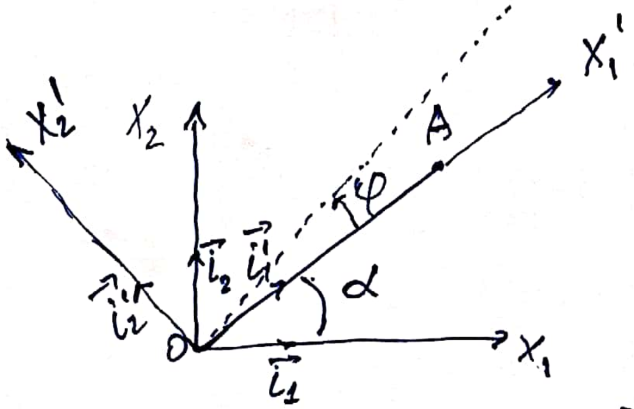
ET ders 3. devam

Noktede şekildeğiştirmeler

$$\varepsilon'_{ij} = \varepsilon_{nm}\,\alpha_{in}\,\alpha_{jm} \quad ; \quad i;j=1,2 \;;\; n;m=1,2$$

$$\varepsilon'_{11} = \varepsilon_{11}\,\alpha_{11}\,\alpha_{11} + \varepsilon_{12}\,\alpha_{11}\,\alpha_{12} + \varepsilon_{21}\,\alpha_{12}\,\alpha_{11} + \varepsilon_{22}\,\alpha_{12}\,\alpha_{12}$$



$$\alpha_{11} = \cos\alpha \quad ; \quad \alpha_{12} = \sin\alpha$$

$$\varepsilon'_{11} = \varepsilon_{11}\cos^2\alpha + 2\varepsilon_{12}\sin\alpha\cos\alpha + \varepsilon_{22}\sin^2\alpha$$

$$\boxed{\varepsilon'_{11} = \varepsilon_{11}\cos^2\alpha + \varepsilon_{12}\sin 2\alpha + \varepsilon_{22}\sin^2\alpha}$$

$$\varphi \approx \frac{\partial u_1'}{\partial x_2'} \;; \qquad u_1' = \alpha_{11}u_1 + \alpha_{12}u_2 =$$

$$\boxed{u_1' = u_1\cos\alpha + u_2\sin\alpha}$$

$$\alpha_{21} = -\sin\alpha \qquad (\vec{i}_2'\,\vec{i}_1) = \cos(\pi/2+\alpha) = \; =$$

$$\alpha_{22} = \cos\alpha$$

$$x_1 = \alpha_{11}x_1' + \alpha_{21}x_2' \;;$$

$$x_2 = \alpha_{12}x_1' + \alpha_{22}x_2' \;;$$

$$x_1 = \cos\alpha\, x_1' - \sin\alpha\, x_2' \;;$$

$$x_2 = \sin\alpha\, x_1' + \cos\alpha\, x_2' \;;$$

$$\varphi \approx \frac{\partial u_1'}{\partial x_2'} = \frac{\partial(u_1\cos\alpha + u_2\sin\alpha)}{\partial x_1}\frac{\partial x_1}{\partial x_2'} + \frac{\partial(u_1\cos\alpha + u_2\sin\alpha)}{\partial x_2}\frac{\partial x_2}{\partial x_2'} =$$

$$= \left[\frac{\partial u_1}{\partial x_1}\cos\alpha + \frac{\partial u_2}{\partial x_1}\sin\alpha\right](-\sin\alpha) + \left[\frac{\partial u_1}{\partial x_2}\cos\alpha + \frac{\partial u_2}{\partial x_2}\sin\alpha\right]\cos\alpha =$$

$$= \frac{\partial u_1}{\partial x_1}(-\sin\alpha\cos\alpha) + \frac{\partial u_2}{\partial x_2}\sin\alpha\cos\alpha +$$

