in the x direction and $\delta_x^n z_i$ the nth central difference of z at i taken in the x direction,

$$2hD_x z_i = z_r - z_l + 2\epsilon_{1x} \qquad \left[\epsilon_{1x} = \mu\left(-\frac{\delta_x^3}{6} + \frac{\delta_x^5}{30} - \dots\right) z_i\right], \quad (5.2.1)$$

$$h^2D_x^2 z_i = z_r - 2z_i + z_l + \epsilon_{2x} \qquad \left[\epsilon_{2x} = \left(-\frac{\delta_x^4}{12} + \frac{\delta_x^6}{90} - \dots\right) z_i\right], \quad (5.2.2)$$

$$2h^3D_x^3 z_i = z_{rr} - 2z_r + 2z_l - z_{ll} + 2\epsilon_{3x}$$
$$\left[\epsilon_{3x} = \mu\left(-\frac{\delta_x^5}{4} + \frac{7\delta_x^7}{120} - \dots\right) z_i\right], \quad (5.2.3)$$

$$h^4D_x^4 z_i = z_{rr} - 4z_r + 6z_i - 4z_l + z_{ll} + \epsilon_{4x}$$
$$\left[\epsilon_{4x} = \left(-\frac{\delta_x^6}{6} + \frac{7\delta_x^8}{240} - \dots\right) z_i\right]. \quad (5.2.4)$$

Similarly, calling k the constant spacing of the pivotal points in the y direction and $\delta_y^n z_i$ the nth central difference of z at i taken in the y direction, and indicating the pivotal points adjoining z_i vertically by z_{aa}, z_a, z_b, and z_{bb}, as shown in Fig. 5.2 (the subscripts a and b stand for "above" and "below"), the partial derivatives with respect to y are given by

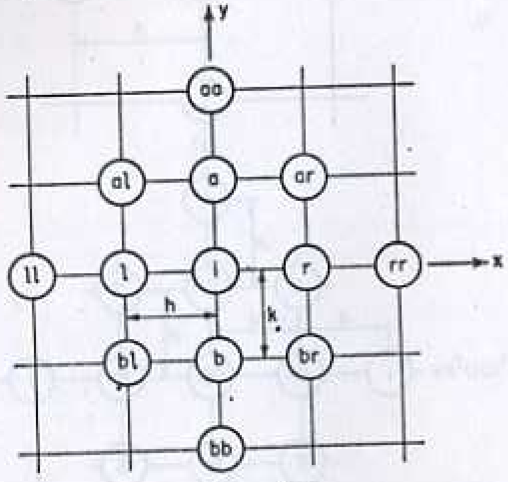

Figure 5.2

$$2kD_y z_i = z_a - z_b + 2\epsilon_{1y} \qquad \left[\epsilon_{1y} = \mu\left(-\frac{\delta_y^3}{6} + \frac{\delta_y^5}{30} - \dots\right) z_i\right], \quad (5.2.5)$$

$$k^2D_y^2 z_i = z_a - 2z_i + z_b + \epsilon_{2y} \qquad \left[\epsilon_{2y} = \left(-\frac{\delta_y^4}{12} + \frac{\delta_y^6}{90} - \dots\right) z_i\right], \quad (5.2.6)$$

$$2k^3D_y^3 z_i = z_{aa} - 2z_a + 2z_b - z_{bb} + 2\epsilon_{3y}$$
$$\left[\epsilon_{3y} = \mu\left(-\frac{\delta_y^5}{4} + \frac{7\delta_y^7}{120} - \dots\right) z_i\right], \quad (5.2.7)$$

$$k^4D_y^4 z_i = z_{aa} - 4z_a + 6z_i - 4z_b + z_{bb} + \epsilon_{4y}$$
$$\left[\epsilon_{4y} = \left(-\frac{\delta_y^6}{6} + \frac{7\delta_y^8}{240} - \dots\right) z_i\right]. \quad (5.2.8)$$

The expression for the second mixed derivative of z with respect to x and y, D_{xy}, is obtained by applying the operator giving D_x to the operator giving D_y, that is, by the "product" D_xD_y:

$$D_{xy} z_i = \frac{1}{2k}\left[\frac{1}{2h}(z_r - z_l)_a - \frac{1}{2h}(z_r - z_l)_b\right] + \frac{1}{2hk}\epsilon_{1,xy}$$

or

$$4hkD_{xy} z_i = z_{ar} - z_{al} - z_{br} + z_{bl} + 2\epsilon_{1,xy}$$
$$[-\tfrac{1}{6}(u\delta_x u\delta_y^3 + u\delta_y u\delta_x^3) + \dots]z_i. \quad (5.2.9)$$

Similarly, the fourth mixed derivative $\partial^4 z/\partial x^2\partial y^2 \equiv D_{xxyy}$ is the operational product of D_x^2 and D_y^2, or

$$h^2k^2D_{xxyy} z_i = (z_r - 2z_i + z_l)_a - 2(z_r - 2z_i + z_l)_i + (z_r - 2z_i + z_l)_b + \epsilon_{2,xy}$$
$$= (z_{ar} + z_{al} + z_{br} + z_{bl}) - 2(z_a + z_b + z_r + z_l) + 4z_i + \epsilon_{2,xy} \qquad [-\tfrac{1}{12}(\delta_x^2\delta_y^4 + \delta_y^2\delta_x^4) + \dots]z_i. \quad (5.2.10)$$

The *Laplacian (or harmonic) operator* ∇^2*

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = D_x^2 + D_y^2$$

becomes by Eqs. (5.2.2), (5.2.6) for a *rectangular lattice of* mesh sizes h, k

$$h^2k^2\nabla^2 z_i = k^2(z_r - 2z_i + z_l) + h^2(z_a - 2z_i + z_b) + k^2\epsilon_{2x} + h^2\epsilon_{2y} \quad (5.2.11)$$

and for the particular case of equal spacing of the pivotal points in the x and y directions, that is, for a *square lattice* of mesh size h,

$$h^2\nabla^2 z_i = z_a + z_b + z_r + z_l - 4z_i + \epsilon_{2x} + \epsilon_{2y}, \quad (5.2.12)$$

where ϵ_{2x} and ϵ_{2y} are given by Eqs. (5.2.2) and (5.2.6), respectively. The *biharmonic operator*

$$\nabla^4 \equiv \nabla^2(\nabla^2) = \frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}\dagger$$

* The operator ∇^2 should not be confused with the second backward difference.
† The operator ∇^4 should not be confused with the fourth backward difference.

