equation are expanded in terms of central differences, since the accuracy of these expansions is greater than the accuracy obtainable by lateral differences.

The derivatives involved in the boundary conditions of the problem may be expressed in terms of lateral or of central differences. For example, the following boundary conditions at the origin are translated by means of Eqs. (2.7.16) into the corresponding central difference conditions by using the first term of their expansions:

$$\begin{aligned} y(0) &= 0; \quad & y_0 &= 0; \\ y'(0) &= 0; \quad & y_1 - y_{-1} &= 0; \\ y''(0) &= 0; \quad & y_1 - 2y_0 + y_{-1} &= 0; \\ y'''(0) &= 0; \quad & y_2 - 2y_1 + 2y_{-1} - y_{-2} &= 0; \\ y^{iv}(0) &= 0; \quad & y_2 - 4y_1 + 6y_0 - 4y_{-1} + y_{-2} &= 0. \end{aligned} \tag{4.1.1}$$

These equations are actually used to define the values y_{-1} and y_{-2}, which lie beyond the interval of definition of y, in terms of y_0, y_1, y_2, and have errors of order h^2.

When central differences of order h^2 are used in the equations, and either forward or backward differences are used in the boundary conditions, these last should also have errors of order h^2, whenever possible. Hence, for example, the conditions of Eqs. (4.1.1) should be expressed in terms of the difference operators of Fig. 2.5b:

$$\begin{aligned} y(0) &= 0; \quad & y_0 &= 0; \\ y'(0) &= 0; \quad & -y_2 + 4y_1 - 3y_0 &= 0; \\ y''(0) &= 0; \quad & -y_3 + 4y_2 - 5y_1 + 2y_0 &= 0; \\ y'''(0) &= 0; \quad & -3y_4 + 14y_3 - 24y_2 + 18y_1 - 5y_0 &= 0; \\ y^{iv}(0) &= 0; \quad & -2y_5 + 11y_4 - 24y_3 + 26y_2 - 14y_1 + 3y_0 &= 0. \end{aligned} \tag{4.1.2}$$

4.2 Step-by-step Integration of Boundary Value Problems

Ordinary boundary value problems of the second order can be solved by the methods of Chapter III, that is, by step-by-step forward integration formulas, in conjunction with trial and error and/or interpolation procedures.

Consider, for example, the simple problem

$$y'' + y^2 = 0; \qquad y(0) = 2; \qquad y(1) = 0. \tag{a}$$

Multiplying the equation by h^2 and using central differences, we obtain the recurrence equation

$$y_{i+1} = 2y_i - h^2y_i^2 - y_{i-1}, \tag{b}$$

which, for example, for $h = \frac{1}{4}$ becomes

$$y_{i+1} = 2y_i - \left(\frac{y_i}{4}\right)^2 - y_{i-1}. \tag{c}$$

To start the solution *assume* a value $y_1 = 1.5$ and integrate forward, as shown in the third column of Table 4.1. With $y_1 = 2.0$ we obtain the results given in the fourth column of Table 4.1. A linear interpolation between $y_1 = 1.5$, $y_4 = -0.52$, and $y_1 = 2.0$, $y_4 = 0.76$ gives [Eq. (1.2.7)]

Table 4.1

i	x_i \ y_i	$y_1 = 1.5$	$y_1 = 2.0$	$y_1 = 1.70$	$y_1 = 1.6825$
0	0	2.00	2.00	2.00	2.0000
1	0.25	1.50	2.00	1.70	1.6825
2	0.50	0.86	1.75	1.22	1.1881
3	0.75	0.17	1.31	0.65	0.6055
4	1.00	−0.52	0.76	0.05	0.0000

$$y_1 = \frac{1.5 \times 0.76 - 2.0 \times (-0.52)}{0.76 - (-0.52)} = 1.70$$

and with $y_1 = 1.70$, Eq. (c) gives the results of the fifth column of Table 4.1. Interpolating between 1.5 and 1.70,

$$y_1 = \frac{1.5 \times 0.05 - 1.70(-0.52)}{0.05 - (-0.52)} = 1.6825$$

and with $y_1 = 1.6825$ the y_i of column 6 in Table 4.1 are the correct answer for $n = 4$.

