= SONLU FARKLAR YÖNTEMİ =

21. 1. 1992

Bölgede. $\bar{\Omega} \times T$

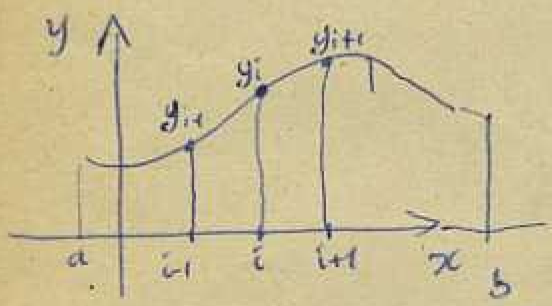
S. K. $\partial\Omega \times T$

B. K. $\Omega(t_0)$

S. F. Y. — Ayrık noktalardaki değktemlere indirgeniyor

Ayrık noktalardaki tüm değerler bilmen sabit değerlerdir.

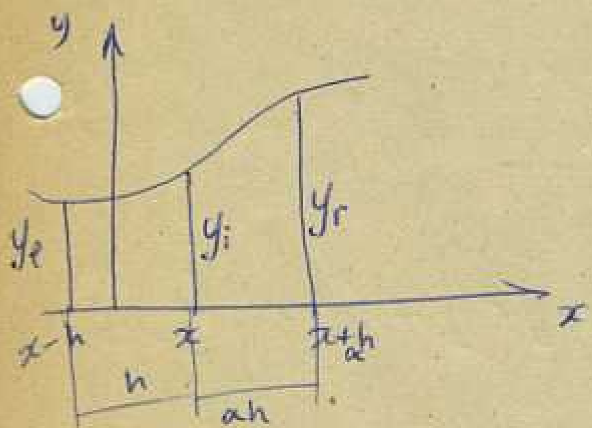
TÜREV



$$\left.\frac{dy(x)}{dx}\right|_{x=x_i} = ?$$

①

TAYLOR AÇILIMI



$$y(x+h) = y(x) + h\,y'(x) + \frac{h^2}{2!}y''(x) + \dots$$

$$= \sum_{n=0}^{\infty} \frac{h^n}{n!} y^{(n)}(x)$$

$0! = 1$

$1! = 1$

$$y^{(n)}(x) = \frac{dy^n}{dx^n}$$

$$y_r = y(x+\alpha h) = y_i + \alpha h\,y_i'(x) + \frac{\alpha^2 h^2}{2!}y_i''(x) + \frac{\alpha^3 h^3}{3!}y_i'''(x) + \dots$$

$$y_l = y(x-h) = y_i - h\,y_i'(x) + \frac{h^2}{2!}y_i''(x) - \frac{h^3}{3!}y_i'''(x)$$

$$y_r - y_l = (1+\alpha)h\,y_i' + \frac{h^2}{2!}(\alpha^2-1)y_i'' + \frac{h^3}{3!}(1+\alpha^3)y_i'''$$

$$y_i'(x) = \frac{y_r - y_i}{(1+\alpha)h} + \left| (1-\alpha)\frac{h}{2} y_i''(x) - (1-\alpha+\alpha^2)\frac{h^2}{3!} y_i'''(x) + \dots \right.$$

$$y_i'(x) = \frac{y_r - y_l}{(1+\alpha)h} \qquad e_t = (1-\alpha)\frac{h}{2} \cdot y_i'' - \frac{1+\alpha^3}{1+\alpha}\frac{h^2}{6} y_i''' + \dots$$

$$e_t = O(h) \qquad \alpha \neq 1$$

$$e_t = O(h^2) \qquad \alpha = 1$$

ikinci türevi elimine etsek (bu şekilde y_i' yü hesaplayalım)

$$y_r - \alpha^2 y_l = y_i(1-\alpha^2) + \alpha h(1+\alpha) y_i' + \dots$$

$$y_i' = \frac{y_r - \alpha^2 y_l - (1-\alpha^2) y_i}{\alpha(1+\alpha)h} - \frac{\alpha^3 - \alpha^2}{6\alpha(1+\alpha)} h^3 y_i''' + \dots$$

$$e_t = O(h^2) \qquad \alpha \neq 1$$

Eğer y_i' yü elimine etsek

$$y_i'' = \frac{1}{h^2}\frac{2}{\alpha(\alpha+1)}\left[\alpha y_l - (1+\alpha) y_i + y_r\right] + (1-\alpha)\frac{h}{3} y_i''' + \dots$$

