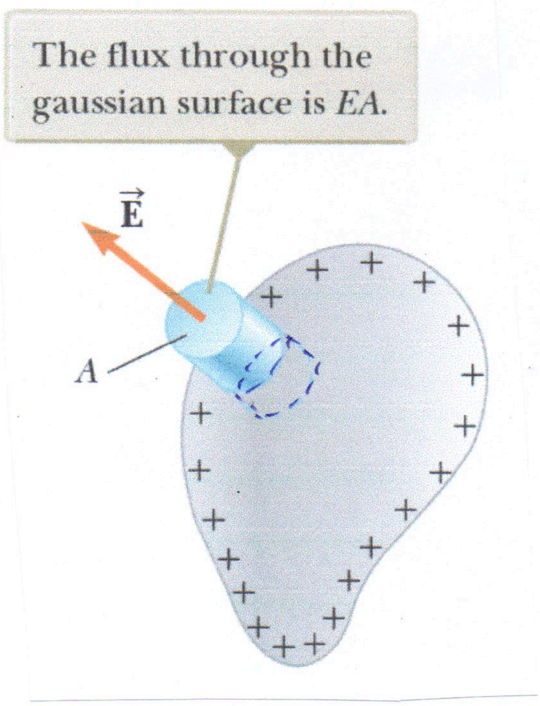


The net charge inside the GS must be zero because the electric field is zero inside the conductor that is at electrostatic equilibrium.

$$\left(\oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0} \right)$$

But if the conductor is charged, these charges must reside on its surface otherwise the net zero electric field condition are not satisfied.

3. The electric field just outside a charged conductor is perpendicular to the surface and it has a magnitude of $\frac{\sigma}{\varepsilon_0}$, where σ is the surface charge density at that point.



To reach the magnitude of electric field just outside the conducting material, the small clydrical GS is chosen. Some part of the cylinder is outside the conductor and the remaining part is inside conductor. Let's consider "+" conducting material.

The non-zero electric flux comes from only the flat face surface which is outside conductor. Because the electric field inside the conductor is zero, so the flat surface inside the conductor does not experience any electric field lines and also electric flux. For the lateral surface, $\vec{E}$ is parallel to the surface and there is always $90°$ between the $\vec{E}$ and $d\vec{A}$ vectors.

$$\Phi_E = \oint \vec{E}\cdot d\vec{A} = \int_{\substack{\text{outside}\\ \text{flat}\\ \text{surface}}} \vec{E}\cdot d\vec{A} + \cancelto{0}{\int_{\substack{\text{lateral}\\ \text{surface}}}} + \cancelto{0}{\int_{\substack{\text{inside}\\ \text{flat}\\ \text{surface}}}} = \frac{q_{in}}{\varepsilon_0}$$

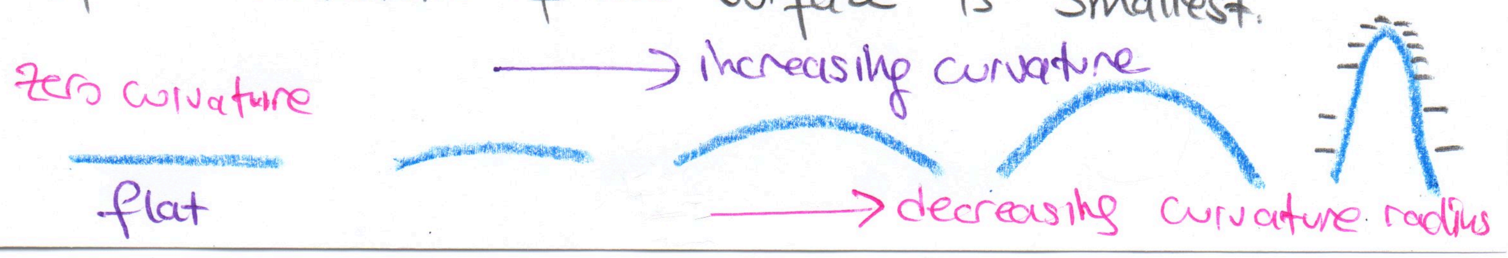
$$\int E \, dA \cos 0 = \frac{q_{in}}{\varepsilon_0}$$

$$E \underbrace{\boxed{\int dA}}_{A} = \frac{q_{in}}{\varepsilon_0} \rightarrow \sigma A$$

$$E \cancel{A} = \frac{\sigma \cancel{A}}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{\varepsilon_0}$$

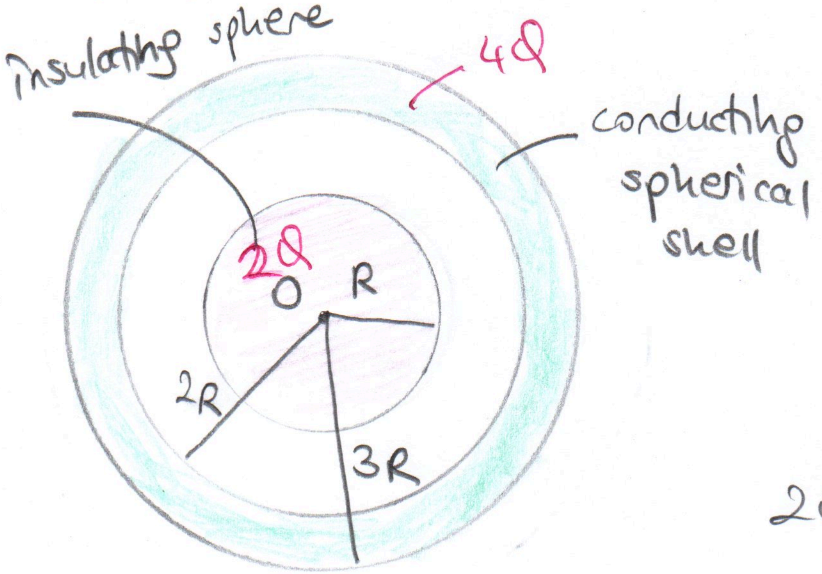
The magnitude of the electric field immediately outside a charged conductor.

4) On a irregularly shaped conductor, the surface charge density is greatest at locations where the radius of the curvature of the surface is smallest.



Exercise A solid insulating sphere of radius R has a non-uniform charge density $\rho = \alpha r$ and has a total charge of $+2Q$ (where α is a positive constant and r is the radial distance from the origin.) Concentric with the sphere, a conducting spherical shell with the inner and outer radii of 2R and 3R carries +4Q charge.

a) find the α constant in terms of Q and R.



