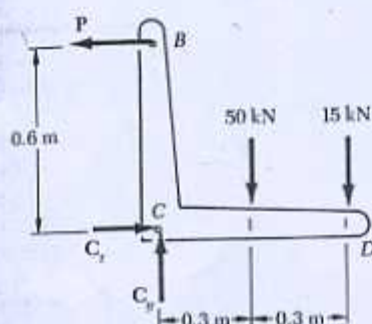


SAMPLE PROBLEM 1.2

Two forces are applied to the bracket *BCD* as shown. (a) Knowing that the control rod *AB* is to be made of a steel having an ultimate normal stress of 600 MPa, determine the diameter of the rod for which the factor of safety with respect to failure will be 3.3. (b) The pin at *C* is to be made of a steel having an ultimate shearing stress of 350 MPa. Determine the diameter of the pin *C* for which the factor of safety with respect to shear will also be 3.3. (c) Determine the required thickness of the bracket supports at *C* knowing that the allowable bearing stress of the steel used is 300 MPa.



Free Body: Entire Bracket. The reaction at *C* is represented by its components C_x and C_y .

$$+\uparrow \Sigma M_C = 0: P(0.6 \text{ m}) - (50 \text{ kN})(0.3 \text{ m}) - (15 \text{ kN})(0.6 \text{ m}) = 0 \quad P = 40 \text{ kN}$$

$$\Sigma F_x = 0: C_x = 40 \text{ kN} \quad C = \sqrt{C_x^2 + C_y^2} = 76.3 \text{ kN}$$

$$\Sigma F_y = 0: C_y = 65 \text{ kN}$$

a. **Control Rod *AB*.** Since the factor of safety is to be 3.3, the allowable stress is

$$\sigma_{\text{all}} = \frac{\sigma_U}{F.S.} = \frac{600 \text{ MPa}}{3.3} = 181.8 \text{ MPa}$$

For $P = 40 \text{ kN}$ the cross-sectional area required is

$$A_{\text{req}} = \frac{P}{\sigma_{\text{all}}} = \frac{40 \text{ kN}}{181.8 \text{ MPa}} = 220 \times 10^{-6} \text{ m}^2$$

$$A_{\text{req}} = \frac{\pi}{4} d_{AB}^2 = 220 \times 10^{-6} \text{ m}^2 \quad d_{AB} = 16.74 \text{ mm} \quad \blacktriangleleft$$

b. **Shear in Pin *C*.** For a factor of safety of 3.3, we have

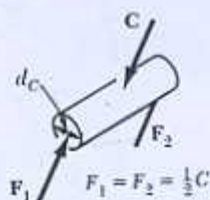
$$\tau_{\text{all}} = \frac{\tau_U}{F.S.} = \frac{350 \text{ MPa}}{3.3} = 106.1 \text{ MPa}$$

Since the pin is in double shear, we write

$$A_{\text{req}} = \frac{C/2}{\tau_{\text{all}}} = \frac{(76.3 \text{ kN})/2}{106.1 \text{ MPa}} = 360 \text{ mm}^2$$

$$A_{\text{req}} = \frac{\pi}{4} d_C^2 = 360 \text{ mm}^2$$

$$d_C = 21.4 \text{ mm} \quad \text{Use: } d_C = 22 \text{ mm} \quad \blacktriangleleft$$



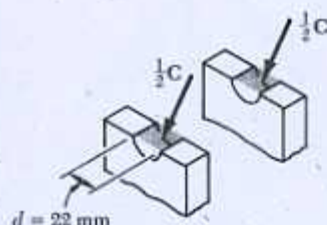
The next larger size pin available is of 22-mm diameter and should be used.

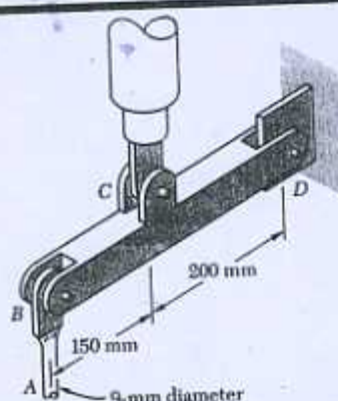
c. **Bearing at *C*.** Using $d = 22 \text{ mm}$, the nominal bearing area of each bracket is $22t$. Since the force carried by each bracket is $C/2$ and the allowable bearing stress is 300 MPa, we write

$$A_{\text{req}} = \frac{C/2}{\sigma_{\text{all}}} = \frac{(76.3 \text{ kN})/2}{300 \text{ MPa}} = 127.2 \text{ mm}^2$$

$$\text{Thus } 22t = 127.2 \quad t = 5.78 \text{ mm}$$

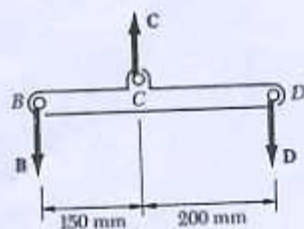
$$\text{Use: } t = 6 \text{ mm} \quad \blacktriangleleft$$