$$+ \frac{\partial u_2}{\partial x_1}(-\sin^2\alpha) + \frac{\partial u_1}{\partial x_2}\cos^2\alpha =$$

$$= \left(\frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_1}\right)\sin\alpha\cos\alpha + \frac{\partial u_1}{\partial x_2}\cos^2\alpha -$$

$$- \frac{\partial u_2}{\partial x_1}\sin^2\alpha = \left(\frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_1}\right)\sin\alpha\cos\alpha +$$

$$+ \underline{\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right)} + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\cos 2\alpha =$$

$$\boxed{\varphi_1 = (\varepsilon_{22} - \varepsilon_{11})\frac{1}{2}\sin 2\alpha + \frac{1}{2}\varepsilon_{12}\cos 2\alpha + \frac{1}{2}\omega_3;}$$

$$\cancel{\varphi_{\alpha+\pi/2} = -(\varepsilon_{22} - \varepsilon_{11})\frac{1}{2}\sin 2\alpha - \frac{1}{2}\varepsilon_{12}\cos 2\alpha + \frac{1}{2}\omega_3}$$

$$\cancel{\varphi_\theta = \frac{\partial u_2}{\partial x_1}\cos^2\theta + \left(\frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_1}\right)\sin\theta\cos\theta - \frac{\partial u_1}{\partial x_2}\sin^2\theta}$$

$$\cancel{\sin(\alpha + \pi/2) = \cos\alpha}$$

$$\cancel{\cos(\alpha + \pi/2) = -\sin\alpha}$$

$$\cancel{\varphi_{\alpha+\pi/2} = \left(\frac{\partial u_2}{}\right.}$$

$$\varphi_2 = \cancel{\frac{\partial u_2'}{\partial x_1'} = \frac{\partial}{\partial x_1}(\alpha_{12}u_1' + \alpha_{21}u_2')\frac{\partial x_1}{\partial x_1'} + \frac{\partial}{\partial x_2}(\alpha_{12}u_1' + \alpha_{21}u_2')\frac{\partial x_2}{\partial x_1'}}$$

$$\cancel{= \frac{\partial u_1}{\partial x_1}\sin\alpha\cos\alpha + \frac{\partial u_2'}{\partial x_1}(-\sin\alpha\cos\alpha) + \frac{\partial u_1}{\partial x_2}\sin^2\alpha - \frac{\partial u_2}{\partial x_2}\sin^2\alpha}$$

$$\cancel{\varepsilon'_{12} = \varepsilon_{nm}\alpha_{1n}\alpha_{2m} = \varepsilon_{11}\cos\alpha(-\sin\alpha) + \varepsilon_{12}\cos\alpha(-\sin\alpha) +}$$

$$\cancel{+\varepsilon_{21}\cos\alpha\cos\alpha) + \varepsilon_{22}\sin\alpha\cos\alpha}$$

$$\cancel{+\varepsilon_{21}\sin\alpha(-\sin\alpha))}$$

$$\varepsilon'_{12} = \varepsilon_{11}\alpha_{11}\alpha_{21} + \varepsilon_{12}\alpha_{11}\alpha_{22} + \varepsilon_{21}\alpha_{12}\alpha_{21} + \varepsilon_{22}\alpha_{12}\alpha_{22} =$$

$$= \varepsilon_{11}\cos\alpha(-\sin\alpha) + \varepsilon_{12}\cos^2\alpha + \varepsilon_{21}\sin\alpha(-\sin\alpha) +$$

$$+ \varepsilon_{22}\sin\alpha\cos\alpha = (\varepsilon_{22}-\varepsilon_{11})\sin\alpha\cos\alpha +$$

$$+ \varepsilon_{12}(\cos^2\alpha - \sin^2\alpha)$$

$$\boxed{\varepsilon'_{12} = (\varepsilon_{22}-\varepsilon_{11})\frac{1}{2}\sin 2\alpha + \varepsilon_{12}(\cos^2\alpha - \sin^2\alpha)}$$