To determine α, let's write the total charge of the insulating sphere.

$$2Q = \int dq = \int \rho \, dV$$

$$2Q = \int_{r=0}^{r=R} \alpha r (4\pi r^2 dr)$$

$$V = \frac{4}{3}\pi r^3$$

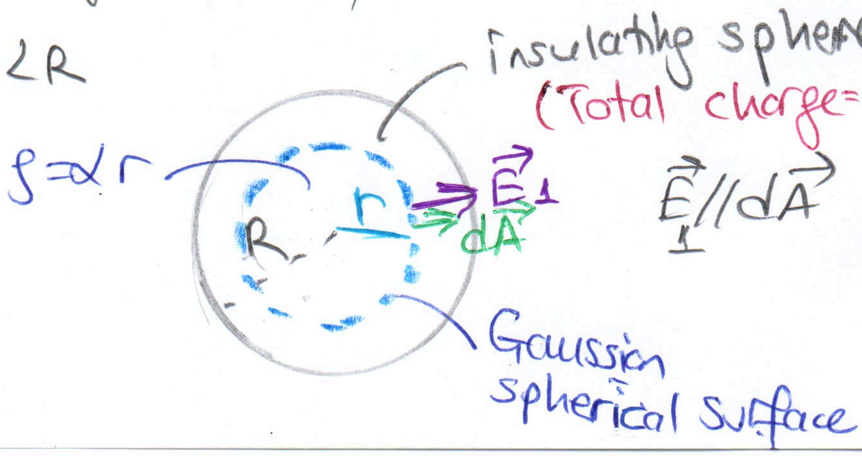
$$dV = 4\pi r^2 dr$$

$$2Q = 4\pi\alpha \int_0^R r^3 dr$$

$$2Q = \cancel{4}\pi\alpha \frac{r^4}{\cancel{4}}\Bigg|_0^R \Rightarrow \boxed{\alpha = \frac{2Q}{\pi R^4}}$$

b) Find the magnitude of the electric field in the regions in terms of k_e, Q, r, and R.

i) $r < R$



$$\vec{E}_1 // d\vec{A}$$

$$\oint \vec{E}_1 \cdot d\vec{A} = \frac{\boxed{q_{in}}}{\varepsilon_0}$$

$$q_{in} = \int \rho \, dV$$

$$= \int_{r=0}^{r=r} \alpha r \, 4\pi r^2 dr$$

$$= 4\pi\alpha \int_0^r r^3 dr$$

$$= 4\pi \boxed{\alpha} \frac{r^4}{4} = \cancel{4\pi} \frac{2Q}{\cancel{\pi} R^4} \frac{r^4}{\cancel{4}}$$

$$\boxed{q_{in} = 2Q \frac{r^4}{R^4}}$$

$$\oint E_1 \, dA \cos 0 = \frac{2Q}{\varepsilon_0} \frac{r^4}{R^4}$$

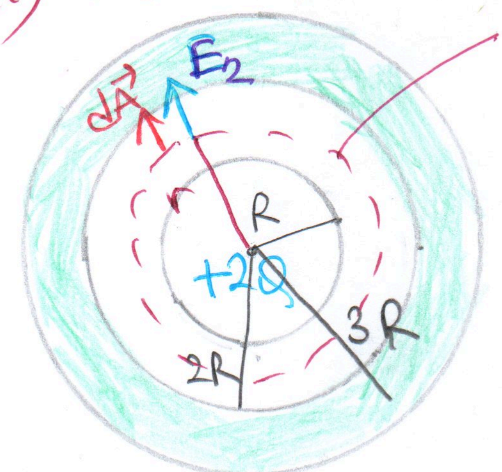
$$E_1 \oint dA = \frac{2Q \, r^4}{\varepsilon_0 R^4}$$

$$E_1 \cancel{4}\pi \cancel{r^2} = \frac{\cancel{2}Q \, r^{\cancel{4}\,2}}{\varepsilon_0 R^4}$$

$$E_1 = \frac{1}{2\pi\varepsilon_0} \frac{Q r^2}{R^4} = 2k_e \frac{Q r^2}{R^4}$$

$$E_1 = 2k_e \frac{Q r^2}{R^4}, \quad r < R$$

(ii) $R < r < 2R$



$$\vec{E}_2 \parallel d\vec{A}$$

$$\oint \vec{E}_2 \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$\oint E_2 \, dA \cos 0 = \frac{q_{in}}{\varepsilon_0}$$

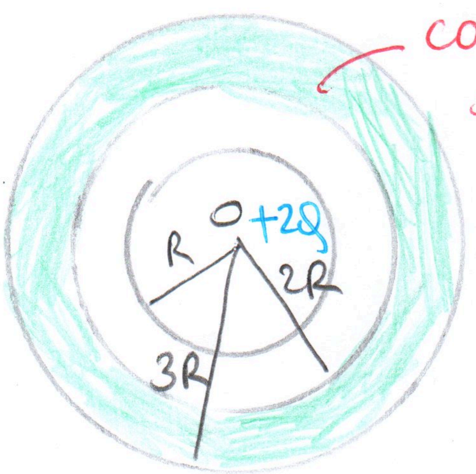
$$E_2 \oint dA = \frac{2Q}{\varepsilon_0}$$

$$E_2 \cdot 4\pi r^2 = \frac{2Q}{\varepsilon_0}$$

$$E_2 = \frac{1}{4\pi\varepsilon_0} \frac{2Q}{r^2}$$

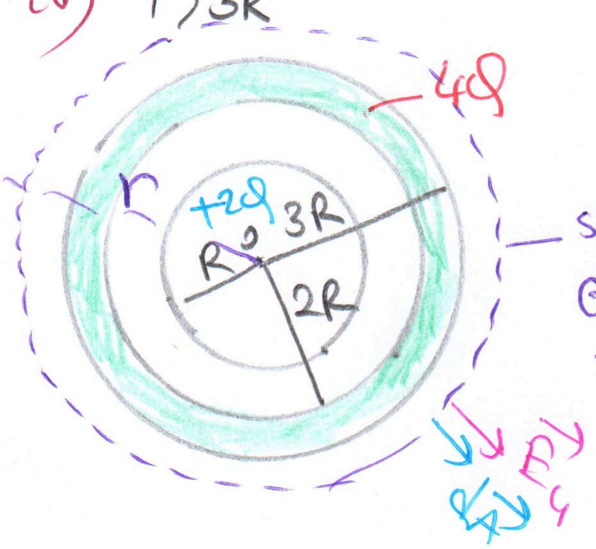
$$E_2 = k_e \frac{2Q}{r^2}, \quad R < r < 2R$$

iii) $2R < r < 3R$



$2R < r < 3R$ region corresponds to the inside of conducting spherical shell. According to the electrostatic equilibrium condition, $E_3 = 0$, $2R < r < 3R$