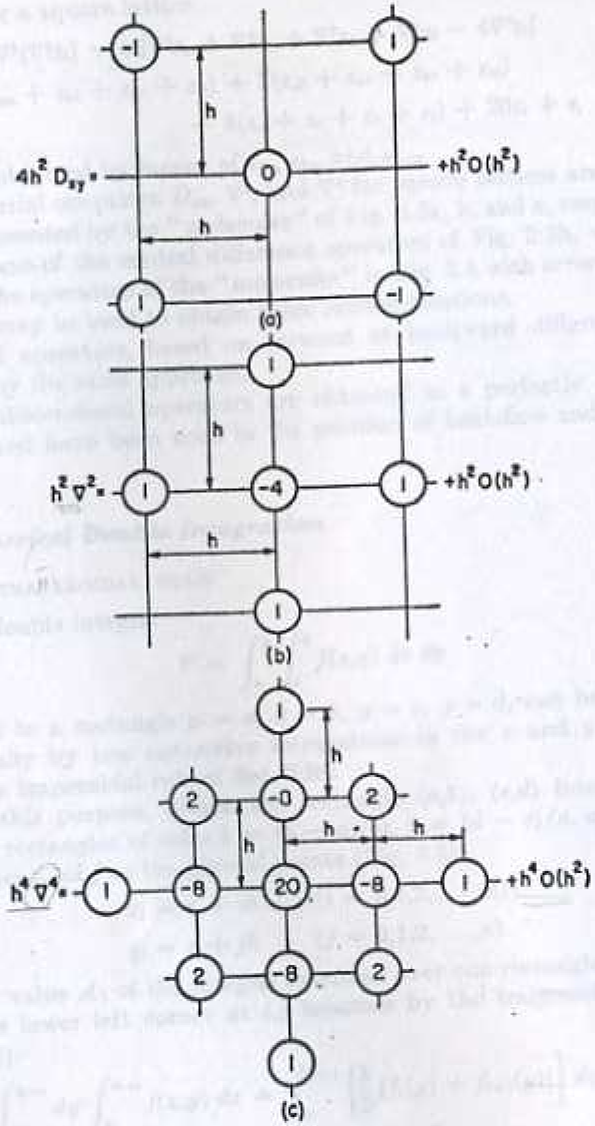

Fig. 5.3. Central difference partial operators.

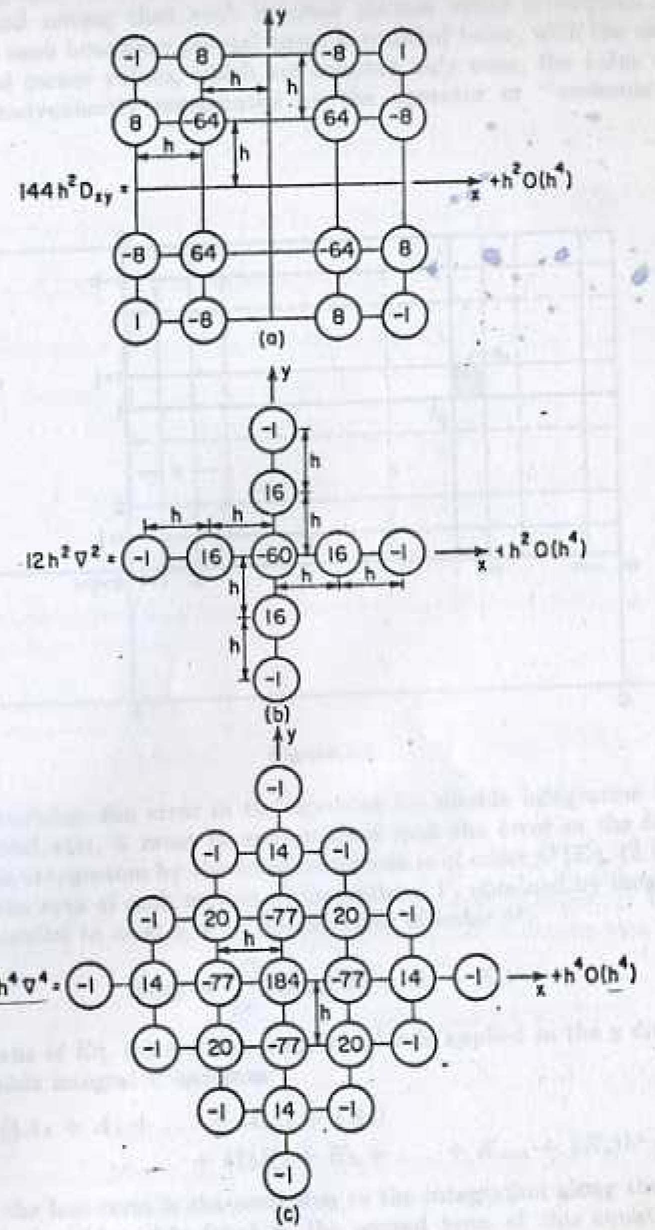

Fig. 5.4. Second-order operators.