Similar procedures may be used to solve higher-order equations, using as parameters the first few ordinates y_i and computing the errors in the boundary conditions at the far end of the interval of integration. This method of solution becomes cumbersome as soon as the order of the equation is higher than third or fourth.

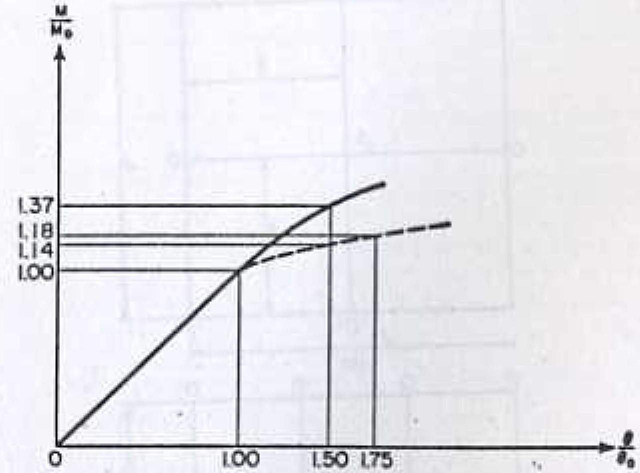


Fig. 5.26. Torque vs. twist in plastic torsion of square section.

Thus the elastic and the plastic torques are proportional to V and

$$\frac{M_1}{M_0} = \frac{V_1}{V_0} = \frac{779.1}{685.8} = 1.136 \qquad \text{for } \frac{\theta_1}{\theta_0} = 1.5$$

$$\frac{M_2}{M_0} = \frac{V_2}{V_0} = \frac{810.3}{685.8} = 1.182 \qquad \text{for } \frac{\theta_2}{\theta_0} = 1.75$$

Fig. 5.26 gives the graph of M/M_0 versus θ/θ_0 for

$$\left|\frac{\partial\phi}{\partial\xi}\right|_{max} = \frac{855}{1/6} = 5130 \quad \text{(broken line)}$$

and for

$$\left|\frac{\partial\phi}{\partial\xi}\right|_{max} = \frac{1106}{1/6} = 6636 \quad \text{(continuous line)},$$

that is, for the more accurate value of the slope at A obtained by Table 5.8. It is seen that the relation between torque and twist is nonlinear in the plastic range and that the numerical approximation of the boundary slope influences considerably the results.

5.9 *Membrane Vibrations*

The technique used in the preceding section to solve torsion problems may be used to solve two-dimensional vibration problems.

The differential equation for the free vibrations of a membrane is obtained by adding to the equilibrium equation (5.6.1) the inertia forces

$-m\frac{\partial^2 z}{\partial t^2}$, where m is the membrane mass per unit of area, and by setting the external pressure p equal to zero:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - \frac{m}{S}\frac{\partial^2 z}{\partial t^2} = 0. \tag{5.9.1}$$

To find the natural frequencies ω of a membrane, the function $z(x,y,t)$ is assumed to represent a harmonic vibration:

$$z(x,y,t) = Z(x,y)\sin\omega t \tag{a}$$

and is substituted in Eq. (5.9.1), which becomes, after division by $\sin\omega t$,

$$\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} + \frac{m\omega^2}{S}Z = 0. \tag{5.9.2}$$