$$\varepsilon'_{12} = 0 \Rightarrow \cancel{\tan 2\alpha = \frac{1}{2}(\varepsilon_{11}-\varepsilon_{22})}$$

$$\boxed{\tan 2\alpha = \frac{\varepsilon_{12}}{\frac{1}{2}(\varepsilon_{11}-\varepsilon_{22})}}$$

x_1^*, x_2^* asal yönler ise, onda $\varepsilon_{12} = 0$ olur

$$\begin{cases} \varepsilon'_{11} = \varepsilon_{11}\cos^2\alpha + \varepsilon_{22}\sin^2\alpha \\ \varepsilon'_{12} = \frac{1}{2}(\varepsilon_{22}-\varepsilon_{11})\sin 2\alpha \end{cases}$$

$$\begin{cases} \varepsilon'_{11} - \frac{1}{2}(\varepsilon_{11}+\varepsilon_{22}) = \frac{1}{2}(\varepsilon_{11}-\varepsilon_{22})\cos 2\alpha \\ \varepsilon'_{12} = \frac{1}{2}(\varepsilon_{22}-\varepsilon_{11})\sin 2\alpha \end{cases}$$

Tekrar düzlem gerilme ve düzlem şekildeğiş-
tirme durumları hakda.

Bu durumlarda ait denklemler takımını tekrar yazalım

Denge denklemleri

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0 \; ; \quad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0 \qquad (1)$$

Hook yasası

$$\varepsilon_{11} = a\,\sigma_{11} - b\,\sigma_{22} \qquad a = \frac{1}{E}, \; b = \frac{\nu}{E} \quad \text{düz. gerilme}$$

$$\varepsilon_{22} = -b\,\sigma_{11} + a\,\sigma_{22} \qquad a = \frac{1-\nu^2}{E}, \; b = \frac{\nu(1+\nu)}{E} \quad \text{düz. şekil.}$$

$$\varepsilon_{12} = \frac{1}{2G}\,\sigma_{12} \qquad (2)$$

~~ve ya~~ Şekildeğiştirme - yerdeğiştirme ilişkileri:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \qquad (3)$$

(3)'den ε_{12}'den $\frac{\partial^2}{\partial x_1 \partial x_2}$ - ye göre türev alalım

$$\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{1}{2}\left(\frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2}\right) \qquad (4)$$

ε_{11}'den $\frac{\partial^2}{\partial x_2^2}$ ye göre, ε_{22}'den ise $\frac{\partial^2}{\partial x_1^2}$ - ye göre türevler alalım

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} = \frac{\partial^3 u_1}{\partial x_2^2 \partial x_1} \; ; \quad \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2} \; ; \qquad (5)$$

(4) ve (5)'den

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2\,\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} \qquad (6)$$

uygunluk (compatibility) ~~şartlar~~ denklemidir.

yukarıdaki denklemlere kuvvetlere göre verilen sınır koşullarını ekleyelim

$$\sigma_{11} n_1 + \sigma_{22} n_2 = P_1 \quad (7)$$

$$\sigma_{12} n_1 + \sigma_{22} n_2 = P_2$$

Yerdeğiştirmelere göre verilen sınır koşulları ise

$$u_1\big|_{S_{II}} = \varphi_1 \quad ; \quad u_2\big|_{S_{II}} = \varphi_2 \quad (8)$$

biçiminde yazarız.