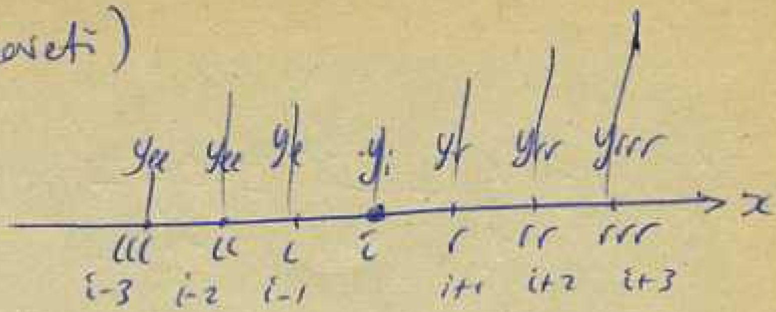
$$\alpha \neq 1 \qquad e_t = O(h)$$

$$\alpha = 1 \qquad e_t = O(h^2)$$

ARKADAN FARKLAR (türev)

∇ (arkadan fark işareti)

$\nabla y_i \triangleq y_i - y_{\iota}$



$$\nabla\nabla y_i = \nabla^2 y_i = \nabla(y_i - y_{\iota}) = y_i - y_{\iota} - (y_{\iota} - y_{\iota\iota}) = y_i - 2y_{\iota} + y_{\iota\iota}$$

$$\nabla^n y_i = \nabla\nabla^{n-1} y_i$$

$$\nabla^3 y_i = y_i - 3y_{i-1} + 3y_{i-2} - y_{i-3}$$

$$\nabla^4 y_i = y_i - 4y_{i-1} + 6y_{i-2} - 4y_{i-3} + y_{i-4}$$

Özellikler

$$\nabla(y_i + y_j) = \nabla y_i + \nabla y_j$$

$$\nabla(c y_i) = c\nabla y_i$$

$$\nabla^m(\nabla^n y_i) = \nabla^{m+n} y_i$$

Taylor açılımı

$$y(x+h) = y(x) + \frac{h}{1!} y'(x) + \cdots \qquad D \triangleq \frac{d}{dx}$$

$$= y(x) + \frac{h}{1!} Dy(x) + \frac{h^2}{2!} D^2 y(x) + \frac{h^3}{3!} D^3 y(x) + \cdots$$

$$y(x+h) = \left(1 + \frac{h}{1!}D + \frac{h^2}{2!}D^2 + \frac{h^3}{3!}D^3 + \cdots \right) y(x)$$

$$e^{\pm x} = 1 \pm \frac{x}{1!} + \frac{x^2}{2!} \pm \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots$$

$$y_r = y(x+h) = e^{hD} y(x)$$

$$y_r = e^{hD} y_i$$

$$y(x-h) = y_l = e^{-hD} y_i$$

$$\nabla y_i = y_i - y_l = \left[1 - e^{-hD}\right] y_i \Rightarrow \boxed{\nabla = 1 - e^{-hD}}$$

$$\nabla^2 = \nabla . \nabla = (\nabla)^2 = 1 - 2e^{-hD} + e^{-2hD}$$

$$= \cancel{1} + \left(\cancel{1} - \frac{2h}{1!}D + \frac{4h^2}{2!}D^2 - \frac{8h^3}{3!}D^3 + \frac{16h^4}{4!}D^4 - \cdots \right)$$

$$-2\left(1 - \frac{hD}{1!} + \frac{h^2 D^2}{2!} - \frac{h^3}{3!}D^3 + \cdots \right)$$

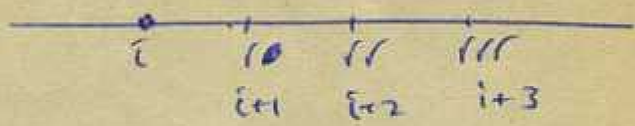
$$\nabla^2 = h^2 D^2 - h^3 D^3 + \frac{7}{12} h^4 D^4 - \cdots$$

$$\nabla^3 = \nabla^2 . \nabla = (\nabla)^3$$

ÖNDEN FARKLAR

Δ önden farklar operatörü

$\Delta y_i = y_I - y_i$



$\Delta^2 y_i = \Delta(\Delta y_i) = \Delta(y_I - y_i) = y_{II} - y_I - (y_I - y_i) = y_{II} - 2y_I + y_i$