To solve the problem in nondimensional form for the case of a square membrane of side L supported on a flat boundary, the usual transformation

$$x = \xi L; \quad y = \eta L \tag{b}$$

is introduced in Eq. (5.9.2), which thus becomes

$$\frac{\partial^2 Z}{\partial \xi^2} + \frac{\partial^2 Z}{\partial \eta^2} + KZ = 0, \tag{5.9.3}$$

with

$$K = \frac{mL^2}{S}\omega^2, \tag{5.9.4}$$

and Eq. (5.9.3) is then transformed into a finite difference equation by the operator of Fig. 5.3b with $h^2 = 1/n^2$:

$$Z_a + Z_b + Z_c + Z_d + \left(\frac{K_n}{n^2} - 4\right)Z_e = 0. \tag{5.9.5}$$

The conditions on the boundary require that

$$Z = 0 \quad \text{on the boundary.} \tag{5.9.6}$$

For $n = 2$ and 3 (Fig. 5.27a, b), Eq. (5.9.5) gives

$n = 2$; $\quad \left(\frac{K_2}{4} - 4\right)Z_1 = 0; \quad K_2 = 16 \quad (e = +19\%)$

$n = 3$; $\quad Z_1 + 0 + Z_1 + 0 + \left(\frac{K_3}{9} - 4\right)Z_1 = 0; \quad K_3 = 18$

$(e = +9\%)$

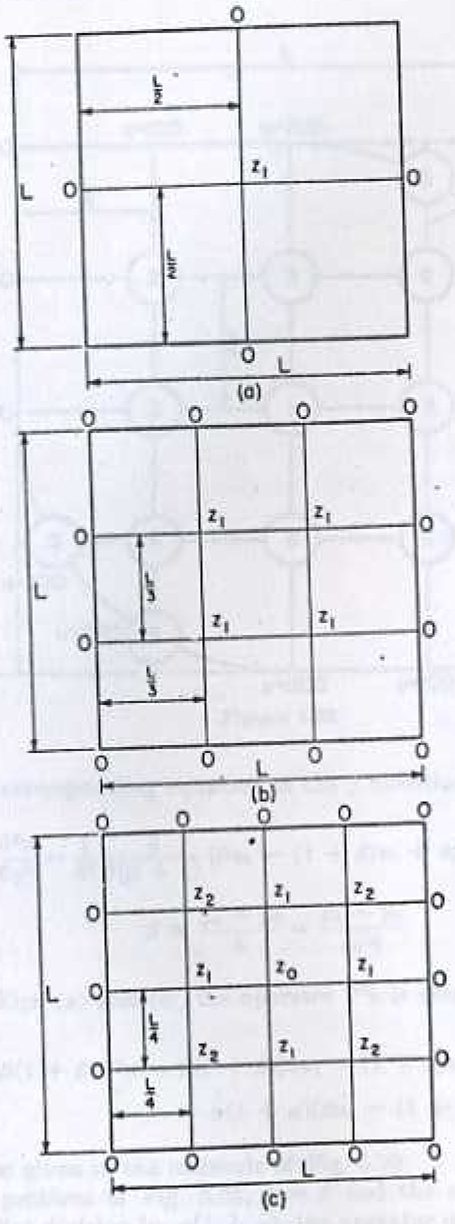


Figure 5.27

The determinantal equation for $n = 4$ (Fig. 5.27c) is

$$\begin{vmatrix} \frac{K_4}{16} - 4 & 2 & 0 \\ 2 & \frac{K_4}{16} - 4 & 1 \\ 0 & 4 & \frac{K_4}{16} - 4 \end{vmatrix} = 0$$

and its smallest root is

$$K_4 = 18.75 \quad (e = +5\%).$$

The extrapolated values of K are

$$K_{2,3} = 19.60 \quad (+0.7\%); \qquad K_{3,4} = 19.71 \quad (+0.15\%);$$
$$K_{2,3,4} = 19.75 \quad (-0.051\%).$$

The true value of K is 19.739.

5.10 *Pivotal Points Near Curved Boundaries*

In all the problems of the preceding sections the pivotal points inside the rectangular domains and on their boundaries fell on the corners of a rectangular lattice and were evenly spaced in the x- and the y-directions. When a two-dimensional domain covered by a rectangular lattice is bounded by curves, instead, some or all of its pivotal boundary points do not fall on the corners of the lattice, and special formulas must be used at pivotal points adjoining the boundaries.