Şimdi ε_{11}, ε_{22} ve ε_{12} lerin (2) Hook yasasındaki ifadelerini (6) birgelik (uygunluk) denkleminde yerine yazalım

$$\frac{\partial^2}{\partial x_2^2}\left(a\sigma_{11} - b\sigma_{22}\right) + \frac{\partial^2}{\partial x_1^2}\left(-b\sigma_{11} + a\sigma_{22}\right) =$$

$$= \frac{2}{2G}\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} \quad (*)$$

Denge denklemlerinden

$$\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = -\frac{\partial^2 \sigma_{11}}{\partial x_1^2} \quad ; \quad \frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = -\frac{\partial^2 \sigma_{22}}{\partial x_2^2}$$

yazarsak

$$2\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = -\frac{\partial^2 \sigma_{11}}{\partial x_1^2} - \frac{\partial^2 \sigma_{22}}{\partial x_2^2} \quad (9)$$

buluruz. (9) ifadesini (*) 'de yerine yazarsak

$$\frac{\partial^2}{\partial x_2^2}\left(a\sigma_{11} - b\sigma_{22}\right) + \frac{\partial^2}{\partial x_1^2}\left(-b\sigma_{11} + a\sigma_{22}\right) =$$

$$= -\frac{1}{2G}\cancel{(\sigma_{11}+\sigma_{22})}\left(\frac{\partial \sigma_{11}}{\partial x_1^2} + \frac{\partial^2 \sigma_{22}}{\partial x_2^2}\right) \quad (10)$$

buluruz

a ve b'nin $1/E$ ve ν/E ifadelerinde

$1/2G$ 'nin $\frac{1+\nu}{E}$ ifadesini (10)'de yazarsak

onda

$$\frac{1}{E}\frac{\partial^2\sigma_{11}}{\partial x_2^2}-\frac{\nu}{E}\frac{\partial^2\sigma_{22}}{\partial x_2^2}-\frac{\nu}{E}\frac{\partial^2\sigma_{11}}{\partial x_1^2}+\frac{1}{E}\frac{\partial^2\sigma_{22}}{\partial x_1^2}=$$

$$=-\frac{2(1+\nu)}{E}\cancel{\Delta(\sigma_{11}+\sigma_{22})}\left(\frac{\partial^2\sigma_{11}}{\partial x_1^2}+\frac{\partial^2\sigma_{22}}{\partial x_2^2}\right)$$

$$\frac{1}{E}\left(\frac{\partial^2\sigma_{11}}{\partial x_2^2}+\frac{\partial^2\sigma_{22}}{\partial x_1^2}\right)-\frac{\nu}{E}\left(\frac{\partial^2\sigma_{22}}{\partial x_2^2}+\frac{\partial^2\sigma_{11}}{\partial x_1^2}\right)=$$

$$\cancel{=-\frac{1+\nu}{E}\left[\frac{\partial^2}{\partial x_1^2}(\sigma_{11}+\sigma_{22})+\frac{\partial^2}{\partial x_2^2}(\sigma_{11}+\sigma_{22})\right]}$$

$$=-\frac{2(1+\nu)}{E}\left(\frac{\partial^2\sigma_{11}}{\partial x_1^2}+\frac{\partial^2\sigma_{22}}{\partial x_2^2}\right)$$

$$\frac{\partial^2\sigma_{11}}{\partial x_2^2}+\frac{\partial^2\sigma_{22}}{\partial x_1^2}-\cancel{\nu\frac{\partial^2\sigma_{22}}{\partial x_2^2}}-\cancel{\nu\frac{\partial^2\sigma_{11}}{\partial x_1^2}}+\cancel{2}(1+\cancel{\nu})\frac{\partial^2\sigma_{11}}{\partial x_1^2}+$$

$$+\cancel{2}(1+\cancel{\nu})\frac{\partial^2\sigma_{22}}{\partial x_2^2}=0$$

$$\frac{\partial^2}{\partial x_2^2}\sigma_{11}+\frac{\partial^2}{\partial x_1^2}\sigma_{11}+\frac{\partial^2\sigma_{22}}{\partial x_1^2}+\frac{\partial^2\sigma_{22}}{\partial x_2^2}=0$$

$$\Delta\sigma_{11}+\Delta\sigma_{22}=0\Rightarrow$$

$\boxed{\Delta(\sigma_{11}+\sigma_{22})=0}$! denklemini buluruz (11)

Düzlem şekildeğiştirme durumunda (11) denkleminin elde edilmesi ödev olarak öğrencilere verilir.