$y_I = e^{hD} y_i$ $\qquad$ $y(x+h) = y_I = y_i(x) + \frac{h}{1!} y_i'(x) + \frac{h^2}{2!} y_i''$

$\Delta = e^{hD} - 1$ $\qquad$ $= \left(1 + \frac{hD}{1!} + \frac{h^2D^2}{2!} + \cdots \right) y_i(x)$

$e^{hD} = 1 + \Delta$ $\qquad$ $y_I = e^{hD} y_i$

$hD = \ln(1+\Delta)$

$hD = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots$

h^2D^2

	i	$i+1$	$i+2$	$i+3$	$i+4$	$e_t = O(h)$
hD	-1	+1				
$2hD$	-3	4	-1			
h^2D^2	1	-2	+1			
h^2D^2	2	-5	4	-1		
h^3D^3	-1	+3	-3	+1		
$2h^3D^3$	-5	18	-24	14	-3	
h^4D^4	1	-4	+6	-4	+1	
h^4D^4	3	-14	26	-24	11	-2

MERKEZSEL FARKLAR

δ operatörü ile gösteriyoruz.

$$\delta y_i \stackrel{\Delta}{=} y(x_i + \tfrac{h}{2}) - y(x_i - \tfrac{h}{2}).$$

$$\delta y_i = y_{i+1/2} - y_{i-1/2}$$

$$\delta^2 y_i = \delta \delta y_i = \delta (y_{i+1/2} - y_{i-1/2})$$

	$i-3$	$i-2$	$i-1$	i	$i+1$	$i+2$	$i+3$
$2hD$			-1	0	$+1$		
$12hD$			$\boxed{-8}$	$\boxed{0}$	$\boxed{8}$	$\boxed{-1}$	
h^2D^2			$+1$	-2	$+1$		
$12h^2D^2$		$\boxed{-1}$	$\boxed{16}$	$\boxed{-30}$	$\boxed{16}$	$\boxed{-1}$	
$2h^3D^3$		-1	$+2$	0	-2	$+1$	
$8h^3D^3$	$\boxed{+1}$	$\boxed{-8}$	$\boxed{+13}$	$\boxed{0}$	$\boxed{-13}$	$\boxed{8}$	$\boxed{-1}$
h^4D^4		$+1$	-4	6	-4	$+1$	
$6h^4D^4$	$\boxed{-1}$	$\boxed{12}$	$\boxed{-39}$	$\boxed{56}$	$\boxed{-39}$	$\boxed{12}$	$\boxed{-1}$

Hata h mertebesinde

Hata h^4 mertebesinde

İNTEGRAL

$$I_1 = \int_x^{x+h} f(x)\,dx = y \Big|_x^{x+h} = y(x+h) - y(x)$$

$f(x) = y'(x)$

$$= h\left(y_i'(x) + \frac{h}{2!} y_i''(x) + \frac{h^2}{3!} y_i'''(x) + \ldots \right.$$

WEDDLE FORMÜLÜ

$$\frac{3h}{10}\left(y_{i-3} + 5y_{i-2} + y_{i-1} + 6y_i + y_{i+1} + 5y_{i+2} + y_{i+3} \right) + O(h^8)$$

$$\nabla^3 = h^3D^3 - \frac{3}{2}h^4D^4 + \frac{5}{4}h^5D^5 - \dots$$

$$\nabla = 1 - e^{-hD} \Rightarrow e^{-hD} = 1 - \nabla \Rightarrow -hD = \ln(1-\nabla)$$

$$\ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots + (-1)^n \frac{x^n}{n}$$

$$-hD = \ln(1-\nabla) = -\left(\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots\right)$$

$$hD = \nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} + \frac{\nabla^4}{4} + \dots$$

$$h^2D^2 = \nabla^2 + \nabla^3 + \nabla^4\left(\frac{1}{4} + \frac{2}{3}\right) + \nabla^5\left(\frac{1}{4} + \frac{1}{6}\right) + \dots$$