Consider, for example, the steady-state temperature problem of the plate of Fig. 5.28, a square plate of sides L with two corners rounded off by arcs of circle of radius $L/2$. The temperature u in the plate satisfies the Laplacian equation $\nabla^2 u = 0$ (Sec. 5.4). This equation can be transformed into a difference equation by the operator of Fig. 5.3b at points (1), (2), and (3), but point (4) is not evenly spaced from the adjoining pivotal points and must be dealt with by means of a special equation.

The difference equation $\nabla^2 u = 0$ at the upper point (4) may be obtained, for example, by means of Eq. (2.2:3), which in the present case becomes

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}\frac{2}{\alpha(\alpha + 1)}\left[\alpha u_l - (1 + \alpha)u_c + u_r\right] + 0(h); \tag{a}$$

$$\alpha = \frac{x_r - x_c}{h} = \frac{x_5 - x_4}{L/4}, \tag{b}$$

5.19 A Boundary Value Problem Involving $\nabla^4 z$

The boundary value problem for the deflections w of a square plate of sides a, built in all around, is given by*
encastre

$$\nabla^4 w = \frac{q}{D}; \qquad w = \frac{\partial w}{\partial n} = 0 \qquad \text{on the boundary,} \tag{5.19.1}$$

where q = load per unit area of plate,

$D = \dfrac{Eh^3}{12(1-\nu^2)}$ = flexural rigidity of plate,

h = plate thickness,

E, ν = Young's modulus and Poisson's ratio, respectively, of the plate material,

n = direction of the normal to the boundary.

The problem (5.19.1) is reduced to nondimensional form by the transformation

$$x = \xi a; \qquad y = \eta a; \qquad w(x,y) = \frac{qa^4}{D} z(\xi,\eta), \tag{a}$$

giving

$$\nabla^4 z = 1; \qquad z = \frac{\partial z}{\partial n} = 0 \qquad \text{on the boundary,} \tag{5.19.2}$$

where the operator ∇^4 is taken with respect to ξ and η.

The numerical solution of the problem (5.19.2) is obtained by substituting for $\nabla^4 z$ the central difference operator of Fig. 5.3c with a square lattice of mesh size $1/n$.

The boundary conditions, by Eq. (4.1.1), state that the values of z at pivotal points immediately outside the boundary are equal to the values of z at points immediately inside the boundary on the same normal.

Starting with $n = 2$ (Fig. 5.55), Eq. (5.19.2) gives

$$(z_0 + z_0 + z_0 + z_0) + 2 \cdot 0 - 8 \cdot 0 + 20 z_0 = \frac{1}{2^4},$$

from which

$$z_0\Big]_2 = \frac{1}{2^4 \cdot 24} = \frac{1}{384}.$$

For $n = 4$, Eq. (5.19.2) applied at the points of Fig. 5.56 gives

$$\text{at} \quad (0) \qquad 20 z_0 - 32 z_1 + 8 z_2 = \frac{1}{4^4};$$

* See, for example, S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, p. 197.