SAMPLE PROBLEM 1.3

The rigid beam BCD is attached by bolts to a control rod at B , to a hydraulic cylinder at C , and to a fixed support at D . The diameters of the bolts used are: $d_B = d_D = 8$ mm, $d_C = 12$ mm. Each bolt acts in double shear and is made from a steel for which the ultimate shearing stress is $\tau_U = 300$ MPa. The 9-mm-diameter control rod AB is made of a steel for which the ultimate tensile stress is $\sigma_U = 450$ MPa. If the minimum factor of safety is to be 3.0 for the entire unit, determine the largest upward force which may be applied by the hydraulic cylinder at C .



Solution. The factor of safety with respect to failure must be 3.0 or more in each of the three bolts and in the control rod. These four independent criteria will be considered separately.

Free Body: Beam BCD . We first determine the force at C in terms of the force at B and in terms of the force at D .

$$+\circlearrowleft \sum M_D = 0: \quad B(350 \text{ mm}) - C(200 \text{ mm}) = 0 \quad C = 1.750 B \quad (1)$$

$$+\circlearrowleft \sum M_B = 0: \quad -D(350 \text{ mm}) + C(150 \text{ mm}) = 0 \quad C = 2.33 D \quad (2)$$

Control Rod. For a factor of safety of 3.0 we have

$$\sigma_{all} = \frac{\sigma_U}{F.S.} = \frac{450 \text{ MPa}}{3.0} = 150 \text{ MPa}$$

The allowable force in the control rod is

$$B = \sigma_{all}(A) = (150 \text{ MPa}) \frac{1}{4} \pi (9 \text{ mm})^2 = 9.54 \text{ kN}$$

Using Eq. (1) we find the largest permitted value of C :

$$C = 1.750 B = 1.750(9.54 \text{ kN}) \quad C = 16.70 \text{ kN} \quad \blacktriangleleft$$

Bolt at B . $\tau_{all} = \tau_U / F.S. = (300 \text{ MPa}) / 3 = 100 \text{ MPa}$. Since the bolt is in double shear, the allowable value of force B is

$$B = \tau_{all}(2A) = (100 \text{ MPa}) \frac{2\pi}{4} (8 \text{ mm})^2 = 10.05 \text{ kN}$$

From Eq. (1): $C = 1.750 B = 1.750(10.05 \text{ kN}) \quad C = 17.59 \text{ kN} \quad \blacktriangleleft$

Bolt at D . Since this bolt is the same as bolt B , the allowable force is $D = B = 10.05 \text{ kN}$. Using Eq. (2):

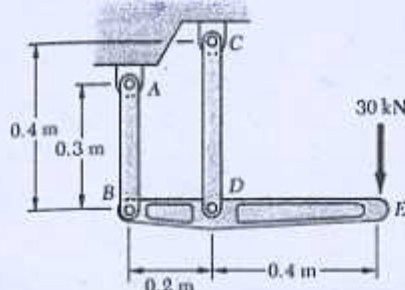
$$C = 2.33 D = 2.33(10.05 \text{ kN}) \quad C = 23.4 \text{ kN} \quad \blacktriangleleft$$

Bolt at C . We again have $\tau_{all} = 100 \text{ MPa}$ and write

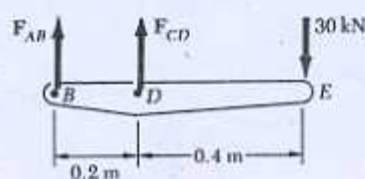
$$C = \tau_{all}(2A) = (100 \text{ MPa}) \frac{2\pi}{4} (12 \text{ mm})^2 \quad C = 22.6 \text{ kN} \quad \blacktriangleleft$$

Summary. We have found separately four maximum allowable values of the force C . In order to satisfy all these criteria we must choose the *smallest* value, namely: $C = 16.70 \text{ kN} \quad \blacktriangleleft$

SAMPLE PROBLEM 2.1



The rigid bar BDE is supported by two links AB and CD . Link AB is made of aluminum ($E = 70$ GPa) and has a cross-sectional area of 500 mm^2 ; link CD is made of steel ($E = 200$ GPa) and has a cross-sectional area of 600 mm^2 . For the 30-kN force shown, determine the deflection (a) of B , (b) of D , (c) of E .