iv) $r > 3R$



$$\oint \vec{E_4} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$E_4 \oint dA = \frac{q_{in}}{\varepsilon_0}$$

$$E_4 . 4\pi r^2 = \frac{(2Q + 4Q)}{\varepsilon_0}$$

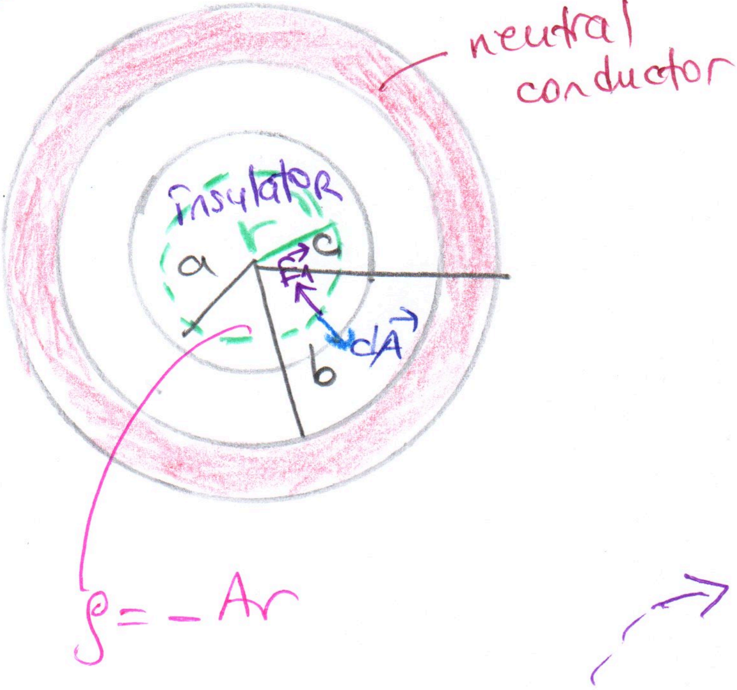
$$E_4 = \frac{1}{4\pi\varepsilon_0} \frac{6Q}{r^2}$$

$$E_4 = k_e \frac{6Q}{r^2} \; , \; r > 3R$$

Exercise A solid insulating sphere of radius a has a non-uniform charge density $\rho = -Ar$ where A is a positive constant and r is the radial distance from the origin. Concentric with the sphere, a neutral conducting spherical shell whose the inner and outer radii are b and c is located.

a) Find the electric field vector in the following regions

i) $r<a$



$\theta = 180°$ between the $\vec{E_1}$ and $d\vec{A}$

$$\oint \vec{E_1} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$\oint E_1 \, dA \cos 180° = \frac{q_{in}}{\varepsilon_0}$$

$$-E_1 \oint dA = \frac{q_{in}}{\varepsilon_0}$$

$$-E_1 4\pi r^2 = \frac{q_{in}}{\varepsilon_0}$$

$$q_{in} = \int \rho \, dV$$

$$= \int_{r=0}^{r=r} -Ar \, 4\pi r^2 \, dr$$

$$= -4\pi A \int_0^r r^3 \, dr$$

$$q_{in} = -\cancel{4}\pi A \frac{r^4}{\cancel{4}}$$

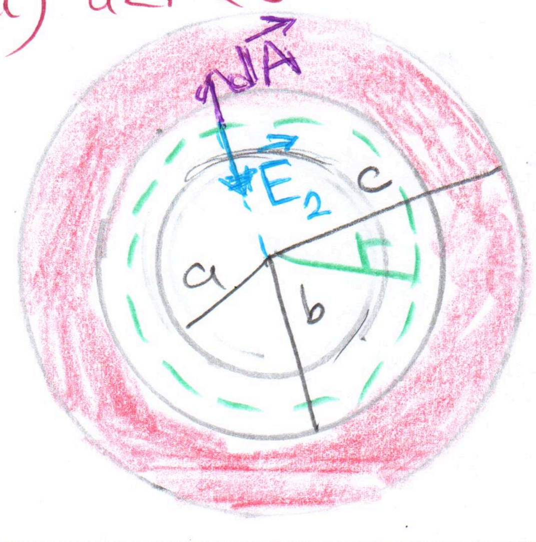
$$\boxed{q_{in} = -\pi A r^4}$$

$$\cancel{-E_1 4\pi r^2 = -\pi A \frac{r^{4\,2}}{\varepsilon_0}}$$

$$* \quad \boxed{E_1 = \frac{A}{4\varepsilon_0} r^2 \, , \; r<a} \quad *$$

$$\vec{E_1} = -\frac{A}{4\varepsilon_0} r^2 \hat{r} \, , \; r<a$$

ii) $a<r<b$



$$\oint \vec{E_2} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$\oint E_2 \, dA \cos 180 = \frac{q_{in}}{\varepsilon_0}$$

$$-E_2 \oint dA = \frac{q_{in}}{\varepsilon_0}$$

$$-E_2 4\pi r^2 = \frac{\boxed{q_{in}}}{\varepsilon_0}$$

$$q_{in} = \int \rho \, dV$$

$$= \int_{r=0}^{r=a} -Ar \, 4\pi r^2 \, dr$$ ← the total charge of the insulating sphere is surrounded by the Gaussian surface!

$$= -4\pi A \int_0^a r^3 \, dr$$

$$= -4\pi A \left. \frac{r^4}{4} \right|_0^a$$

$$= -\cancel{4}\pi A \frac{a^4}{\cancel{4}}$$

$$\boxed{q_{in} = -\pi A a^4}$$

$$-E_2 \, 4\pi r^2 = \boxed{\frac{q_{in}}{\varepsilon_0}}$$

$$-E_2 \, 4\cancel{\pi} r^2 = -\cancel{\pi} \frac{A a^4}{\varepsilon_0}$$

$$* \boxed{E_2 = \frac{A}{4\varepsilon_0} \frac{a^4}{r^2}, \quad a<r<b} *$$

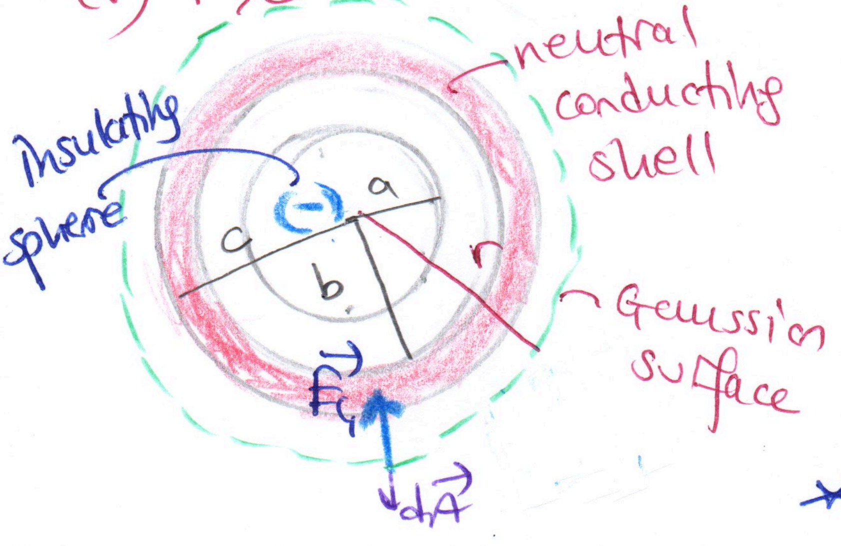
$$\vec{E}_2 = -\frac{A}{4\varepsilon_0} \frac{a^4}{r^2} \hat{r}, \quad a<r<b$$