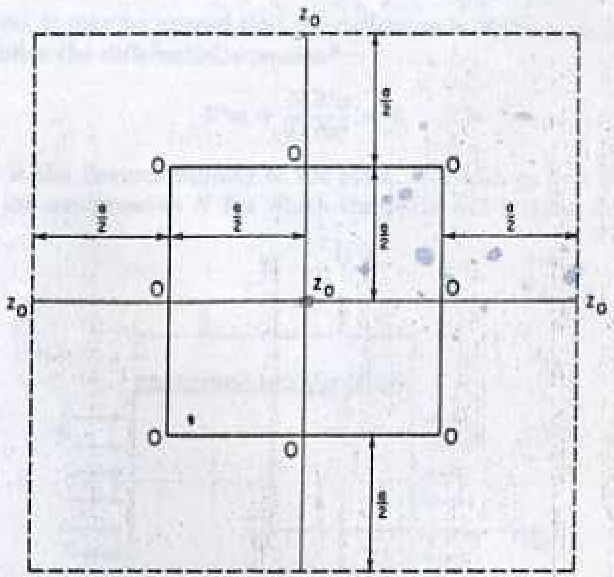


Figure 5.55

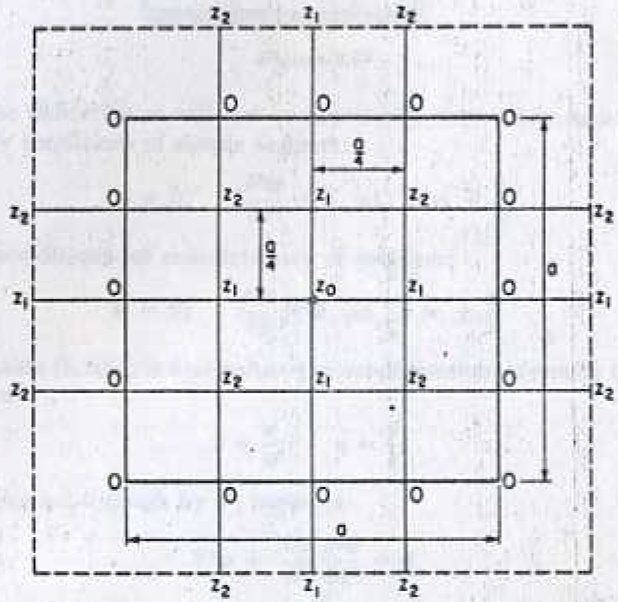


Figure 5.56

at (1) $$-8z_0 + 26z_1 - 16z_2 = \frac{1}{4^4};$$

at (2) $$2z_0 - 16z_1 + 24z_2 = \frac{1}{4^4},$$

from which, for example, by Gauss's scheme,

$$z_0\Big]_4 = \frac{0.461}{4^4}; \qquad z_1\Big]_4 = \frac{0.309}{4^4}; \qquad z_2\Big]_4 = \frac{0.209}{4^4}.$$

For $n = 8$, it is similarly found that $z_0\Big]_8 = 5.857/8^4$.

The deflection at the center of the plate for $\nu = 0.3$ and with $n = 2$, 4, 8 becomes, by Eq. (a),

$$w_0\Big]_2 = \frac{12(1-0.3^2)}{384}\frac{qa^4}{Eh^3} = 0.0284\frac{qa^4}{Eh^3} \qquad (e = -106\%),$$

$$w_0\Big]_4 = \frac{12(1-0.3^2)\cdot 0.461}{256}\frac{qa^4}{Eh^3} = 0.0197\frac{qa^4}{Eh^3} \qquad (e = -43\%),$$

$$w_0\Big]_8 = \frac{12(1-0.3^2)\cdot 5.857}{4096}\frac{qa^4}{Eh^3} = 0.0156\frac{qa^4}{Eh^3} \qquad (e = -13\%).$$

The percentage errors e are computed from the series solution value ($w_0 = 0.0138qa^4/Eh^3$) given by Timoshenko.* Extrapolations of the h^2-type can be used on the approximate values (see Sec. 2.13) and give

$$w_0\Big]_{2,4} = 0.0167\frac{qa^4}{Eh^3} \qquad (e = -21\%);$$

$$w_0\Big]_{4,8} = 0.0142\frac{qa^4}{Eh^3} \qquad (e = -2.9\%),$$

$$w_0\Big]_{2,4,8} = 0.0140\frac{qa^4}{Eh^3} \qquad (e = -1.4\%).$$

5.20 Two-dimensional Characteristic Value Problems

Consider a square plate of side a, simply supported along two opposite edges and built in along the other two edges. The plate is acted upon by a uniform compressive force N per unit of length, perpendicular to the simply supported edges and acting in the plane of the plate (Fig. 5.57). Choosing the x-axis parallel to the built-in edges, with origin at the center

* See S. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, p. 202.

