Numerical Methods KOM2722-AVE3842

Week 3

Dr. Bilal Erol

Rolle’s Theorem

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If

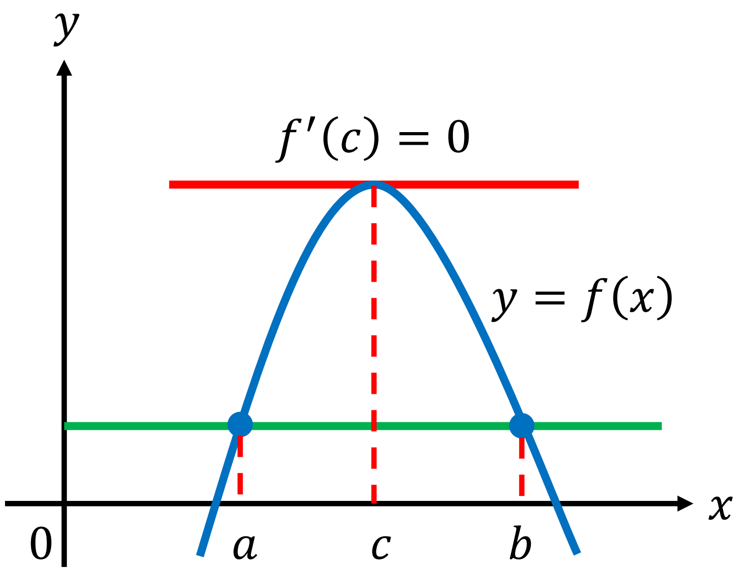
$$f(a) = f(b)$$

then there is at least one number c in (a, b) at which

$$f'(c) = 0.$$

Thus, there should be one place where tangent line should be horizontal

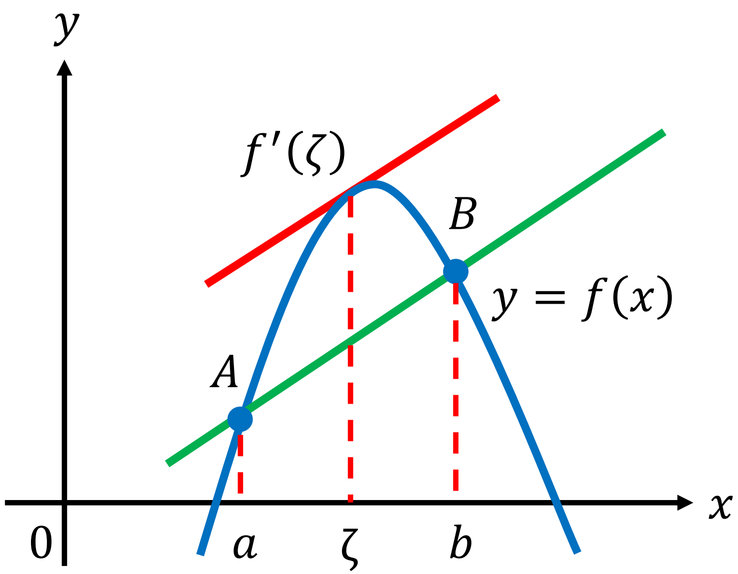
Note that it is also satisfied for functions intersect the x axis at two points; $f(a) = f(b) = 0$.



The Mean Value Theorem

Let f be a given function, continuous on closed interval $[a, b]$ and differentiable at every point of its interior (a, b). Then there exists a point $\zeta \in [a, b]$ such that

$$f'(\zeta) = \frac{f(b) - f(a)}{b - a}$$



The mean value theorem (MVT) states that between any two points on the graph of a differentiable function f there at least one place where the tangent line to the graph is parallel to line joining.

Intermediate Value Theorem

Let $f \in C([a,b])$ be given, and assume that W is a value between $f(a)$ and $f(b)$, that is, either $\mathrm{f}(a) \leq W \leq f(b)$, or $f(b) \leq W \leq f(a)$. Then there exists a point $c \in [a,b]$ such that $f(c) = W$.

* This theorem says that a certain point exists does not give us much information about its numerical value.