(iii) $b<r<c$

The electric must be zero, because this region coincides with the inside of the conducting shell

$$* \boxed{E_3 = 0, \quad b<r<c} *$$

(v) $r>c$



$$\oint \vec{E}_4 \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

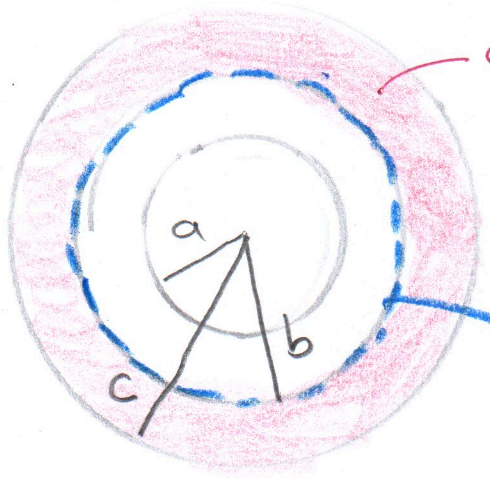
$$\oint E_4 \, dA \cos 180° = \frac{q_{in}}{\varepsilon_0}$$

$$-E_4 \oint dA = \frac{-\pi A a^4}{\varepsilon_0}$$

$$-E_4 \, 4\cancel{\pi} r^2 = -\cancel{\pi} \frac{A a^4}{\varepsilon_0}$$

$$* \boxed{E_4 = -\frac{A a^4}{4\varepsilon_0} \frac{a^4}{r^2} \hat{r}, \quad r>c} *$$

b) Determine the induced charge densities on the inner and outer surfaces of the conductive shell.



$r=b$ the inner surface

$r=c$ the outer surface

Let's write the Gauss's law for the inner surface, As is known, $r=b$ corresponds to zero electric field region.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$\downarrow$

$0 \longrightarrow q_{in} = 0 = q_{\text{insulting sphere}} + q_{\text{inner surface of the conducting shell}}$

$q_{\text{insulting sphere}} \rightarrow -\pi A a^4$

$$0 = -\pi A a^4 + q_{inner}$$

$$q_{inner} = \pi A a^4$$

Since the conducting shell is neutral

$$q_{inner} + q_{outer} = 0 \implies q_{outer} = -\pi A a^4$$

To reach σ_{inner} and σ_{outer}, let's write charge equations:

$$q_{inner} = \int \underbrace{\sigma_{inner}}_{\text{uniform}} dA = \sigma_{inner} \int dA = \sigma_{inner} 4\pi b^2$$

$$q_{inner} = \sigma_{inner} 4\cancel{\pi} b^2 = \cancel{\pi} A a^4$$

$$\boxed{\sigma_{inner} = \frac{Aa^4}{4b^2}}$$

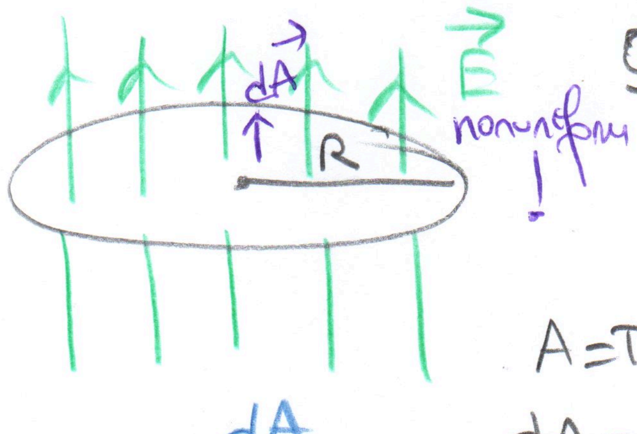
$$q_{outer} = \int \underbrace{\sigma_{outer}}_{\text{uniform}} dA = \sigma_{outer} \int dA = \sigma_{outer} 4\pi c^2$$

$$q_{outer} = \sigma_{outer} 4\cancel{\pi} c^2 = -\cancel{\pi} A a^4$$

$$\boxed{\sigma_{outer} = -\frac{Aa^4}{4c^2}}$$

ADDITIONAL EXERCISES

Exercise An electric field with constant direction is perpendicular to a plane of a circle with radius R. The magnitude of the electric field at a distance r from the center of the circle is given by $E_0(1-\frac{r}{R})$. Find the electric flux passes through the circle.



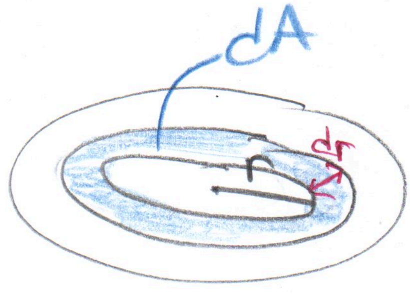
$$\Phi_E = \int \vec{E} \cdot \vec{dA} = \int E\, dA \cos 0$$

$$= \int E\, dA = \int E_0 \left(1 - \frac{r}{R}\right) 2\pi r\, dr$$

$A = \pi r^2$

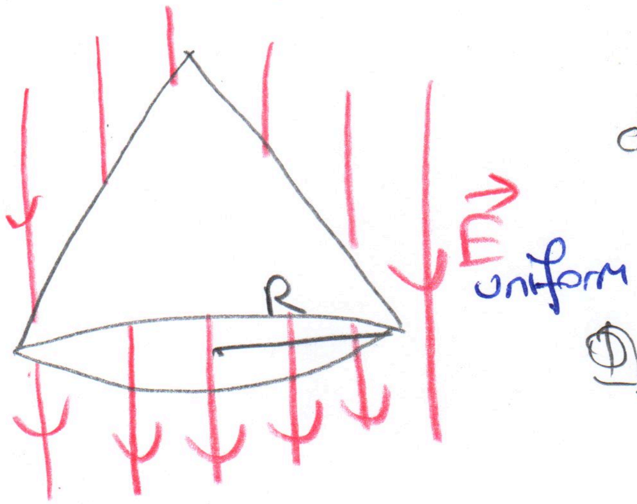
$dA = 2\pi r\, dr$

$$\Phi_E = 2\pi E_0 \int_{r=0}^{r=R} \left(r - \frac{r^2}{R}\right) dr$$



$$\boxed{\Phi_E = 2\pi E_0 \left(\frac{r^2}{2} - \frac{r^3}{3R}\right)\Bigg|_{r=0}^{R} = \frac{1}{3}\pi E_0 R^2}$$