of the plate, it may be proved that the deflection w of the plate at a point (x,y) satisfies the differential equation*

$$\nabla^4 w + \frac{N}{D}\frac{\partial^2 w}{\partial x^2} = 0, \tag{5.20.1}$$

where D is the flexural rigidity of the plate. We wish to find the lowest value of the compression N for which the plate will buckle, that is, for

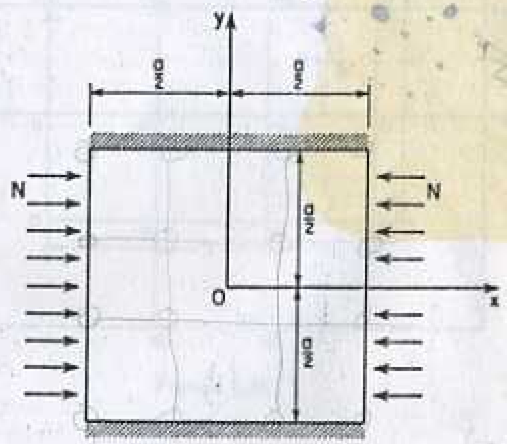


Figure 5.57

which the deflection w will not be identically zero, while satisfying the boundary conditions of simple support:

$$w = 0; \qquad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = \pm\frac{a}{2} \tag{5.20.2}$$

and the conditions for complete lack of rotation:

$$w = 0; \qquad \frac{\partial w}{\partial y} = 0 \quad \text{at} \quad y = \pm\frac{a}{2} \tag{5.20.3}$$

Equation (5.20.1) is first reduced to nondimensional form by the transformation

$$\xi = \frac{x}{a}; \qquad \eta = \frac{y}{a} \tag{a}$$

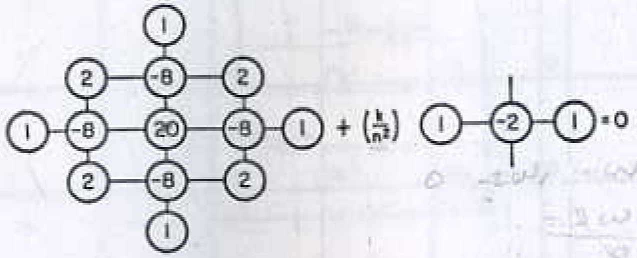
and, multiplied through by a^4, becomes

$$\nabla^4 w + \frac{Na^2}{D}\frac{\partial^2 w}{\partial \xi^2} = 0, \tag{5.20.4}$$

* See, for instance, S. Timoshenko, *Theory of Elastic Stability*, p. 337.

$h^4 \nabla^4 w$ T.

where $\nabla^4 w$ is taken with respect to ξ and η. The plate is then covered with a square lattice of mesh size $1/n$. Multiplying the equation through by $h^4 = 1/n^4$, and using the operator of Fig. 5.3c for $h^4\nabla^4 w$ and the operator of Eq. (5.2.2) for $\partial^2 w/\partial\xi^2$, the difference equation corresponding to Eq. (5.20.4) is obtained in molecular form:


(5.20.5)

where

$$k = \frac{Na^2}{D}. \tag{5.20.6}$$

The application of Eq. (5.20.5) at the internal pivotal points gives rise to a set of homogeneous linear algebraic equations in the unknown pivotal displacements w_i, whose determinant must be identically zero if the w_i are to be different from zero. The lowest value of N making the determinant of Eqs. (5.20.5) equal to zero is the lowest critical value of the compressive force.

In order to apply Eq. (5.20.5) at the pivotal points immediately inside the boundary, the deflection must be known at fictitious pivotal points immediately *outside* the boundary. These deflections are given in terms of the deflections at the pivotal points immediately inside the boundary by the boundary conditions. In fact, by Eqs. (4.1.1), Eqs. (5.20.2) become

$$w_i = 0; \qquad w_l = -w_r \qquad \text{at } \xi = \pm\tfrac{1}{2}, \tag{5.20.7}$$

while Eqs. (5.20.3) give

$$w_i = 0; \qquad w_a = w_b \qquad \text{at } \eta = \pm\tfrac{1}{2}. \tag{5.20.8}$$

Starting with $n = 3$ and the antisymmetrical deflections of Fig. 5.58,* we obtain

$$(-w_1 + w_1 + 0 + 0) + 2(0 + 0 + 0 - w_1) - 8(0 + 0 + w_1 - w_1)$$
$$+ 20w_1 + \frac{k_1}{9}(0 - 2w_1 - w_1) = 0,$$

* Symmetrical deflections in the x direction give a higher buckling load.