-> We will use this theorem as the basis for finding the roots.

Note: $C([a,b])$ — The set of functions f which are defined on the interval $[a,b]$, continuous on all of (a,b), and continuous from the interior of $[a,b]$ at the endpoints.

Extreme Value Theorem

Let $f \in C([a,b])$ be given; then there exists a point $m \in [a,b]$ such that $f(m) \leq f(x)$ for all $x \in [a,b]$, and a point $M \in [a,b]$ such that $f(M) \geq f(x)$ for all $x \in [a,b]$. Moreover, f achieves its maximum and minimum values on $[a,b]$ either at the endpoints a or b, or at a critical point.

The student should recall that a critical point is a point where the first derivative is either undefined or equal to zero.

Integral Mean Value Theorem

Let f and g both be in $C([a,b])$, and assume further that g does not change sign on $[a,b]$. Then there exists a point $\zeta \in [a,b]$ such that

$$\int_a^b g(t)f(t)dt = f(\zeta)\int_a^b g(t)dt$$

Discrete Average Value Theorem

Let $f \in C([a,b])$ and consider the sum,

$$S = \sum_{k=1}^{n} a_k f(x_k)\,,$$

Where each point $x_k \in [a,b]$, and the coefficients satisfy

$$a_k \geq 0, \qquad \sum_{k=1}^{n} a_k = 1$$

Then there exists a point $\eta \in [a,b]$ such that $f(\eta) = S$, i.e.,

$$f(\eta) = \sum_{k=1}^{n} a_k f(x_k)$$

Proof:

$$f(x_k) \leq f_M \Rightarrow S = \sum_{k=1}^{n} a_k f(x_k) \leq f_M \sum_{k=1}^{n} a_k = f_M$$

Linear Interpolation Methods

Tells how approximating the function with straight lines for example, almost all graphs produced by computers are actually the results of piecewise linear interpolation in which the machine draws very large number of very small straight lines represent the curve.

Given a set of data points x_k: (called as nodes),

We say the function p interpolates the function f at these nodes,

if $\boldsymbol{p}(\boldsymbol{x}_{\boldsymbol{k}}) = \boldsymbol{f}(\boldsymbol{x}_{\boldsymbol{k}})$ for **all** $\boldsymbol{k}$.

We are most interested in the extent to $\boldsymbol{p} \approx \boldsymbol{f}$

Linear interpolation methods based on using straight line to approximate a given function.

$\Rightarrow$ 2 points to determine a straight line

x_0, x_1 and a function f $\Rightarrow$ we want to find the equation of a straight line that passes through these two points: $(x_0, f(x_0))$, $(x_1, f(x_1))$

$$p_1(x) = \frac{x_1 - x}{x_1 - x_0} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

Let us investigate how accurate the linear interpolation using Rolle's Theorem $\Rightarrow$ Special case of mean value theorem if $f(c) = f(b) \Rightarrow$ there exists ζ such that $f'(\zeta) = 0$.

Define,

$$E(x) = f(x) - p_1(x),$$

$$w(x) = (x - x_0)(x - x_1),$$

$$G(x) = E(x) - \frac{w(x)}{w(t)} E(t)$$

t some fixed value in (x_0, x_1) where ($(x_0 < t < x_1)$)

$$G(x_0) = 0, G(x_1) = 0, G(t) = 0$$

Then the Rolle's Theorem states that;

There exists a point z_0 between x_0 and t such that $G'(z_0) = 0$

and a point z_1, between x_1 and t, such that $G'(z_1) = 0$.

Let's apply Rolle's Theorem to G' and assert that there exists

a point ζ between z_0 and z_1 such that $G''(\zeta) = 0$.

But,

$$G''(x) = F''(x) - \frac{2}{w(t)} E(t)$$

$$G''(\zeta) = 0 \quad => \quad F''(\zeta) - \frac{2}{w(t)} E(t) = 0$$

And we have $\underbrace{f(t) - p_1(t)}_{= E(t) \text{ because we defined } (E(x) = f(x) - p_1(x))} = \frac{1}{2}\overbrace{(t - x_0)(t - x_1)}^{w(t)} f''(\zeta)$

For any $t \in [x_0, x_1]$,

The error in the approximation will grow rapidly outside the interval $[x_0, x_1]$.