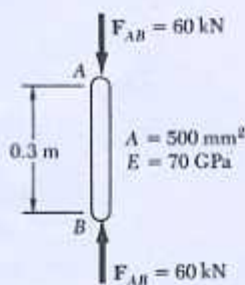
Free Body: Bar BDE

$$+\circlearrowleft \sum M_B = 0: \quad -(30 \text{ kN})(0.6 \text{ m}) + F_{CD}(0.2 \text{ m}) = 0$$

$$F_{CD} = +90 \text{ kN} \quad F_{CD} = 90 \text{ kN} \text{ tension}$$

$$+\circlearrowleft \sum M_D = 0: \quad -(30 \text{ kN})(0.4 \text{ m}) - F_{AB}(0.2 \text{ m}) = 0$$

$$F_{AB} = -60 \text{ kN} \quad F_{AB} = 60 \text{ kN} \text{ compression}$$

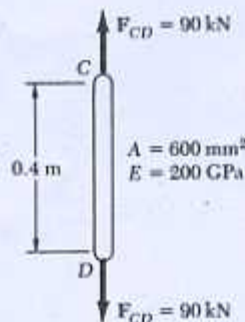


a. Deflection of B . Since the internal force in link AB is compressive, we have $P = -60 \text{ kN}$

$$\delta_B = \frac{PL}{AE} = \frac{(-60 \times 10^3 \text{ N})(0.3 \text{ m})}{(500 \times 10^{-6} \text{ m}^2)(70 \times 10^9 \text{ Pa})} = -514 \times 10^{-6} \text{ m}$$

The negative sign indicates a contraction of member AB , and, thus, an upward deflection of end B :

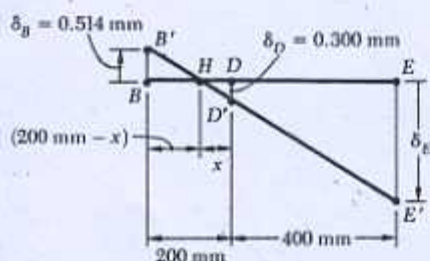
$$\delta_B = 0.514 \text{ mm} \uparrow$$



b. Deflection of D . Since in rod CD , $P = 90 \text{ kN}$, we write

$$\delta_D = \frac{PL}{AE} = \frac{(90 \times 10^3 \text{ N})(0.4 \text{ m})}{(600 \times 10^{-6} \text{ m}^2)(200 \times 10^9 \text{ Pa})} = 300 \times 10^{-6} \text{ m}$$

$$\delta_D = 0.300 \text{ mm} \downarrow$$



c. Deflection of E . We denote by B' and D' the displaced positions of points B and D . Since the bar BDE is rigid, points B' , D' , and E' lie in a straight line and we write

$$\frac{BB'}{DD'} = \frac{BH}{HD} \quad \frac{0.514 \text{ mm}}{0.300 \text{ mm}} = \frac{(200 \text{ mm}) - x}{x} \quad x = 73.7 \text{ mm}$$

$$\frac{EE'}{DD'} = \frac{HE}{HD} \quad \frac{\delta_E}{0.300 \text{ mm}} = \frac{(400 \text{ mm}) + (73.7 \text{ mm})}{73.7 \text{ mm}}$$

$$\delta_E = 1.928 \text{ mm} \downarrow$$

2.9. STATICALLY INDETERMINATE PROBLEMS

In the problems considered in the preceding section, we could always use free-body diagrams and equilibrium equations to determine the internal forces produced in the various portions of a member under given loading conditions. The values obtained for the internal forces were then substituted into Eqs. (2.8) or (2.9) to compute the deformation δ of the member.

There are many problems, however, in which the internal forces cannot be determined from statics alone. In fact, in most of these problems the reactions themselves—which are external forces—cannot be determined by simply drawing a free-body diagram of the member and writing the corresponding equilibrium equations. The equilibrium equations must be complemented by relations involving deformations obtained by considering the geometry of the problem. Because statics is not sufficient to determine either the reactions or the internal forces, problems of this type are said to be *statically indeterminate*. The following examples will show how to handle this type of problem.