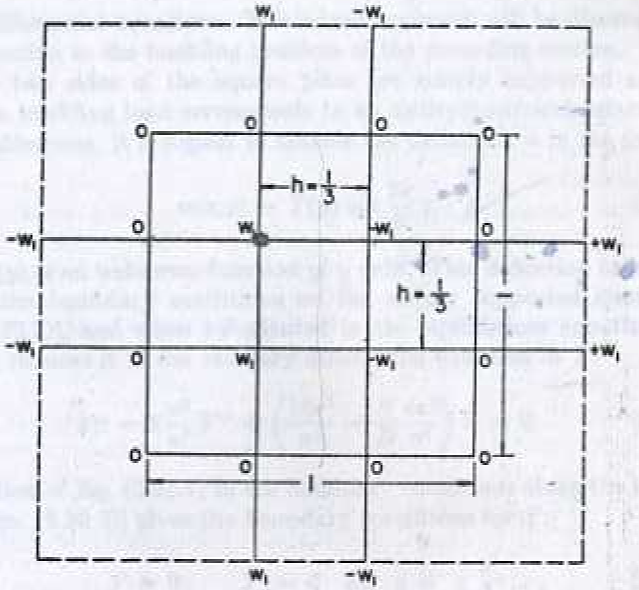


Figure 5.58

$2w_1$, $-2w_1$

or

$$w_1\left(18 - \frac{k_1}{3}\right) = 0,$$

from which

$$k_1 = 54 = 5.471\pi^2. \tag{b}$$

With $n = 4$ and the antisymmetrical deflections of Fig. 5.59, we obtain the four equations of Table 5.17, in which

$$\gamma = \frac{k_4}{16}.$$

Table 5.17

w_1	w_2	w_3	w_4
$20 - 2\gamma$	-16	0	0
-8	$22 - 2\gamma$	0	0
2	$\gamma - 8$	$20 - 2\gamma$	-8
$\gamma - 8$	4	-16	$18 - 2\gamma$

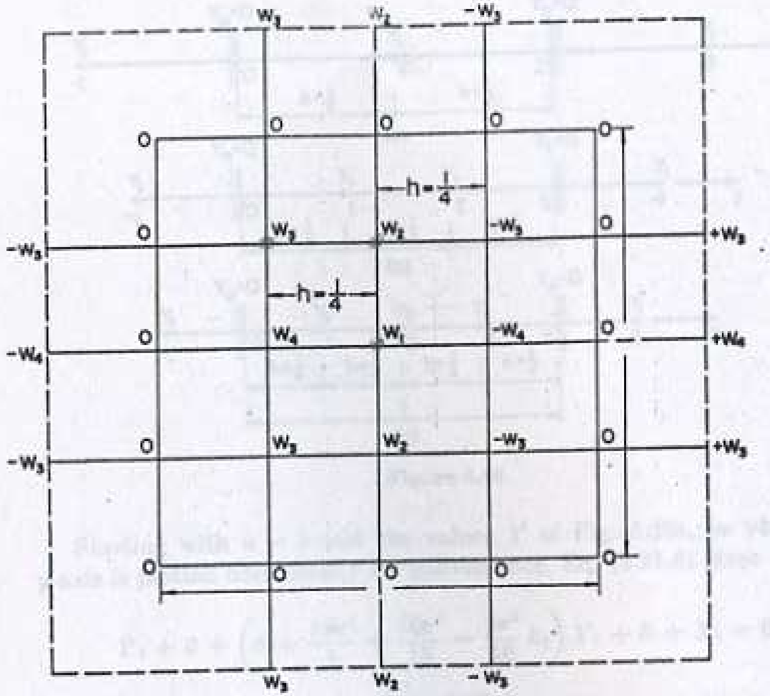


Figure 5.59

Setting the determinant of this set of equations equal to zero,

$$(20 - 2\gamma)[(20 - 2\gamma)(22 - 2\gamma) - 8 \times 16][(20 - 2\gamma)(18 - 2\gamma) - 8 \times 16] = 0,$$