Exercise A right circular cone of radius R and height h is placed with its axis parallel to a uniform electric field of E. The direction of the electric field lines is in the −y direction. Determine the electric flux entering the curved surface of the cone

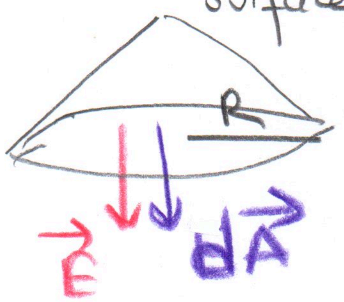


Since there is no charge inside closed surface of the cone, Φ_E must be zero.

$$\Phi_E = (\Phi_E)_{\substack{\text{circular}\\ \text{surface}}} + (\Phi_E)_{\substack{\text{curved}\\ \text{(lateral)}\\ \text{surface}}} = 0$$

$$(\Phi_E)_{\substack{\text{circular}\\ \text{surface}}} = -(\Phi_E)_{\substack{\text{curved}\\ \text{surface}}}$$

$$(\Phi_E)_{\substack{\text{circular}\\ \text{surface}}} = \int \vec{E} \cdot d\vec{A} = \int E\, dA \cos 0 = E \int dA$$



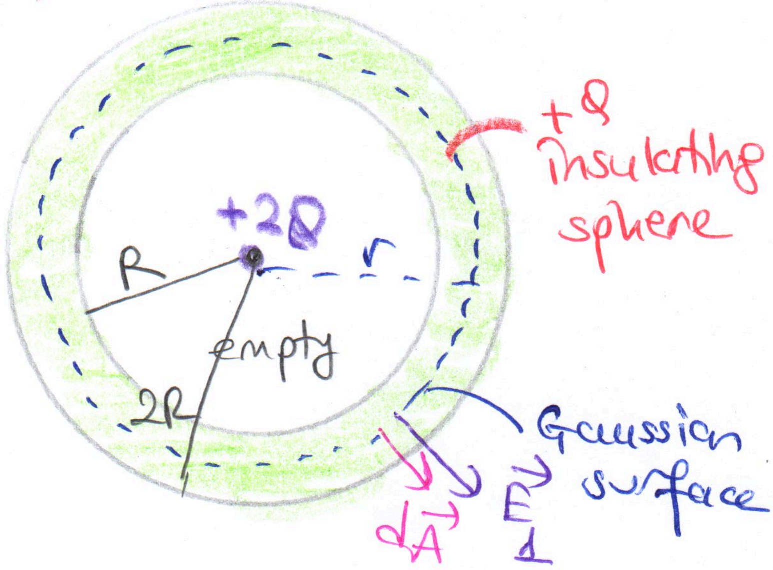
$$(\Phi_E)_{\substack{\text{circular}\\ \text{surface}}} = E \cdot \pi R^2$$

$$\boxed{(\Phi_E)_{\substack{\text{curved}\\ \text{surface}}} = -E\pi R^2}$$

Exercise A point charge of +2Q is located at the center of a spherical insulating shell whose the inner and outer radii are R and 2R. The shell carries +Q charge and has a uniform charge density ρ.

a) Find the electric fields in the following regions

i) $R < r < 2R$



$$\oint \vec{E_1} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$\oint E_1 \, dA \cos\theta = \frac{q_{in}}{\varepsilon_0}$$

$$E_1 \oint dA = \frac{\boxed{q_{in}}}{\varepsilon_0}$$

$$E_1 \, 4\pi r^2 = \frac{1}{\varepsilon_0}\boxed{\left[2Q + \frac{Q}{7R^3}(r^3 - R^3)\right]}$$

$$\boxed{q_{in}} = q_{\text{point charge}} + \underbrace{q_{\text{insulating sphere}}}$$

$q_{\text{point charge}} \downarrow 2Q$

Homogenous charge distribution $\Rightarrow$ Between the r and R radius!

$$* \boxed{E_1 = k_e \frac{Q}{7}\left[\frac{13}{r^2} + \frac{r}{R^3}\right]} *$$

The total charge of Q is distributed $\frac{4}{3}\pi\left[(2R)^3 - (R)^3\right]$ volume

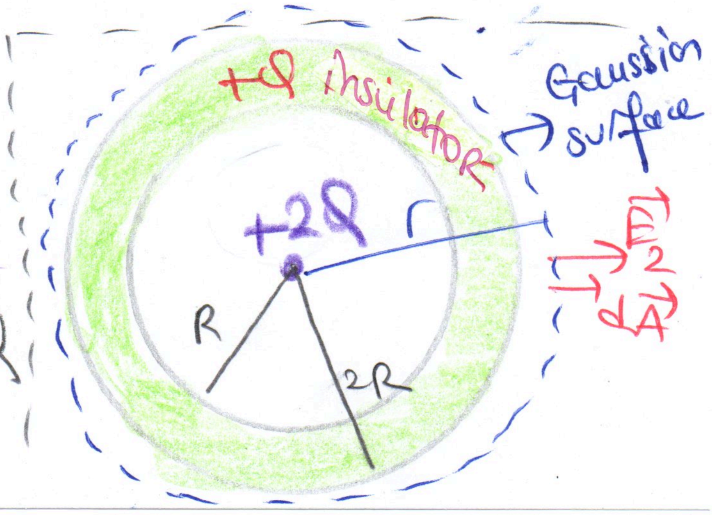
$q_{\text{insulating sphere}}$ $\qquad \frac{4}{3}\pi\left[r^3 - (R)^3\right]$ volume