Example 2.02

A rod of length L , cross-sectional area A_1 , and modulus of elasticity E_1 , has been placed inside a tube of the same length L , but of cross-sectional area A_2 and modulus of elasticity E_2 (Fig. 2.23a). What is the deformation of the rod and tube when a force P is exerted on a rigid end plate as shown?

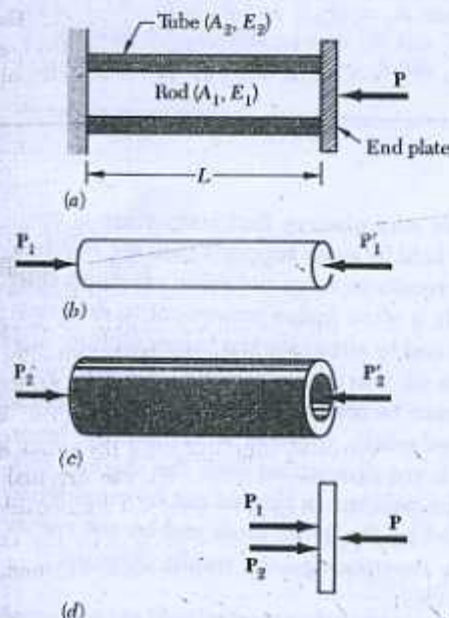


Fig. 2.23

Denoting by P_1 and P_2 , respectively, the axial forces in the rod and in the tube, we draw free-body diagrams of all three elements (Fig. 2.23b, c, d). Only the last of the diagrams yields any significant information, namely:

$$P_1 + P_2 = P \quad (2.11)$$

Clearly, one equation is not sufficient to determine the two unknown internal forces P_1 and P_2 . The problem is statically indeterminate.

However, the geometry of the problem shows that the deformations δ_1 and δ_2 of the rod and tube must be equal. Recalling Eq. (2.7), we write

$$\delta_1 = \frac{P_1 L}{A_1 E_1} \quad \delta_2 = \frac{P_2 L}{A_2 E_2} \quad (2.12)$$

Equating the deformations δ_1 and δ_2 , we obtain:

$$\frac{P_1}{A_1 E_1} = \frac{P_2}{A_2 E_2} \quad (2.13)$$

Equations (2.11) and (2.13) may be solved simultaneously for P_1 and P_2 :

$$P_1 = \frac{A_1 E_1 P}{A_1 E_1 + A_2 E_2} \quad P_2 = \frac{A_2 E_2 P}{A_1 E_1 + A_2 E_2}$$

Either of Eqs. (2.12) may then be used to determine the common deformation of the rod and tube.

Example 2.03

A bar AB of length L and uniform cross section is attached to rigid supports at A and B before being loaded. What are the stresses in portions AC and BC due to the application of a load P at point C (Fig. 2.24a)?

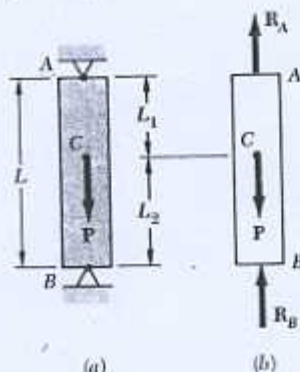


Fig. 2.24

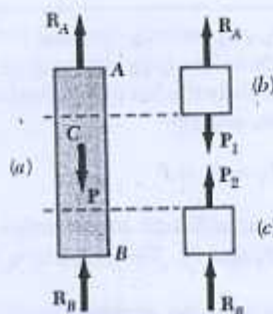


Fig. 2.25

Drawing the free-body diagram of the bar (Fig. 2.24b), we obtain the equilibrium equation

$$R_A + R_B = P \quad (2.14)$$

Since this equation is not sufficient to determine the two unknown reactions R_A and R_B , the problem is statically indeterminate.

However, the reactions may be determined if we observe from the geometry that the total elongation δ of the bar must be zero. Denoting by δ_1 and δ_2 , respectively, the elongations of the portions AC and BC , we write

$$\delta = \delta_1 + \delta_2 = 0$$

or, expressing δ_1 and δ_2 in terms of the corresponding internal forces P_1 and P_2 :

$$\delta = \frac{P_1 L_1}{AE} + \frac{P_2 L_2}{AE} = 0 \quad (2.15)$$

But we note from the free-body diagrams shown respectively in parts b and c of Fig. 2.25 that $P_1 = R_A$ and $P_2 = -R_B$. Carrying these values into (2.15), we write

$$R_A L_1 - R_B L_2 = 0 \quad (2.16)$$

Equations (2.14) and (2.16) may be solved simultaneously for R_A and R_B ; we obtain $R_A = PL_2/L$ and $R_B = PL_1/L$. The desired stresses in AC and BC may be obtained by dividing, respectively, $P_1 = R_A$ and $P_2 = -R_B$ by the cross-sectional area of the bar.