we obtain from its third factor the lowest root, $\gamma = 3.821$, and hence,

$$k_4 = 61.136 = 6.193\pi^2. \quad \text{(c)}$$

The extrapolated value $k_{2,4} = 7.121\pi^2$ is slightly higher than the minimum value of k given by Timoshenko* (which occurs for a plate with ratio of sides a/b between 0.6 and 0.7 and a symmetrical deflection) and hence cannot differ appreciably from the true value of k.

5.21 *The Solution of Partial Differential Equations by Separation of the Variables and Finite Differences*

The classical method of the separation of the variables and modern numerical procedures may often be advantageously combined to solve partial differential equations. This mixed approach will be illustrated by its application to the buckling problem of the preceding section.

* See S. Timoshenko, *Theory of Elastic Stability*, p. 345.

Since two sides of the square plate are simply supported and the minimum buckling load corresponds to an antisymmetrical deformation in the x-direction, it is logical to assume the deflection w in the form

$$w(x,y) = Y(y) \sin \frac{2\pi}{a} x, \quad (5.21.1)$$

where $Y(y)$ is an unknown function of y only. This deflection function w satisfies the boundary conditions on the simply supported boundaries [Eqs. (5.20.2)], and when substituted in the equilibrium equation [Eq. (5.20.1)], reduces it to the ordinary differential equation in Y:

$$Y^{iv} - 8\frac{\pi^2}{a^2} Y'' + \left(\frac{16\pi^4}{a^4} - \frac{N}{D}\frac{4\pi^2}{a^2}\right) Y = 0. \quad (5.21.2)$$

Substitution of Eq. (5.21.1) in the boundary conditions along the built-in edges [Eqs. (5.20.3)] gives the boundary conditions for Y:

$$Y = 0; \quad Y' = 0 \quad \text{at} \quad y = \pm \frac{a}{2}. \quad (5.21.3)$$

The partial characteristic value problem of Eqs. (5.20.1), (5.20.2), and (5.20.3) has thus been reduced to the ordinary characteristic value problem of Eqs. (5.21.2) and (5.21.3).

To solve this last problem in nondimensional form, we use the transformation $y = a\eta$, obtaining

$$\frac{d^4Y}{d\eta^4} - 8\pi^2\frac{d^2Y}{d\eta^2} + \left(16\pi^4 - 4\pi^2\frac{Na^2}{D}\right) Y = 0; \quad (5.21.4)$$

$$Y(\eta) = 0; \quad \frac{dY}{d\eta} = 0 \quad \text{at} \quad \eta = \pm\tfrac{1}{2}. \quad (5.21.5)$$

Multiplying Eq. (5.21.4) by $h^4 = 1/n^4$ and substituting central difference expressions for the derivatives of Y [Eqs. (2.7.16)], we obtain the difference equation in Y:

$$Y_{bb} - \left(\frac{8\pi^2}{n^2} + 4\right) Y_b + \left(6 + \frac{16\pi^2}{n^2} + \frac{16\pi^4}{n^4} - \frac{4\pi^2 k}{n^4}\right) Y_c - \left(\frac{8\pi^2}{n^2} + 4\right) Y_a + Y_{aa} = 0, \quad (5.21.6)$$

where k is given by Eq. (5.20.6). The boundary conditions require that

$$Y_a = Y_b \quad \text{at} \quad \eta = \pm\tfrac{1}{2}. \quad (5.21.7)$$