**Optimization**

[This is a simplified version of Simon and Blume’s “Mathematics for Economists” Chapter 17-18]

Consider a function where and . This means and is a scalar. Suppose that we are interested in finding which satisfies the following condition:

for any .

This problem can be stated as

where the solution is denoted by , if there is any.

**Theorem:** Suppose that solves Then,

This is called “First Order Necessary Condition” (fonc). Here is the derivative of with respect to , and is known as the Jacobian of . As usual, is the value of the derivative at point .

**Remark:** Other forms of denoting the derivative are

**Example:** Suppose that

Note that at But Therefore, does not give us the maximum for this example.

This example shows that FONC is not sufficient. Therefore, let us ask what the sufficient conditions are. To see this, let us first define

as the “Hessian of ” which is simply the matrix of second order derivatives, where

**Theorem:** and for all is negative definite if and only if is a local-maximum of .

**Definition:**  is negative definite if principle minors of change sign and the first principle minor is negative.

**Example:** Therefore,

This means:

The first principle minor is and the second principle minor is

**Definition:** If is a local maximum of , then there is a neighborhood around where is the solution to the maximization problem:

**Remark:** If is a concave function, then gives the global maximum. That is because, concavity ensures that is negative definite for all .

Now let us analyze a typical constraint optimization problem:

where . The first order necessary conditions for this problem can be calculated by using the Lagrangian:

where and is the number of constraints.

**Theorem:** If solves

then

for some .

Example: and . If we want to solve

we can write the Lagrangian

Note that

The solution is

**Remark:** If is concave and is linear, then the Lagrange conditions are also sufficient.

**Remark:** gives us the increase in if the constraint was slackened by 1 unit.