Gerilme fonksiyonu.

gerilmelere göre denklemleri yazalım

denge denklemleri

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}=0;\quad \frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}=0$$

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)(\sigma_{11}+\sigma_{22})=0 \qquad (12)$$

gerilme fonksiyonu adlandırılan ϕ fonksiyonu ile gerilmeleri

$$\sigma_{11}=\frac{\partial^2\phi}{\partial x_2^2},\quad \sigma_{22}=\frac{\partial^2\phi}{\partial x_1^2},\quad \sigma_{12}=-\frac{\partial^2\phi}{\partial x_1\partial x_2} \qquad (13)$$

biçiminde ~~yazarsak~~ yazarsak, onda (12)'deki denge denklemleri otomatikman sağlanır:

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}=0\Rightarrow\frac{\partial}{\partial x_1}\left(\frac{\partial^2\phi}{\partial x_2^2}\right)-\frac{\partial}{\partial x_2}\left(\frac{\partial^2\phi}{\partial x_1\partial x_2}\right)=0;$$

$$\frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}=0\Rightarrow-\frac{\partial}{\partial x_1}\left(\frac{\partial^2\phi}{\partial x_1\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial^2\phi}{\partial x_1^2}\right)=0; \quad (14)$$

(12)'deki sonuncu denklem ise

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\left(\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_1^2}\right)=\frac{\partial^4\phi}{\partial x_1^4}+2\frac{\partial^4\phi}{\partial x_1^2\partial x_2^2}+\frac{\partial^4\phi}{\partial x_2^4}=0$$

yani,

$$\boxed{\frac{\partial^4\phi}{\partial x_1^4}+2\frac{\partial^4\phi}{\partial x_1^2\partial x_2^2}+\frac{\partial^4\phi}{\partial x_2^4}=0} \qquad (15)$$

biçiminde olur

Bu denkleme bi-harmonik denklem denir.

Kartezyen koordinatlarda ikiölçülü problemlerin çözümleri

Polinomlarla çözüm

Şimdi işlemlerin kolaylığı açısından

$$x_1 = x, \quad x_2 = y \qquad (16)$$

$$\sigma_{11} = \sigma_x \; ; \; \sigma_{22} = \sigma_y \; , \; \sigma_{12} = \tau_{xy}$$

işaretlemelerini kullanalım ve (15) denklemini

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \qquad (17)$$

biçiminde yazalım ve gerilmeleri $\sigma_x = \frac{\partial^2 \phi}{\partial y^2}$, $\sigma_y = \frac{\partial^2 \phi}{\partial x^2}$, $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$ formülleri ile bulalım.

(17) 'denkleminin polinom çözümü denildiğinde ϕ gerilme fonksiyonunu

$$\phi = \sum_{n,m=0}^{\infty} A_{nm} x^n y^m \qquad (18)$$

biçiminde ~~olan~~ polinom olarak gösterip, A_{nm} bilinmeyen katsayıları (17) denkleminden ve uygun sınır koşullarından bulmak demektir.

Şimdi (18)'den yararlanarak özel hallere bakalım ve $n = m = 0$, $A_{nm} = 0 \; n \geq 1, \; m \geq 1$ durumunda

$\phi = A_{00}$ olur. $\phi = A_{00}$ (17) denklemini sağlıyor, yalnız bu çözümden elde edilen gerilmeler sıfır olur: $\sigma_x = 0$, $\sigma_y = 0$; $\tau_{xy} = 0$ ve bu yüzden pratik önem taşımıyor. Aynı durum

$$\phi = A_{00} + A_{10}x + A_{01}y$$

durum için de geçerlidir, yani bu durumda da $\sigma_x = \sigma_y = \tau_{xy} = 0$ olur.