---

$$q_{\text{insulating sphere}} = \frac{\cancel{\frac{4}{3}\pi}\left[r^3 - R^3\right] Q}{\cancel{\frac{4}{3}\pi}(7R^3)} = \frac{Q}{7R^3}(r^3 - R^3)$$

$$\boxed{q_{in} = 2Q + \frac{Q}{7R^3}(r^3 - R^3)}$$

ii) $r > 2R$ $\qquad q_{in} = +2Q + Q = +3Q$



$$\oint \vec{E_2} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$\oint E_2\, dA \cos 0 = \frac{3Q}{\varepsilon_0}$$

$$E_2 \oint dA = \frac{3Q}{\varepsilon_0}$$

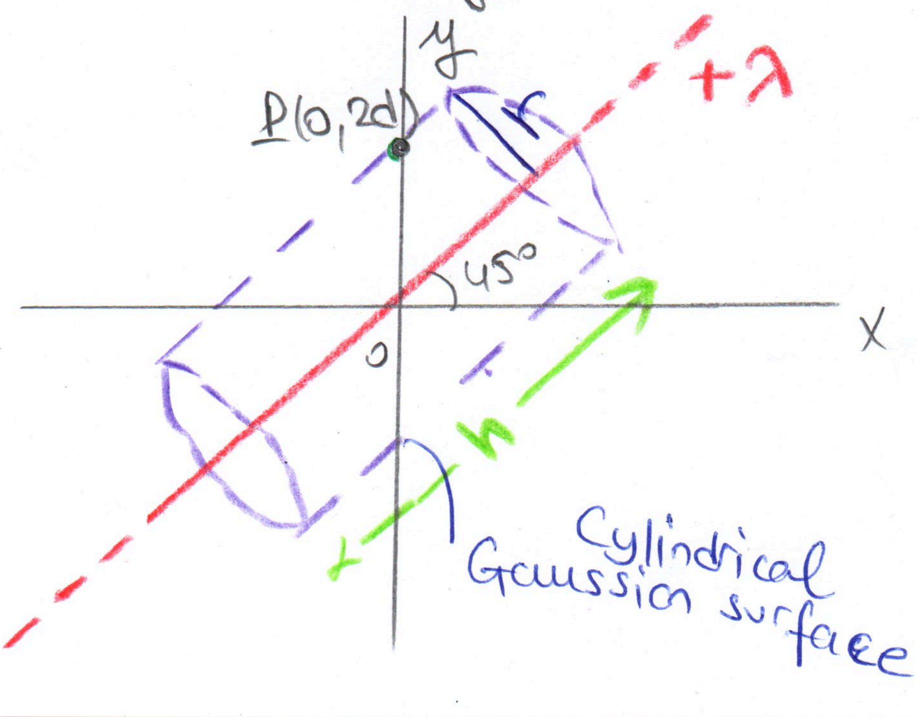
$$E_2 . 4\pi r^2 = \frac{3Q}{\varepsilon_0} \Rightarrow \boxed{E_2 = k_e \frac{3Q}{r^2} \,,\; r > 2R} \;*$$

b) Find the electric fields for the same regions, if the spherical shell is conducting

for $R < r < 2R$, $E = 0$ because this region is inside the conductor.

for $r > 2R$ q_{in} does not change. $E = k_e \frac{3Q}{r^2}$, $r > 2R$

Exercise) A long straight wire with linear charge density of λ is located in the xy-plane that makes $45°$ with the x-axis as shown in the figure. The charge is distributed along the wire uniformly.

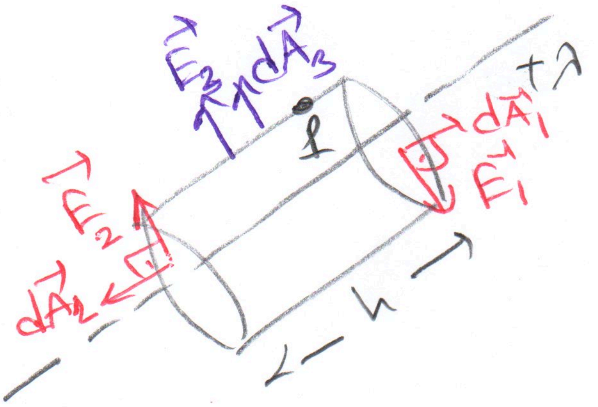


a) Find the electric field vector at point P (0,2d) in terms of d, λ, π, ε_0, and the unit vectors by using Gauss's law.

P point must coincide on the Gaussian surface!

Every point on the lateral surface of the Gaussian surface including P point must have the same electric field magnitude

$$\oint \vec{E} \cdot d\vec{A} = \underbrace{\cancel{\int}}_{\text{upper flat circular surface}} \overset{0}{} + \underbrace{\cancel{\int}}_{\text{lower flat circular surface}} \overset{0}{} + \underbrace{\int}_{\text{lateral surface}} = \frac{q_{in}}{\varepsilon_0}$$



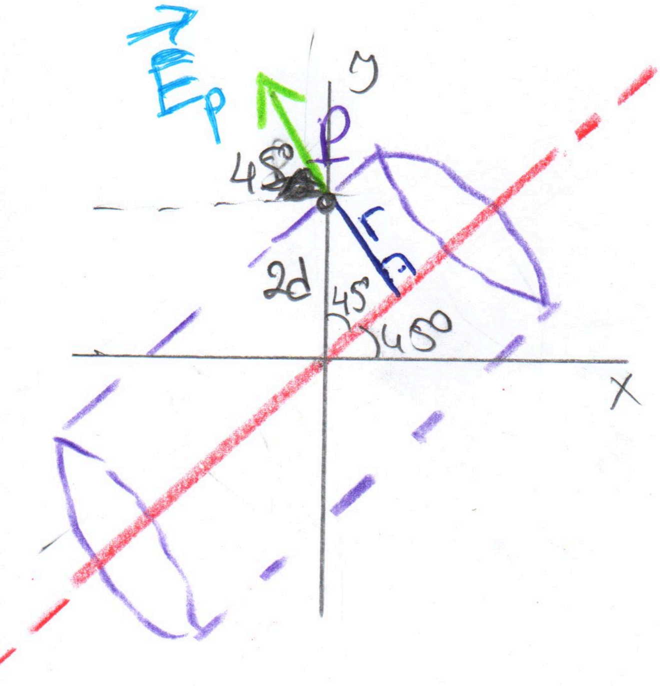
$$\int \vec{E}_3 \cdot d\vec{A}_3 = \frac{q_{in}}{\varepsilon_0}$$

$$\int E_p \cdot dA \cos 0 = \frac{\lambda h}{\varepsilon_0}$$

$$E_p \boxed{\int dA} = \frac{\lambda h}{\varepsilon_0}$$

$$E_p \cdot 2\pi r \cancel{h} = \frac{\lambda \cancel{h}}{\varepsilon_0}$$

$$E_p = \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{\boxed{r}}$$



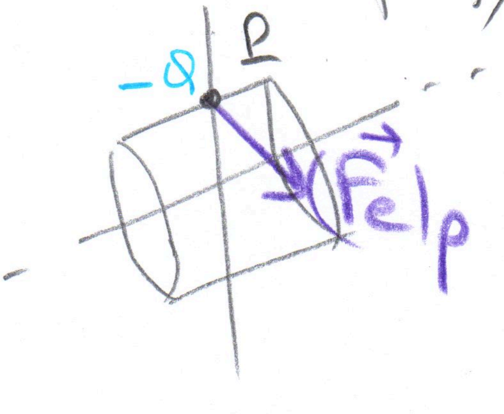
$$\sin 45 = \frac{r}{2d} = \frac{\sqrt{2}}{2} \qquad r = d\sqrt{2}$$