$$h^3D^3 = \nabla^3 + \frac{3}{2}\nabla^4 + \frac{7}{4}\nabla^5 + \dots$$

$$h^4D^4 = \nabla^4 + 2\nabla^5 + \frac{17}{6}\nabla^6 +$$

$$D = \frac{\nabla}{h} + \frac{hD^2}{2} - \frac{h^3D^3}{6}$$

$$D^2 = \frac{\nabla^2}{h^2} + hD^3 -$$

$$D^3 = \frac{\nabla^3}{h^3} + \frac{3}{2}hD^4$$

$$Dy_i = \frac{\nabla}{h}(y_i) + \mathcal{O}(h)$$

$$Dy_i = \frac{y_i - y_{i-1}}{h} + \mathcal{O}(h)$$

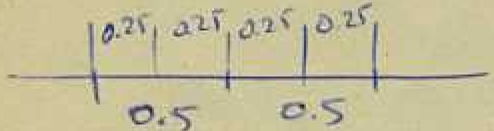
$$D^2 y_i = \frac{\nabla^2}{\cancel{y}h^2} y_i + \mathcal{O}(h) =$$

						i
hD					-1	$+1$
$2hD$				$\boxed{+1}$	$\boxed{-4}$	$\boxed{+3}$
h^2D^2				$+1$	-2	$+1$
			$\boxed{-1}$	$\boxed{+4}$	$\boxed{-5}$	$\boxed{+2}$
h^3D^3			-1	$+3$	-3	$+1$
$2h^3D^3$		$\boxed{+3}$	$\boxed{-14}$	$\boxed{+24}$	$\boxed{-16}$	$\boxed{3}$
h^4D^4		$+1$	-4	$+6$	-4	$+1$
	$\boxed{-2}$	$\boxed{+11}$	$\boxed{-24}$	$\boxed{26}$	$\boxed{-14}$	$\boxed{3}$

Hata mertebesini yükseltmek istesek $\mathcal{O}(h)$ dan $\downarrow$ $\mathcal{O}(h^2)$ ye daha çok terim kullanarak.

[Katsayıların toplamının sıfıra eşit olduğu görülüyor.]

Aynı hatayı elde etmek için iki yol var.

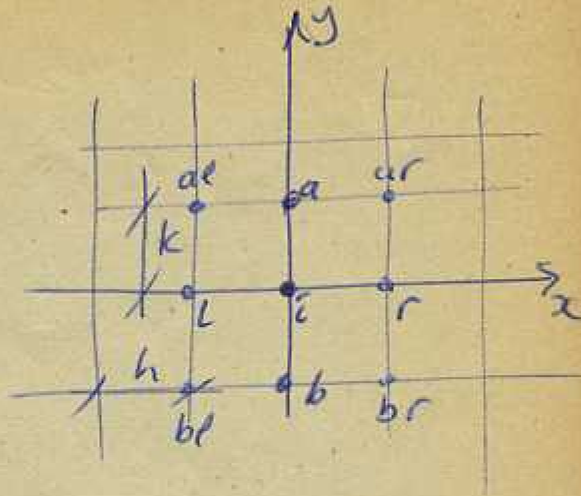


ya aralık küçük tutulur

ya da yüksek mertebeden operatörler kullanılabilir.

KISMİ TÜREV

Merkezsel farka göre



$$2h\,D_x z_i = z_r - z_l + \ldots$$

$$h^2 D_x^2 z_i = z_r - 2z_i + z_l$$

$$2h^3 D_x^3 z_i = z_{rr} - 2z_r + 2z_l - z_{ll}$$

$$h^4 D_x^4 z_i = z_{rr} - 4z_r + 6z_i - 4z_l + z_{ll}$$

$$2k\,D_y z_i = z_a - z_b$$

$$k^2 D_y^2 z_i = z_a - 2z_i + z_b$$

$$D_{xy} z_i = D_y(D_x z_i) = D_y\left(\frac{z_r - z_l}{2h}\right) = \frac{1}{2h}\left[\left(\frac{z_{ar} - z_{br}}{2k}\right) - \left(\frac{z_{al} - z_{bl}}{2k}\right)\right]$$

$$4hk\,D_{xy} z_i = z_{ar} - z_{al} - z_{br} + z_{bl}$$

özel bir durum olarak $h = k$ alınırsa

$$4h^2 D_{xy} \bullet \quad \begin{array}{ccc} -1 & & +1 \\ & 0 & \\ +1 & & -1 \end{array} \quad + h^2 O(h^2)$$