⇒ Let us take absolute values, investigate for the worst case for the second derivative term.

$$
\begin{aligned}
|f(x) - p_1(x)| &\leq \frac{1}{2}|(x - x_0)(x - x_1)| \max_{x_0 \leq t \leq x_1} |f''(t)| \\
&\leq \frac{1}{2}\left(\max_{x_0 \leq t \leq x_1} |(t - x_0)(t - x_1)|\right)\left(\max_{x_0 \leq t \leq x_1} |f''(t)|\right)
\end{aligned}
$$

So the upper bound on the error depends on the maximum of the function:

$$g(x) = |(x - x_0)(x - x_1)| \;\; = (x_1 - x)(x - x_0)$$

$g(x_0) = g(x_1) = 0$, the Extreme Value Theorem says that the maximum value of g on the interval $[x_0, x_1]$ will be on critical point.

$$g'(x) = x_1 - x - x + x_0 = (x_1 + x_0) - 2x$$

$$x_c = \frac{1}{2}(x_0 + \; x_1) \qquad => g(x_c) = \frac{1}{4}(x_1 - \; x_0)^2$$

Finally our error is bounded

$$f(x) - p_1(x) \leq \frac{1}{8}(x_1 - x_0)^2(\max_{x_0 \leq t \leq x_1} |f''(x)|)$$

<u>Theorem</u> (Linear Interpolation Error)

Let $f \in C^2([x_0, x_1])$ and let $p_1(x)$ be linear polynomials that interpolates f at x_0 and x_1, then for all $x \in [x_0, x_1]$,

$$\begin{aligned} |f(x) - p_1(x)| &\leq \frac{1}{2}|(x - x_0)(x - x_1)| \max_{x_0 \leq t \leq x_1} |f''(x)| \\ &\leq \frac{1}{8}(x_1 - x_0)^2 \max_{x_0 \leq t \leq x_1} |f''(x)| \end{aligned}$$

Note: $C^k([a, b])$ — The set of functions f such that f and its first k derivatives are all in C($[a, b]$)

Example: Consider the problem of constructing a piecewise linear approximation to $f(x) = \log_2(x)$ using the nodes 1/4, 1/2, 1.

$$Q_1(x) = \left(\frac{\frac{1}{2} - \mathrm{x}}{\frac{1}{2} - \frac{1}{4}}\right) log_2\left(\frac{1}{4}\right) + \left(\frac{x - \frac{1}{4}}{\frac{1}{2} - \frac{1}{4}}\right) log_2\left(\frac{1}{2}\right) = 4x - 3$$

$$Q_2(x) = \left(\frac{1 - \mathrm{x}}{1 - \frac{1}{2}}\right) log_2\left(\frac{1}{2}\right) + \left(\frac{x - \frac{1}{2}}{1 - \frac{1}{2}}\right) log_2(1) = 2x - 2$$

$$q(x) = \begin{cases} 4x - 3, & \frac{1}{4} \le x \le \frac{1}{2} \\ 2x - 2, & \frac{1}{2} \le x \le 1 \end{cases}$$

The error:

$$|log_2(x) - \ Q_1(x)\ | \leq \frac{1}{8}\left(\frac{1}{2} - \frac{1}{4}\right)^2 \max_{t\in[\frac{1}{4},\frac{1}{2}]} |log_2(e)t^{-2}| \quad = 0.1803368801 \ \ for\ x \in [\frac{1}{4},\frac{1}{2}]$$

not always the same

$$|\ log_2(x) - \ Q_2(x)\ | \leq \frac{1}{8}\left(1 - \frac{1}{2}\right)^2 \max_{t\in[\frac{1}{2},1]} \ |log_2(e)t^{-2}| = 0.1803368801 for\ x \in [\frac{1}{2},1]$$

$$\Rightarrow \ |\log_2(x) - q(x)| \leq 0.1803368801$$