$$\boxed{E_p = \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{d\sqrt{2}}}$$

$$\vec{E}_p = E_p(-\cos 45\,\hat{i} + \sin 45\,\hat{j})$$

$$\vec{E}_p = \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{d\sqrt{2}} \left[-\frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}\right] \Rightarrow \boxed{\vec{E}_p = \frac{\lambda}{4\pi\varepsilon_0 d}\left[-\hat{i} + \hat{j}\right]}$$

b) Find the electric force vector acting on a charge of $-Q$ which is located at point P in terms of Q, d, λ, π, and ε_0.



$$\vec{F}_e = q\vec{E}$$

$$(\vec{F}_e)_P = -Q\,\vec{E}_P = \frac{Q\lambda}{4\pi\varepsilon_0 d}\left[\hat{i} - \hat{j}\right]$$

Exercise) An infinitely long insulating cylinder of radius R has a volume charge density that varies with the radius

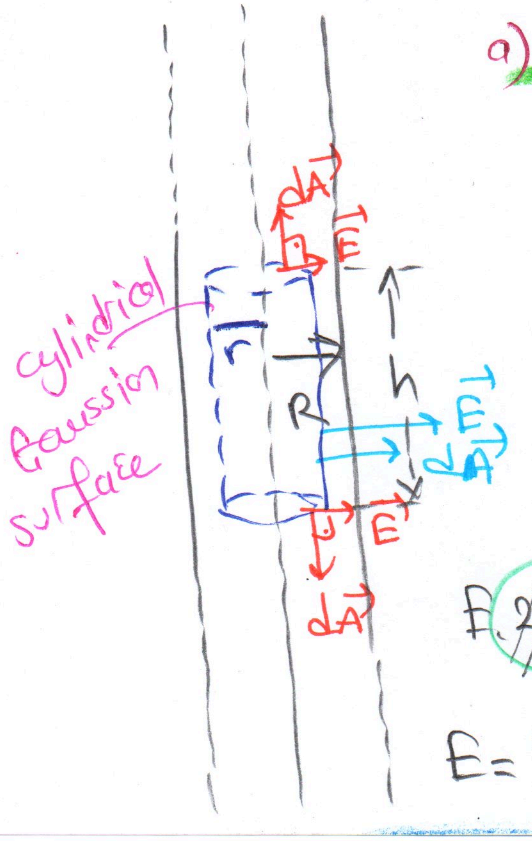
$$\rho = \rho_0\left(a - \frac{r}{b}\right)$$

where ρ_0, a, and b are positive constants and r is the distance from the axis of the cylinder. Use Gauss's law to determine the magnitude of the electric field at radial distances a) $r < R$ and b) $r > R$

$V = \pi r^2 h$

$dV = 2\pi r (dr) h$

a) $r < R$



$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$\int E\,dA \cos 0 = \frac{q_{in}}{\varepsilon_0}$$

$$E \boxed{\int dA} = \frac{q_{in}}{\varepsilon_0}$$

$$q_{in} = \int \rho\, dV$$

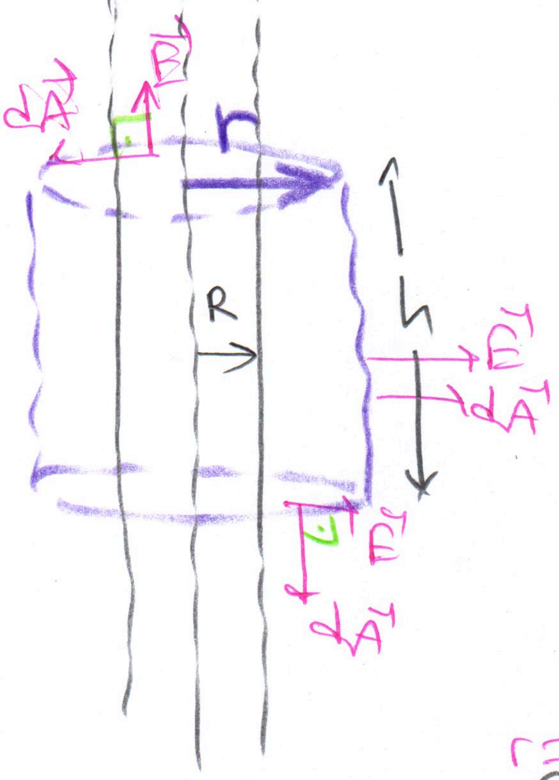
$$q_{in} = \int_{r=0}^{r=r} \rho_0\left(a - \frac{r}{b}\right) 2\pi r h\, dr$$

$$E \cdot \cancel{2\pi r h} = \frac{1}{\varepsilon_0}\cancel{2\pi}\rho_0 \cancel{h}\left[a\frac{r^2}{2} - \frac{r^3}{3b}\right]$$

$$q_{in} = 2\pi\rho_0 h\left[a\frac{r^2}{2} - \frac{r^3}{3b}\right]$$

$$E = \frac{\rho_0 r}{\varepsilon_0}\left[\frac{a}{2} - \frac{r}{3b}\right]$$

b) $r > R$



$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$\int E \, dA \cos 0 = \frac{q_{in}}{\varepsilon_0}$$