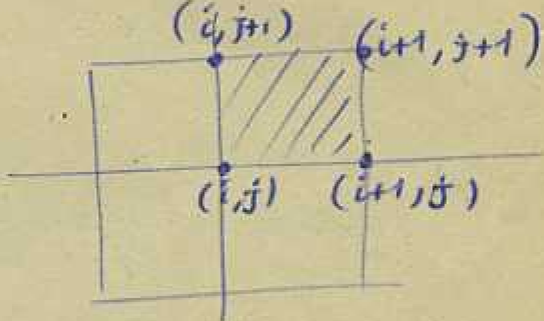
$$h^2\Big(\underbrace{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}_{\nabla^2}\Big) = h^2 \nabla^2 \qquad \begin{array}{ccc} & +1 & \\ +1 & -4 & +1 \\ & +1 & \end{array} \qquad h^2 O(h^2)$$

$$\nabla^4 \equiv \nabla^2(\nabla^2) = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y} + \frac{\partial^4}{\partial y^4}$$

$$h^4 \nabla^4 \qquad \begin{array}{ccccc} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{array} \qquad h^4(\Theta(h^2))$$

iki katlı integrasyon (hacim bulmak için)

$$A_1 = \int_{y_j}^{y_{j+1}} dy \int_{x_i}^{x_{i+1}} f(x,y)\, dx$$



$$= \frac{4}{hk} \int_a^b \int_c^d f(x,y)\, dx dy$$

$$\begin{array}{cccccc} 1 & 2 & 2 & 2 & \cdots & 1 \\ 2 & 4 & 4 & 4 & & \\ 2 & 4 & 4 & 4 & & \\ \vdots & & & & & \\ 2 & & & & & \\ 1 & 2 & 2 & \cdots & & 1 \end{array} \qquad + \frac{1}{h^2}\Theta(h^2)$$

Örnek

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2}$$

$$+ f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

(i) yakınsaklık
(ii) stabilite
(iii) ekonomiklik
(iv) hata

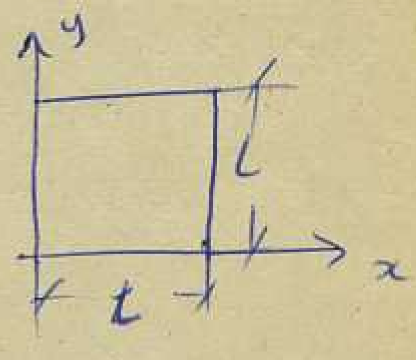
$B^2 - 4AC < 0$ eliptik
$B^2 - 4AC = 0$ parabolik
$B^2 - 4AC > 0$ hiperbolik

$\nabla^2 u = f(x,y)$ (eliptik türden d.v.)

Örnek

$$\nabla^2 z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f(x,y) = \frac{P}{S}$$

$z = 0$ sınırda



$x = \xi L$
$y = \eta L$

$$z = -\frac{PL^2}{S}\phi$$

boyutsuzlaştırma

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} + 1 = 0$$

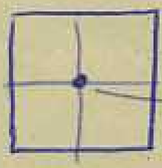
Bir sabun köpüğü problemi veya
burulma problemi

ϕ

$h^2 D^2$ $\quad \begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array}$ $\qquad h^2 O(h^2)$.

(i) $h = \frac{1}{n}$ $\quad n = 2$

$h = \frac{1}{2}$



Bu nokta ana denklemi yazalım.

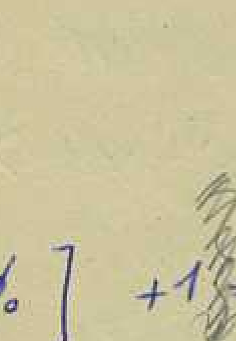
$$\frac{1}{(1/2)^2}\left[-4\phi_0\right] + 1 = 0 \quad \Rightarrow \phi_0 = \frac{1}{16} = 0.0625$$

Hata %25

$$h^2\nabla^2 = h^2\left[0.0 + 0.0 + 0.0 + 0.0 - 4\phi_0\right] + 1 \quad ?$$

(ii) $n = 3$

$h = \frac{1}{3}$



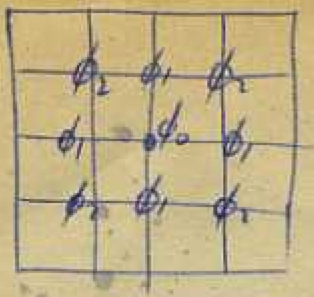
simetriden dolayı tek bilinmeyen var.

$$\frac{1}{(1/3)^2}\left[\phi_1 + \phi_1 - 4\phi_1\right] + 1 = 0$$

$$\phi_1 = \frac{1}{18} = 0.0556$$

Hata $\sim$ %11

(iii) $n = 4$ $\quad h = 1/4$ alalım.



simetri özelliği ile

$$\frac{1}{(1/4)^2}\left[4\phi_1 - 4\phi_0\right] + 1 = 0$$

$$\phi_1 - \phi_0 = -1/16$$

Bunu ϕ_0 noktası için yazdık.