Ondan dolayı,

$$\phi = A_{20}x^2 + A_{11}xy + A_{02}y^2$$

kuadratik polinomdan başlayalım. Basitlik olsun diye,

$$A_{20} = \frac{a_2}{2}, \quad A_{11} = b_2; \quad A_{02} = \frac{c_2}{2} \qquad (19)$$

işaretlemelerini kabul edelim ve yukarıdaki polinomu

$$\phi = \frac{a_2}{2}x^2 + b_2xy + \frac{c_2}{2}y^2 \qquad (20)$$

biçiminde yazalım.

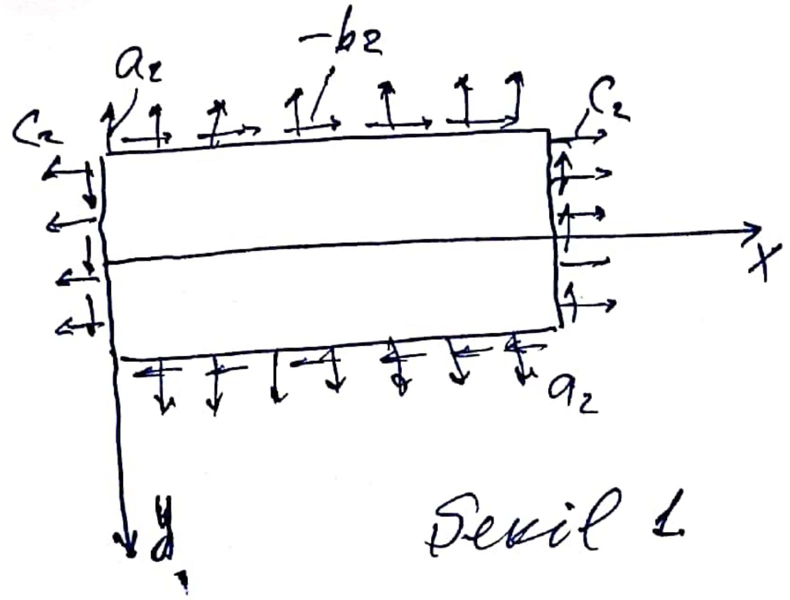
$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\phi}{\partial x^2} \quad \text{ve} \quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \qquad (21)$$

formüllerinden

$$\sigma_x = c_2; \quad \sigma_y = a_2 \quad \text{ve} \quad \tau_{xy} = -b_2 \qquad (22)$$

buluruz. Hatırlatalım ki, (20) (ve ya (19)) polinomu (17) denklemlerini otomatikmen sağlıyor.

Şimdi (22) gerilmelerinin uygulamasını bir dörtgen şeritde gösterelim. Bu şerit Şekil 1-de verilir.



Şekil 1

üçüncü mertebeden polinoma bakalım

$$\Phi_3 = A_{03}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3 \quad (23)$$

Basitlik için

$$A_{03} = \frac{a_3}{3\cdot 2}, \quad A_{21} = \frac{b_3}{2}, \quad A_{12} = \frac{c_3}{2}xy^2; \quad A_{03} = \frac{d_3}{3\cdot 2} \quad (24)$$

işaretlemelerini kabul edelim ve (23)-ü

$$\Phi_3 = \frac{a_3}{3\cdot 2}x^3 + \frac{b_3}{2}x^2y + \frac{c_3}{2}xy^2 + \frac{d_3}{3\cdot 2}y^3 \quad (25)$$

biçiminde yazalım. (25) polinomu (17) denklemini otomatikmen sağlıyor ve (21) den

$$\sigma_x = c_3x + d_3y, \quad \sigma_y = a_3x + b_3y$$

$$\tau_{xy} = -b_3x - c_3y \quad (26)$$

gerilmelerini buluruz.