$$E \int dA = \frac{q_{in}}{\varepsilon_0}$$

$$E . 2\pi r h = 2\pi h \rho_0 \left[a\frac{R^2}{2} + \frac{R^3}{3b} \right]$$

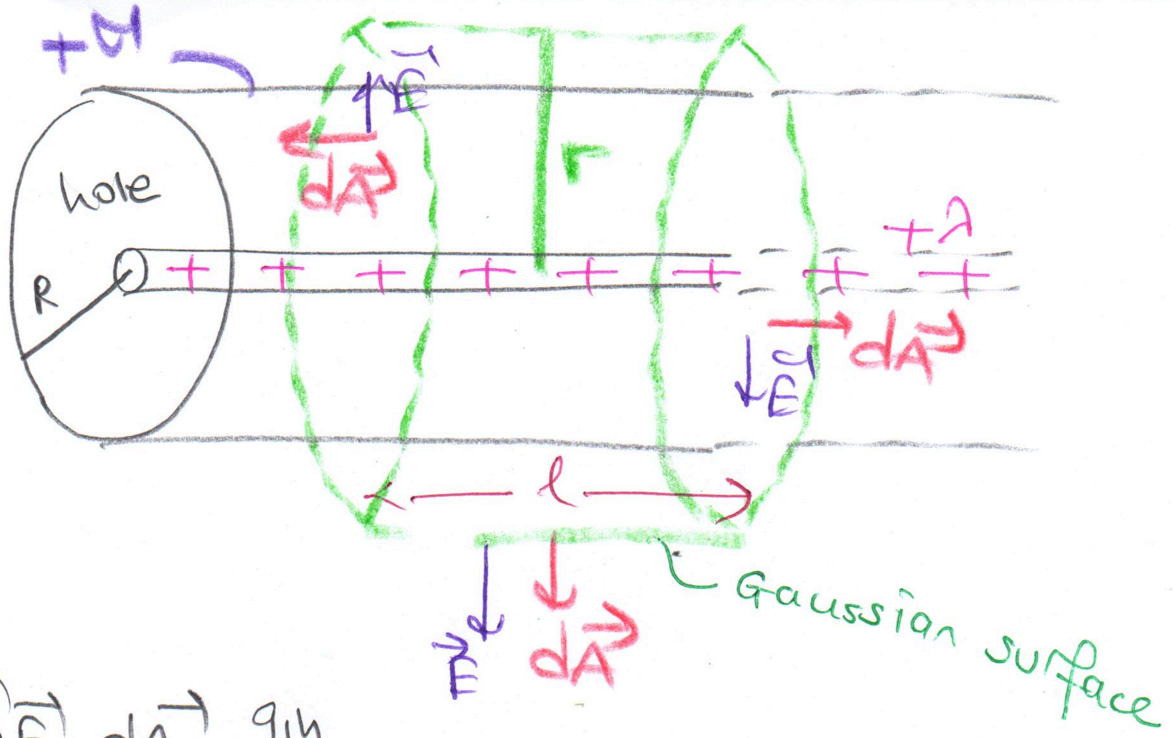
$$\boxed{E = \frac{\rho_0 R^2}{\varepsilon_0 r} \left[\frac{a}{2} - \frac{R}{3b} \right], \quad r > R}$$

$$q_{in} = \int \rho \, dV = \int_{r=0}^{r=R} \rho_0 \left(a - \frac{r}{b} \right) 2\pi r h \, dr$$

$$q_{in} = 2\pi h \rho_0 \left(a\frac{R^2}{2} - \frac{R^3}{3b} \right)$$

Exercise A long straight wire with the uniform linear charge density λ is lying at the center of the cylindrical long insulating shell with radius R. The cylindrical shell carries uniform surface charge density, $+\sigma$.

a) Find the electric field in the region $r > R$, where r is the radial distance from the linear charge distribution.



$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$\int E\,dA \cdot \cos 0 = \frac{q_{in}}{\varepsilon_0}$$

$q_{in} = q_{\text{linear charge}} + q_{\text{cylindrical shell}}$

$$E \int dA = \frac{1}{\varepsilon_0}\left[\lambda l + \sigma 2\pi R l\right]$$

$q_{in} = \lambda l + \sigma A_{\text{lateral surface area of the shell}}$

$$E \cdot 2\pi r l = \frac{1}{\varepsilon_0}\left[\lambda l + \sigma 2\pi R l\right]$$

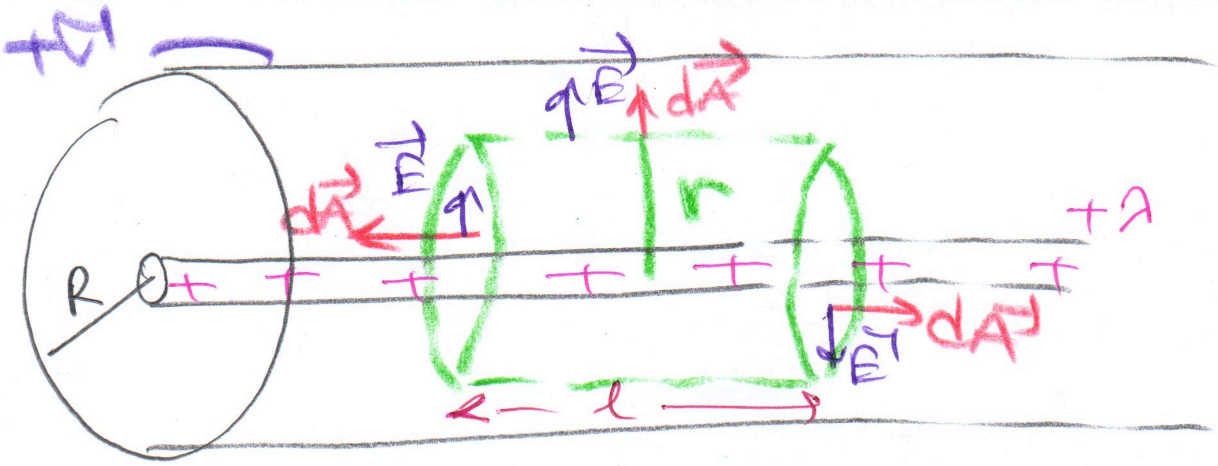
$q_{in} = \lambda l + \sigma 2\pi R l$

$$\boxed{E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r} + \frac{\sigma}{\varepsilon_0}\frac{R}{r}, \quad r > R}$$

b) A point charge of q and mass of m is released at $r = \frac{R}{2}$. Find the acceleration of the point charge when it is released in terms of $\varepsilon_0, \lambda, R, q, m,$ and π.

Since the point charge is located at $r = \frac{R}{2}$, the electric field at $r = \frac{R}{2}$ must be calculated.

$$F_e = qE = ma$$



For $r<R$ $\quad \oint \vec{E}\cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$

$$\int E\,dA\cos 0 = \frac{\lambda \ell}{\varepsilon_0}$$

$$E\int dA = \frac{\lambda \ell}{\varepsilon_0}$$

$$E.\,2\pi r \cancel{\ell} = \frac{\lambda \cancel{\ell}}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0}\,\frac{1}{r} \quad , \; r<R$$

At $r=\frac{R}{2}$ $\quad E_{\frac{R}{2}} = \frac{\lambda}{\cancel{2}\pi\varepsilon_0}\,\frac{\cancel{2}}{R} = \frac{\lambda}{\pi\varepsilon_0 R}$

$$F_e = qE_{\frac{R}{2}} = q\frac{\lambda}{\pi\varepsilon_0 R} = ma$$

$$a = \frac{q\lambda}{\pi\varepsilon_0 R m}$$