ϕ_1 : $\quad \phi_0 + 2\phi_2 - 4\phi$

$$\phi_0 + 2\phi_2 - 4\phi_1 = -\frac{1}{16}$$

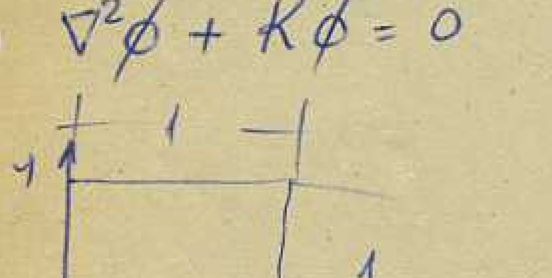
ϕ_2 : $\quad 2\phi_1 - 4\phi_2 = -\frac{1}{16}$

$\boxed{\nabla^2\phi + 1 = 0}$ çöz. galerkin ile ve diğer yöntemlerle.

ÖRNEK

Zar probleminin titreşimine karşılık gelen denklem.

$$\nabla^2\phi + K\phi = 0$$



$$h^2\nabla^2\phi + h^2K\phi = 0$$

ϕ_0'ın sıfırdan farklı olması için K ne olmalı?

$h = 1/2 \quad n = 2$ için $\quad (-4\phi_0) + \frac{1}{4}K\phi_0 = 0$

$\phi_0 \neq 0$ için $\quad K_0 = 16$

$K_E = 19.739$

$n=3$ için $h = 1/3$ $\quad (2\phi_1 - 4\phi_1) + \frac{1}{9}K\phi_1 = 0$

$K = 18$.

$n=4$ için $h = 1/4$ $\quad \phi_0: (4\phi_1 - 4\phi_0) + \frac{1}{16}K\phi_0 = 0$

$\phi_1: (\phi_0 + 2\phi_2 - 4\phi_1) + \frac{1}{16}K\phi_1 = 0$

$\phi_2: (2\phi_1 - 4\phi_2) + \frac{1}{16}K\phi_2 = 0$

$$\begin{array}{ccc} \phi_0 & \phi_1 & \phi_2 \end{array}$$

$$\begin{vmatrix} (\frac{K}{16}-4) & 4 & 0 \\ 1 & (\frac{K}{16}-4) & 2 \\ 0 & 2 & (\frac{K}{16}-4) \end{vmatrix} = 0$$

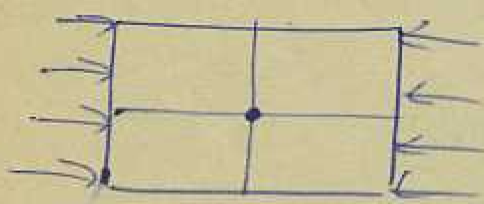
olan determinantı sıfır yapan en küçük K değeri

$K = 18.75$ hatayı 5% buluruz.

$$\nabla^4 w + \frac{N}{D}\frac{\partial^2 w}{\partial x^2} = 0$$

$\left(\frac{K}{16}-4\right)\left[\left(\frac{K}{16}-4\right)^2 - 4\right]\left(4-\left(\frac{K}{16}\ldots\right)^2\right)$

$-4\left(\frac{K}{16}-4\right)$



N

$\frac{K^2}{16} - \frac{8}{16}K + 16 - 8 = 0$

8

$y(0) = 0$; $y_0 = 0$

$y'(0) = 0$; $y_1 - y_{-1} = 0$

$y''(0) = 0$; $y_1 - 2y_0 + y_{-1} = 0$

$y'''(0) = 0$; $y_2 - 2y_1 + 2y_{-1} - y_{-2} = 0$

$\frac{1}{4} - 4 \cdot 8 \cdot \frac{1}{16}$