Superposition Method. We may observe that a structure is statically indeterminate whenever it is held by more supports than are required to maintain its equilibrium. This results in more unknown reactions than available equilibrium equations. It is often found convenient to designate one of the reactions as *redundant* and to eliminate the corresponding support. Since the stated conditions of the problem cannot be arbitrarily changed, the redundant reaction must be maintained in the solution. But it will be treated as an *unknown load* which, together with the other loads, must produce deformations which are compatible with the original constraints. The actual solution of the problem is carried out by considering separately the deformations caused by the given loads and by the redundant reaction, and by adding—or *superposing*—the results obtained.†

† The general conditions under which the combined effect of several loads may be obtained in this way are discussed in Sec. 2.12.

Example 2.04

Determine the reactions at A and B for the steel bar and loading shown in Fig. 2.26, assuming a close fit at both supports before the loads are applied.

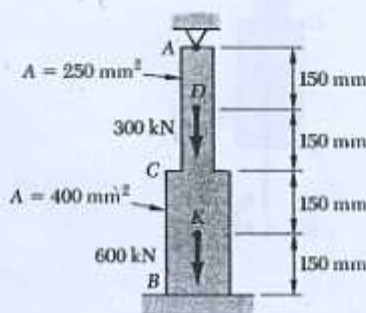


Fig. 2.26

We shall consider the reaction at B as redundant and release the bar from that support. The reaction R_B is now considered as an unknown load (Fig. 2.27a) and will be determined from the condition that the deformation δ of the rod must be equal to zero. The solution is carried out by considering separately the deformation δ_L caused by the given loads (Fig. 2.27b) and the deformation δ_R due to the redundant reaction R_B (Fig. 2.27c).

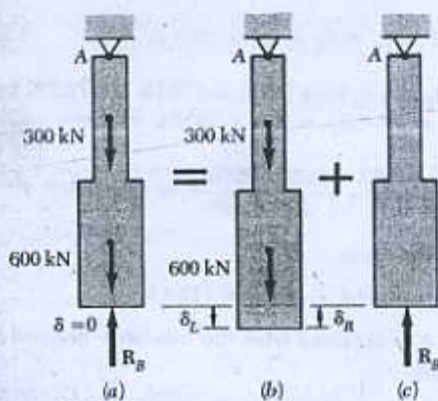


Fig. 2.27

The deformation δ_L is obtained from Eq. (2.8) after the bar has been divided into four portions, as shown in Fig. 2.28.

Following the same procedure as in Example 2.01, we write

$$\begin{aligned} P_1 &= 0 & P_2 &= P_3 = 600 \times 10^3 \text{ N} & P_4 &= 900 \times 10^3 \text{ N} \\ A_1 &= A_2 = 400 \times 10^{-6} \text{ m}^2 & A_3 &= A_4 = 250 \times 10^{-6} \text{ m}^2 \\ L_1 &= L_2 = L_3 = L_4 = 0.150 \text{ m} \end{aligned}$$

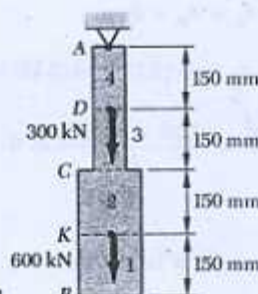


Fig. 2.28

Substituting these values into Eq. (2.8), we obtain

$$\begin{aligned} \delta_L &= \sum_{i=1}^4 \frac{P_i L_i}{A_i E} = \left(0 + \frac{600 \times 10^3 \text{ N}}{400 \times 10^{-6} \text{ m}^2} \right. \\ &\quad \left. + \frac{600 \times 10^3 \text{ N}}{250 \times 10^{-6} \text{ m}^2} + \frac{900 \times 10^3 \text{ N}}{250 \times 10^{-6} \text{ m}^2} \right) \frac{0.150 \text{ m}}{E} \\ \delta_L &= \frac{1.125 \times 10^9}{E} \end{aligned} \quad (2.17)$$

Considering now the deformation δ_R due to the redundant reaction R_B , we divide the bar into two portions, as shown in Fig. 2.29, and write

$$\begin{aligned} P_1 &= P_2 = -R_B \\ A_1 &= 400 \times 10^{-6} \text{ m}^2 & A_2 &= 250 \times 10^{-6} \text{ m}^2 \\ L_1 &= L_2 = 0.300 \text{ m} \end{aligned}$$

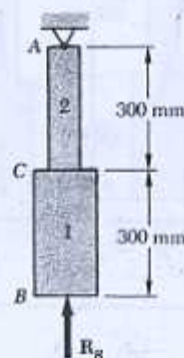


Fig. 2.29

Substituting these values into Eq. (2.8), we obtain

$$\delta_R = \frac{P_1 L_1}{A_1 E} + \frac{P_2 L_2}{A_2 E} = -\frac{(1.95 \times 10^3) R_B}{E} \quad (2.18)$$

Expressing that the total deformation δ of the bar must be zero, we write

$$\delta = \delta_L + \delta_R = 0 \quad (2.19)$$

and, substituting for δ_L and δ_R from (2.17) and (2.18) into (2.19),

$$\delta = \frac{1.125 \times 10^9}{E} - \frac{(1.95 \times 10^3) R_B}{E} = 0$$

Solving for R_B , we have

$$R_B = 577 \times 10^3 \text{ N} = 577 \text{ kN}$$

The reaction R_A at the upper support is obtained from the free-body diagram of the bar (Fig. 2.30). We write

$$\begin{aligned} \Sigma F_y = 0: \quad R_A - 300 \text{ kN} - 600 \text{ kN} + R_B &= 0 \\ R_A &= 900 \text{ kN} - R_B = 900 \text{ kN} - 577 \text{ kN} = 323 \text{ kN} \end{aligned}$$

Example 2.05

Determine the reactions at A and B for the steel bar and loading of Example 2.04, assuming now that a 4.50-mm clearance exists between the bar and the ground before the loads are applied (Fig. 2.31). Assume $E = 200 \text{ GPa}$.

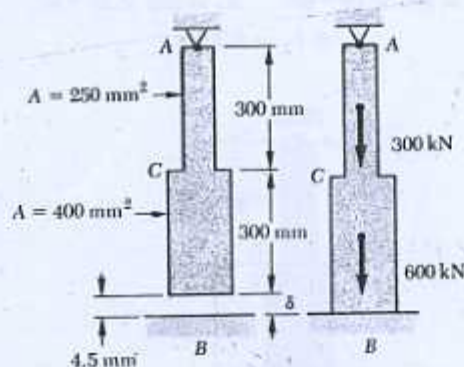


Fig. 2.31

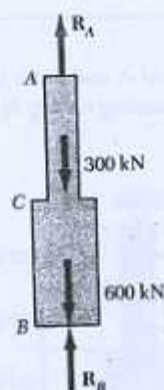


Fig. 2.30

Once the reactions have been determined, the stresses and strains in the bar may easily be obtained. It should be noted that, while the total deformation of the bar is zero, each of its component parts *does* deform under the given loading and restraining conditions.

We follow the same procedure as in Example 2.04. Considering the reaction at B as redundant, we compute the deformations δ_L and δ_R caused respectively by the given loads and by the redundant reaction R_B . However, in this case the total deformation is not zero, but $\delta = 4.5 \text{ mm}$. We write therefore

$$\delta = \delta_L + \delta_R = 4.5 \times 10^{-3} \text{ m} \quad (2.20)$$

Substituting for δ_L and δ_R from (2.17) and (2.18) into (2.20), and recalling that $E = 200 \text{ GPa} = 200 \times 10^9 \text{ Pa}$, we have

$$\delta = \frac{1.125 \times 10^9}{200 \times 10^9} - \frac{(1.95 \times 10^3) R_B}{200 \times 10^9} = 4.5 \times 10^{-3}$$

Solving for R_B , we obtain

$$R_B = 115.4 \times 10^3 \text{ N} = 115.4 \text{ kN}$$

The reaction at A is obtained from the free-body diagram of the bar (Fig. 2.30):

$$\begin{aligned} \Sigma F_y = 0: \quad R_A - 300 \text{ kN} - 600 \text{ kN} + R_B &= 0 \\ R_A &= 900 \text{ kN} - R_B = 900 \text{ kN} - 115.4 \text{ kN} = 785 \text{ kN} \end{aligned}$$